DM8

Existence

It is enough to prove it exists for every interval $[0,\alpha]$, $\alpha>0$. This because the uniqueness tells us that the decompositions must agree where they should — for differing α_1, α_2 — and we can therefore define the decomposition for any particular $t \in \mathbb{R}^+$ simply by refering to the decomposition for any $\alpha > t$.

So fix α. For $0 \leq t \leq \alpha$, $(M_t(X_\alpha))$, is a martingale with respect to the (appropriate) filtration. It admits a right continuous modification. Choose one. Consider

$$Y_t = X_t - M_t(X_\alpha) \qquad 0 \leq t \leq \alpha,$$

where $(M_t(X_\alpha))$ denotes the chosen modification.

Let $\pi^n = \{t_i^n\}$ denote a sequence of partitions of $[0,\alpha]$. Using the discrete analysis from the begining; for each $n \in \mathbb{N}$ we have the (Doob) decompositions,

$$Y_{t_i^n} = U^n_{t_i^n} + A^n_{t_i^n}$$

where $(A^n_{t_i^n})$ is a predictable increasing process.

Now $Y_\alpha = 0$ so $U^n_{t_n^n} = -A^n_{t_n^n}$ $\quad$ (t_n^n is just α!)

as a consequence,

$$Y_{t_i^n} = A^n_{t_i^n} - M_{t_i^n}(A^n_\alpha)$$

We can take π^n to be an increasing sequence of partitions with $\|\pi^n\| \to 0$ as $n \to \infty$. For instance, $\pi^n = \{j\alpha/2^n\}$, $0 \leq j \leq 2^n$.

DM9

What we do now is to show the sequence $\{A^n_\alpha : n \in \mathbb{N}\}$ is uniformly integrable[(†)]. Let $\lambda > 0$, and

$$T^n_\lambda = \alpha \wedge \min\{t^n_{i-1} : A^n_{t^n_i} > \lambda \text{ for some } i\}.^{**}$$

Then for a particular index, i, $\omega \in \{T^n_\lambda(\omega) \le t^n_{i-1}\} \Leftrightarrow A^n_{t^n_j}(\omega) > \lambda$ for some $j \le i-1$ and therefore — A^n is increasing — $A^n_{t^n_i}(\omega) > \lambda$. On the other hand, if $\omega \in \{A^n_{t^n_i}(\omega) > \lambda\}$ then $T^n_\lambda(\omega) \le t^n_{i-1}$. So $\{T^n_\lambda(\omega) \le t^n_{i-1}\} = \{A^n_{t^n_i}(\omega) > \lambda\}$ and since A^n is predictable $\{T^n_\lambda(\omega) \le t^n_{i-1}\} \in \mathcal{F}_{t^n_{i-1}}$.

[[So T^n_λ is a stopping time, $\mathbb{P}$-a.s. less than α.]] Whoops!

Note also $\{T^n_\lambda < \alpha\} = \{A^n_{t^n_n} > \lambda\} \equiv \{A^n_\alpha > \lambda\}$ (cos' $t^n_n = \alpha$)

This bit goes here.

Recall the definition of the σ-field $\mathcal{F}_{T^n_\lambda}$; $\{E \in \mathcal{F} : E \cap \{T^n_\lambda \le t\} \in \mathcal{F}_t, \text{ for all } t\}$; it is not difficult to prove that the conditional expectation $M_{T^n_\lambda}$ may be written

$$M_{T^n_\lambda} = \sum_{\tau \in \tau^n} M_{t^n_i}(\cdot) I_{\{T^n_\lambda = t^n_i\}}$$

So, on $\{T^n_\lambda = t^n_i\}$ the expectation $M_{T^n_\lambda}$ agrees with $M_{t^n_i}$. Now, recall that earlier we wrote

$$Y_{t^n_i} = A^n_{t^n_i} - M_{t^n_i}(A^n_\alpha)$$

(†) Actually, what is used is weak relative compactness. I've got a direct proof, somewhere. Can't remember how it goes.

** T^n_λ is the "last time" A^n is less than λ and is α if it's never greater.

DM10

It follows that

$$Y_{T^n_\lambda} = \sum_{\pi^n} Y_{t^n_i} I_{\{T^n_\lambda = t^n_i\}} = \sum_{\pi^n} A^n_{t^n_i} I_{\{T^n_\lambda = t^n_i\}} - \sum_{\pi^n} M_{t^n_i}(A^n_\alpha) I_{\{T^n_\lambda = t^n_i\}}$$

so that

$$Y_{T^n_\lambda} = A^n_{T^n_\lambda} - M_{T^n_\lambda}(A^n_\alpha)$$

$$\overset{(\dagger)}{\leq} \lambda - M_{T^n_\lambda}(A^n_\alpha)$$

So, as $\{T^n_\lambda < \alpha\} = \{A^n_\alpha > \lambda\}$,

$$\int_{\{T^n_\lambda < \alpha\}} Y_{T^n_\lambda} \, d\mathbb{P} \leq \lambda \mathbb{P}\{T^n_\lambda < \alpha\} - \int_{\{T^n_\lambda < \alpha\}} M_{T^n_\lambda}(A^n_\alpha) \, d\mathbb{P}$$

since $\{T^n_\lambda < \alpha\} \in \mathcal{F}_{T^n_\lambda}$ we can remove $M_{T^n_\lambda}$ and rewrite and rearrage to get;

$$\int_{\{A^n_\alpha > \lambda\}} A^n_\alpha \, d\mathbb{P} = \lambda \mathbb{P}\{T^n_\lambda < \alpha\} - \int_{\{T^n_\lambda < \alpha\}} Y_{T^n_\lambda} \, d\mathbb{P}$$

The term on the left is what we're interested in. Uniform integrability is, roughly speaking, that the integrals of the family, $(A^n_\alpha)_{n \in \mathbb{N}}$, over a 'small' set should be (uniformly) small. If we can show that the left side above is small – uniformly in $n \in \mathbb{N}$, while λ gets large then this is enough......

($\dagger$) Remember, T^n_λ is the last time A^n is less than λ

To do this we need some estimate of the right side of the last equation. Well, recall

$$Y_{T^n_\lambda} = A^n_{T^n_\lambda} - M_{T^n_\lambda}(A^n_\alpha) .$$

<u>Trick</u> : Replace λ with $\frac{\lambda}{2}$ (λ was simply a positive value) and integrate over the $\mathcal{F}_{T^n_{\lambda/2}}$ set, $\{T^n_{\lambda/2} < \alpha\}$, to get

$$-\int_{\{T^n_{\lambda/2} < \alpha\}} Y_{T^n_{\lambda/2}} \, d\mathbb{P} = \int_{\{T^n_{\lambda/2} < \alpha\}} \overbrace{\left(A^n_\alpha - A^n_{T^n_{\lambda/2}}\right)}^{\text{this is non-negative}} d\mathbb{P}$$

Now $\{T^n_{\lambda/2} < \alpha\} \supseteq \{T^n_\lambda < \alpha\}$ (because; if $T^n_\lambda(\omega) = t^n_i$ – the last time $A^n_\bullet(\omega)$ is less than λ – then the last time it was less than $\lambda/2$ occurred no later than this). So,

$$-\int_{\{T^n_{\lambda/2} < \alpha\}} Y_{T^n_{\lambda/2}} \, d\mathbb{P} \geq \int_{\{T^n_\lambda < \alpha\}} \left(A^n_\alpha - A^n_{T^n_{\lambda/2}}\right) d\mathbb{P}$$

$$= \int_{\{A^n_\alpha > \lambda\}} A^n_\alpha \, d\mathbb{P} - \int_{\{T^n_\lambda < \alpha\}} A^n_{T^n_{\lambda/2}} \, d\mathbb{P}$$

The first of the right hand terms is greater than $\lambda \mathbb{P}\{T^n_\lambda < \alpha\}$ while the second is smaller than $\frac{\lambda}{2} \mathbb{P}\{T^n_\lambda < \alpha\}$ so the difference on the right side above is greater than

$$\lambda \mathbb{P}\{T^n_\lambda < \alpha\} - \frac{\lambda}{2} \mathbb{P}\{T^n_\lambda < \alpha\} = \frac{\lambda}{2} \mathbb{P}\{T^n_\lambda < \alpha\}.$$

So we now have

$$-\int_{\{T^n_{\lambda/2} < \alpha\}} Y_{T^n_{\lambda/2}} \, d\mathbb{P} \geq \frac{\lambda}{2} \mathbb{P}\{T^n_\lambda < \alpha\}$$

This gives us an estimate for $\lambda \mathbb{P}\{T^n_\lambda < \alpha\}$! :

$$\lambda \mathbb{P}\{T^n_\lambda < \alpha\} \leq -2 \int_{\{T^n_{\lambda/2} < \alpha\}} Y_{T^n_{\lambda/2}} \, d\mathbb{P},$$

giving the estimate

$$\int_{\{A^n_\alpha > \lambda\}} A^n_\alpha \, d\mathbb{P} \leq -2 \int_{\{T^n_{\lambda/2} < \alpha\}} Y_{T^n_{\lambda/2}} \, d\mathbb{P} - \int_{\{T^n_\lambda < \alpha\}} Y_{T^n_\lambda} \, d\mathbb{P}$$

By hypothesis $\{X_T : T \text{ is a stop time} \leq \alpha\}$ is uniformly integrable. Since $Y_t = X_t - M_t(X_\alpha)$ then $\{Y_T : T \text{ is a stop time} \leq \alpha\}$ is also uniformly integrable, so the sequences $(Y_{T^n_{\lambda/2}})$ and $(Y_{T^n_\lambda})$ are uniformly integrable. Finally,

$$\mathbb{P}\{T^n_\lambda < \alpha\} = \mathbb{P}\{A^n_\alpha > \lambda\} \leq \frac{\mathbb{E}(A^n_\alpha)}{\lambda} = \frac{\mathbb{E}(Y_0)}{\lambda}$$

(the last bit is because $\lambda I_{\{A^n_\alpha > \lambda\}} \leq A^n_\alpha I_{\{A^n_\alpha > \lambda\}}$ (integrate) and $Y_{t^n_i} = A^n_{t^n_i} - M_{t^n_i}(A^n_\alpha)$ so $Y_0 = A^n_0 - M_0(A^n_\alpha)$, $A^n_0 \equiv 0$, take expectations)

This last inequality shows that $\mathbb{P}\{T^n_\lambda < \alpha\} \to 0$ as $\lambda \to \infty$ uniformly in n, this estimates the term,

$$\int_{\{A^n_\alpha > \lambda\}} A^n_\alpha \, d\mathbb{P} ,$$

showing it to be small (uniformly in $n \in \mathbb{N}$) for large values of λ. As we remarked previously this proves that (A^n_α) is uniformly integrable. Now uniform integrability entails weak relative compactness(†) which means that there is an integrable random variable, A_α, and a subsequence $(A^{n_j}_\alpha)$ converging weakly, in L^1, to A_α. This means

$$\lim_j \mathbb{E}(A^{n_j}_\alpha B) = \mathbb{E}(A_\alpha B)$$

for $B \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Now we define

$$A_t = Y_t + M_t(A_\alpha) \qquad 0 \leq t \leq \alpha.$$

Now let $\pi = \bigcup_n \pi^n$, a dense subset of $[0, \alpha]$. Recalling that

$$Y_{t^n_i} = A^n_{t^n_i} - M_{t^n_i}(A^n_\alpha)$$

for every t^n_i (and every $n \in \mathbb{N}$) we note that for any

(†) See Dunford & Schwartz : Linear Operators Vol 1, but don't linger, the point is what follows next.

DM14

$t \in \pi$, $\lim_{n_j} \mathbb{E}(A^{n_j}_t B) = \mathbb{E}(A_t B)$ for $B \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

This is obvious but the notation obscures the fact: The partitions π^n get larger and larger as n gets big. So a $t \in \pi^n$ occurs in π^{n+1}, π^{n+2}, etc, but it will have different 'names' for each $n \in \mathbb{N}$, it will be t^n_i, say, in π^n, but it will be $t^{n+1}_{i'}$ in π^{n+1}, $t^{n+2}_{i''}$ in π^{n+2}, and so on. So for this bit of the argument lets say that $t = t^n_{i(n)}$ in π^n. By definition,

take the right cts modification

$$A_t = Y_t + M_t(A_\alpha)$$

$$= A^n_{t^n_{i(n)}} - M_{t^n_{i(n)}}(A^n_\alpha) + M_t(A_\alpha)$$

of course $\underline{t^n_{i(n)} \text{ is always } t}$, but, a big but, $A^n_{t^n_{i(n)}}$ changes with n. Now for $B \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, we have, in particular

$$\mathbb{E}(A_t B) = \mathbb{E}(A^{n_j}_{t^{n_j}_{i(n_j)}} B) - \mathbb{E}(M_{t^{n_j}_{i(n_j)}}(A^{n_j}_\alpha) B)$$

$$\mathbb{E}(M_t(A_\alpha) B)$$

lets drop the notational difference between t and $t^n_{i(n)}$, we get,

$$\mathbb{E}(A_t B) = \mathbb{E}(A^{n_j}_t B) - \mathbb{E}(M_t(A^{n_j}_\alpha) B) - \mathbb{E}(M_t(A_\alpha) B)$$

$$= \mathbb{E}(A^{n_j}_t B) - \mathbb{E}((A^{n_j}_\alpha - A_\alpha) M_t(B))$$

Of course $M_t(B) \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ so,

$$\lim_{n_j} \mathbb{E}\left((A^{n_j}_\alpha - A_\alpha) M_t(B) \right) = 0$$

therefore $\mathbb{E}(A_t B) = \lim_{n_j} \mathbb{E}(A^{n_j}_t B)$. So now let $0 \leq s \leq t \leq \alpha$, $s, t \in \Pi$, and B as before but non negative

$$\mathbb{E}((A_t - A_s) B) = \lim_{n_j} \mathbb{E}\left((A^{n_j}_t - A^{n_j}_s) B \right) \geq 0^{(\dagger)}.$$

Setting $B = I_{\{A_s > A_t\}}$ then $\mathbb{E}\left((A_t - A_s) I_{\{A_s > A_t\}} \right) \geq 0$

but, clearly $\mathbb{E}\left((A_t - A_s) I_{\{A_s > A_t\}} \right) \leq 0$. So $\mathbb{E}\left((A_t - A_s) I_{\{A_s > A_t\}} \right) = 0$ and $(A_t - A_s) I_{\{A_s > A_t\}} = 0$ $\mathbb{P}$ a.s. If $\mathbb{P}\{A_s > A_t\} > 0$ then $\mathbb{E}\left((A_t - A_s) I_{\{A_s > A_t\}} \right) < 0$ so $\mathbb{P}\{A_s > A_t\} = 0$ and $\mathbb{P}\{A_t \geq A_s\} = 1$. So $A_t \geq A_s$ $\mathbb{P}$ a.s. Now, Π is countable, by throwing away a large enough null set we have, $\mathbb{P}$ a.s.

$$\Pi \ni t \longmapsto A_t(\omega)$$

is monotone increasing. But (Y_t) and $(M_t(A_\alpha))$ are $\mathbb{P}$ a.s. right cts and so, therefore, is (A_t). It follows by right continuity that (A_t) is $\mathbb{P}$ a.s. monotone increasing on $[0, \alpha]$. By definition

$$\begin{aligned} A_0 &= Y_0 + M_0(A_\alpha) \\ &= X_0 - M_0(X_\alpha) + M_0(A_\alpha). \end{aligned}$$

($\dagger$) This is enough to show $A_t \geq A_s$ $\mathbb{P}$ a.s. because it says $A_t - A_s \geq 0$ in $\mathbb{L}^1$, but read on.

DM16

But remember " $Y_{t^n_i} = A^n_{t^n_i} - M_{t^n_i}(A^n_\alpha)$ " so that $Y_0 = A^n_0 - M_0(A^n_\alpha)$ for every n, and $A^n_0 \equiv 0$ $\mathbb{P}$ a.s..

So $Y_0 = M_0(A_\alpha)$ $\mathbb{P}$ a.s. and we see that $A_0 = 0$ $\mathbb{P}$ a.s.

Is (A_t) natural? Consider, a rt cts modification of $(M_t(B))$ for $B \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$; using a telescoping argument and predictability of $(A^{nj}_\cdot)$ we get;

$$\mathbb{E}(A^{nj}_\alpha M_\alpha(B)) = \mathbb{E}\Big(\sum_{\pi^n} M_{t^{nj}_{i-1}}(B)\big(A^{nj}_{t^{nj}_i} - A^{nj}_{t^{nj}_{i-1}}\big)\Big)$$

$$= \mathbb{E}\Big(\sum_{\pi^n} M_{t^{nj}_{i-1}}(B)\big(Y_{t^{nj}_i} - Y_{t^{nj}_{i-1}}\big)\Big)$$

because $A^{nj}_{t^{nj}_i} = Y_{t^{nj}_i} + M_{t^{nj}_i}(A^{nj}_\alpha)$ and the martingale part will vanish inside the expectation.

$$= \mathbb{E}\Big(\sum_{\pi^n} M_{t^{nj}_{i-1}}(B)\big(A_{t^{nj}_i} - A_{t^{nj}_{i-1}}\big)\Big)$$

because $A_{t^{nj}_i} = Y_{t^{nj}_i} + M_{t^{nj}_i}(A_\alpha)$ and, once again, when we have written Y in terms of A and M the martingale part, $(M_{t^{nj}_i}(A_\alpha))$, will vanish inside the expectation. Taking a limit gives

$$\mathbb{E}(A_\alpha M_\alpha(B)) = \mathbb{E}\int_{(0,\alpha]} M_{s-}(B)\,dA_s .$$

(Proposition 2).

Finally: $A_t = Y_t + M_t(A_\alpha) = X_t - M_t(X_\alpha) + M_t(A_\alpha)$

so, $$X_t = M_t(X_\alpha - A_\alpha) + A_t \text{ on } [0,\alpha].$$

QED