The Doob-Meyer Decomposition

Let $X = (X_n)$ be a stochastic process adapted to the stochastic base, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n), n \in \mathbb{N})$. We will assume that X is an $\mathbb{L}^1$-process, that is, $\forall n \; X_n \in \mathbb{L}^2(\Omega, \mathcal{F}_n, \mathbb{P}) \subseteq \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. As usual, we will take $\mathcal{F} = \sigma\left(\bigcup_n \mathcal{F}_n\right)$. As you know, there is a conditional expectation map, M_n, for each $n \in \mathbb{N}$;

$$M_n : \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{L}^1(\Omega, \mathcal{F}_n, \mathbb{P})$$

With the usual properties. At this point it is worth remarking that the restriction of M_n to $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is simply the orthogonal projection of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ onto the closed subspace $\mathbb{L}^2(\Omega, \mathcal{F}_n, \mathbb{P})$. In this case the conditional expectation has many wonderful properties(†). One of these, that is retained by the expectation on $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ albeit in a weaker form, is the orthogonality of $M_{n-1}(X_n)$ and $M_{n-1}^{\perp}(X_n)$(*) for $X_n \in \mathbb{L}^2(\Omega, \mathcal{F}_n, \mathbb{P})$. This 'orthogonality' remains for X_n in $\mathbb{L}^1(\Omega, \mathcal{F}_n, \mathbb{P})$ in the form,

$$M_{n-1}^{\perp}(M_{n-1}(X_n)) = M_{n-1}\left(M_{n-1}^{\perp}(X_n)\right) = 0.$$

Of course we must have $n \geq 2$ above, although by defining $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and seeing $M_0(X) = \mathbb{E}(X) I_{\Omega}$ we can 'extend' this to $n \geq 1$.

(†) See the notes; "Families of σ-fields, Theorem M1".

(*) $M_n^{\perp} = \mathbb{1} - M_n$

One consequence of this is that

$$M_1 \left(M_1^\perp(X_2) + X_1 \right) = X_1$$

indeed

$$M_2 \left(M_2^\perp(X_3) + X_2 \right) = X_2$$

and

$$M_3 \left(M_3^\perp(X_4) + X_3 \right) = X_3$$

and so on! Now it is easy to see that

$$M_1 \left(M_1^\perp(X_2) + X_1 \right) = X_1 ,$$

and

$$M_2 \left(M_2^\perp(X_3) + M_1^\perp(X_2) + X_1 \right) = M_1^\perp(X_2) + X_1 ,$$

Indeed

$$M_n \left(M_n^\perp(X_{n+1}) + M_{n-1}^\perp(X_n) + \cdots + X_1 \right) = M_{n-1}^\perp(X_n) + M_{n-2}^\perp(X_{n-1}) + \cdots \cdots + M_1^\perp(X_2) + X_1 .$$

What this is showing is that the <u>process</u>, U, defined by,

$$U_1 = X_1$$

$$U_2 = M_1^\perp(X_2) + X_1$$

$$U_3 = M_2^\perp(X_3) + M_1^\perp(X_2) + X_1$$

$$\vdots$$

$$U_n = M_{n-1}^\perp(X_n) + M_{n-2}^\perp(X_{n-1}) + \cdots + M_1^\perp(X_2) + X_1$$

is a <u>martingale</u>.

Notice that U is <u>adapted</u>, ie $U_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$.

This observation also allows us to write the original process X in a particular way, starting slowly; $U_1 = X_1$ so

$$A_1 \triangleq X_1 - U_1 = 0,$$

bear with me! Now,

$$U_2 = M_1^{\perp}(X_2) + X_1 = X_2 - M_1(X_2) + X_1$$

so that

$$A_2 \triangleq X_2 - U_2 = M_1(X_2) - X_1 .$$

More generally,

$$U_n = M_{n-1}^{\perp}(X_n) + M_{n-2}^{\perp}(X_{n-1}) + \dots + M_1^{\perp}(X_2) + X_1$$

$$= X_n - M_{n-1}(X_n) + M_{n-2}^{\perp}(X_{n-1}) + \dots + M_1^{\perp}(X_2) + X_1 .$$

We are now going to rearrange the terms on the right side of the last equation as a prelude to writing X in the 'particular way' described above. What we do is to gather $-M_{n-1}(X_n)$ and X_{n-1} (from $M_{n-2}^{\perp}(X_{n-1})$) together, this leaves $-M_{n-2}(X_{n-1})$ "spare". We put this together with X_{n-2} (from $M_{n-3}^{\perp}(X_{n-2})$) leaving $M_{n-3}(X_{n-2})$ spare... and so on. Writing this out formally:

$$X_n - M_{n-1}(X_n) + M_{n-2}^{\perp}(X_{n-1}) + \dots + M_1^{\perp}(X_2) + X_1$$

$$= X_n - \{M_{n-1}(X_n) - X_{n-1}\} - M_{n-2}(X_{n-1}) + M_{n-3}^{\perp}(X_{n-2}) + \dots + M_1^{\perp}(X_2) + X_1$$

$$= X_n - \{M_{n-1}(X_n) - X_{n-1}\} - \{M_{n-2}(X_{n-1}) - X_{n-2}\} - M_{n-3}(X_{n-2}) + \dots + M_1^{\perp}(X_2) + X_1$$

4/

So that eventually we get

$$U_n = X_n - M_{n-1}(X_n) + M_{n-2}^{\perp}(X_{n-1}) + M_{n-3}^{\perp}(X_{n-2}) + \cdots + M_1^{\perp}(X_2) + X_1$$

$$= X_n - \{M_{n-1}(X_n) - X_{n-1}\} - \{M_{n-2}(X_{n-1}) - X_{n-2}\} - \{M_{n-3}(X_{n-2}) - X_{n-3}\} - \cdots$$

$$\cdots - \{M_2(X_3) - X_2\} - \{M_1(X_2) - X_1\}.$$

Hence,

$$X_n = U_n + \sum_{k=1}^{n-1} \{M_k(X_{k+1}) - X_k\}$$

$$= U_n + A_n \quad \text{—— Equation 1}$$

where $A_n = \sum_{k=1}^{n-1} \{M_k(X_{k+1}) - X_k\}$. Notice that $A \triangleq (A_n)$ is a <u>predictable</u> process, that is, $A_n \in L^1(\Omega, \mathcal{F}_{n-1}, \mathbb{P})$, it is adapted "one time step earlier" than (X_n).

Equation 1 is the Doob-Meyer decomposition of the process X. Should X be such that

$$\sup_n \|A_n\|_1 = \sup_n \sum_{k=1}^{n-1} \|M_k(X_{k+1}) - X_k\|_1 < \infty$$

we can describe X as the sum of an L^1-martingale and an L^1-bounded process. This is a matter we may take further.....

The Doob - Meyer Decomposition

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n), n \in \mathbb{N}_0)^{+}$ be a discrete (time-) parameter stochastic base. An _adapted_ sequence, (A_n), is _increasing_ if $\mathbb{P}$-a.s. we have $0 = A_0 \leq A_1 \leq A_2 \ldots$, and (A_n) is called integrable if $\mathbb{E}(\sup_n A_n) < \infty$. We will require, always, that $\forall n$, $\mathbb{E}(A_n) < \infty$. An adapted sequence is called _predictable_ if $\xi_n \in \mathcal{M}(\mathcal{F}_{n-1})$ for $n \geq 1$. (CB Define Super as well)

Now let (X_n) be a _submartingale_ adapted to this stochastic base. Recall that this means that

$$M_n(X_{n+1}) \geq X_n \quad \text{for } n = 0, 1, 2, \ldots$$

By defining;

$$A_0 = 0 \quad , \quad A_{n+1} = A_n + M_n(X_{n+1}) - X_n \quad , n \geq 0$$

we see that,

(i) $A_n \geq 0$, $n \geq 0$,

(ii) $A_n \leq A_{n+1}$,

(iii) (A_n) is predictable.

Moreover, defining $U_n = X_n - A_n$, we see that

$$\begin{aligned} M_n(U_{n+1}) &= M_n(X_{n+1} - A_{n+1}) \overset{(A_n) \text{ is predictable}}{=} M_n(X_{n+1}) - A_{n+1} \\ &= M_n(X_{n+1}) - (A_n + M_n(X_{n+1}) - X_n) \\ &= -A_n + X_n \\ &= U_n . \end{aligned}$$

$^{+}$ $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$

So (U_n) is a martingale. We can write

$$X = U + A$$

where U is a martingale and A is a predictable increasing process.

Predictability has an equivalent formulation:

An increasing process, A, is called <u>natural</u> if for every bounded martingale, Y, we have

$$\mathbb{E}(Y_n A_n) = \mathbb{E}\Big(\sum_{k=1}^{n} Y_{k-1} \Delta A_k \Big), \quad n \geq 1.$$

<u>Proposition 1</u>

An increasing process, A, is predictable iff it is natural.

<u>Pf</u> If A is natural and Y a bounded martingale. Then

$$\begin{aligned}
\mathbb{E}\big((A_n - M_{n-1}(A_n)) Y_n \big) &= \mathbb{E}(A_n Y_n) - \mathbb{E}(A_n Y_{n-1}) \\
&= \mathbb{E}(A_n Y_n) - \mathbb{E}(A_n Y_{n-1}) + \mathbb{E}(A_{n-1} Y_{n-1}) \\
&\qquad - \mathbb{E}(A_{n-1} Y_{n-1}) \\
&= \mathbb{E}(A_n Y_n) - \mathbb{E}(\Delta A_n Y_{n-1}) - \mathbb{E}(A_{n-1} Y_{n-1})
\end{aligned}$$

because A is natural

$$\begin{aligned}
&= \mathbb{E}\Big(\sum_{k=1}^{n} Y_{k-1} \Delta A_k \Big) - \mathbb{E}(Y_{n-1} \Delta A_n) - \mathbb{E}\Big(\sum_{k=1}^{n-1} Y_{k-1} \Delta A_k \Big) \\
&= \mathbb{E}\Big(\sum_{k=1}^{n} Y_{k-1} \Delta A_k \Big) - \mathbb{E}\Big(\sum_{k=1}^{n} Y_{k-1} \Delta A_k \Big) \\
&= 0
\end{aligned}$$

Since Y is arbitrary[(†)], $A_n = M_{n-1}(A_n)$.

(†) The dual of L^1 is L^∞ and for $Z \in L^1$ $\|Z\|_1 = \sup_{Y \in L^\infty_1} \left| \int_\Omega X Y \, d\mathbb{P} \right|$

DMS

So A is predictable. Conversely, suppose that A is predictable and consider for $n \geq 1$, and bounded mart Y,

$$\mathbb{E}\left(\sum_{k=1}^{n} Y_{k-1} \Delta A_k \right) = \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_{k-1} A_k) - \mathbb{E}(Y_{k-1} A_{k-1}) \right\}$$

$$= \sum_{k=1}^{n} \left\{ \mathbb{E}\left(M_{k-1}(Y_k) A_k \right) - \mathbb{E}(Y_{k-1} A_{k-1}) \right\}$$

since A is predictable $$= \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_k A_k) - \mathbb{E}(Y_{k-1} A_{k-1}) \right\}.$$

This is a telescoping sum, $$= \mathbb{E}(Y_n A_n) - \mathbb{E}(Y_0 A_0)$$

since $A_0 = 0$ $$= \mathbb{E}(Y_n A_n).$$

So A is natural. [CB 1999 Now Discuss Elementary Stoch Int]

We move now to the question of Doob-Meyer decomposition for continuous parameter processes. So, ...

$\boxed{\text{STOCHASTIC BASE}}$ CB

Definition 1 (a) An adapted process, A, is called increasing iff $\mathbb{P}$ a.s. –

(i) $A_0(\omega) = 0$

(ii) $t \longmapsto A_t(\omega)$ is monotone increasing right continuous.

If, further, $\mathbb{E}(\sup_t A_t) < \infty$ then A is called integrable. We require, always, $\mathbb{E}(A_t) < \infty$, $t \in [0, \infty)$.

(b) An increasing process is called natural if for every bounded martingale, $(Y_t) \equiv Y$, we have,

$$\mathbb{E}\left(\int_{(0,t]} Y_s \, dA_s \right) = \mathbb{E}\left(\int_{(0,t]} Y_{s-} \, dA_s \right) \qquad t \in (0, \infty).$$

The integrals used in this definition are Lebesgue-Stieltjes integrals. For a bounded martingale, Y, the paths $t \mapsto Y_t(\omega)$ are $\mathbb{P}$ a.s. bounded measurable functions, so these integrable with respect to $dA_s(\omega)$.

Proposition 2

An increasing process, A, is natural iff

$$\mathbb{E}(Y_t A_t) = \mathbb{E}\left(\int_{(0,t]} Y_{s-}\, dA_s \right),$$

for each bounded martingale Y.

Pf

Let $\pi = \{t_i\}$ partition $[0,t]$. For a bounded martingale Y, set,

$$Y_s^{\pi} = \sum_{k=1}^{n} Y_{t_k} I_{(t_{k-1}, t_k]}(s).$$

Then,

$$\mathbb{E}\left(\int_{(0,t]} Y_s^{\pi}\, dA_s \right) = \mathbb{E}\left(\sum_{k=1}^{n} Y_{t_k}(A_{t_k} - A_{t_{k-1}}) \right)$$

$$= \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_{t_k} A_{t_k}) - \mathbb{E}(Y_{t_k} A_{t_{k-1}}) \right\}$$

$$= \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_{t_k} A_{t_k}) - \mathbb{E}\left(M_{t_{k-1}}(Y_{t_k} A_{t_{k-1}}) \right) \right\} \quad (\dagger)$$

$$= \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_{t_k} A_{t_k}) - \mathbb{E}\left(M_{t_{k-1}}(Y_{t_k}) A_{t_{k-1}} \right) \right\}$$

$$= \sum_{k=1}^{n} \left\{ \mathbb{E}(Y_{t_k} A_{t_k}) - \mathbb{E}(Y_{t_{k-1}} A_{t_{k-1}}) \right\}$$

This last sum is a telescoping sum with value $\mathbb{E}(Y_{t_n}A_{t_n})$, which is simply $\mathbb{E}(Y_t A_t)$. Now as mesh $\pi \to 0$ then $Y^{\pi}_{s}(\omega) \to Y_s(\omega)$ for each $s \in (0,t]$ and $\mathbb{P}$ almost every $\omega \in \Omega$. Using Bounded Convergence,

$$\int_{(0,t]} Y^{\pi}_s \, dA_s \xrightarrow[\|\pi\|\to 0]{} \int_{(0,t]} Y_s \, dA_s \quad \mathbb{P} \text{ a.s.}$$

and
$$\sup_{\pi} \left| \int_{(0,t]} Y^{\pi}_s \, dA_s \right| \le \sup_{s\in(0,t]} |Y_s(\omega)| \, A_t(\omega)$$

$$\le K A_t(\omega) \quad \mathbb{P} \text{ a.s.}$$

where K is a bound for Y. Again bounded convergence shows,

$$\mathbb{E}\left(\int_{(0,t]} Y^{\pi}_s \, dA_s\right) \xrightarrow[\|\pi\|\to 0]{} \mathbb{E}\left(\int_{(0,t]} Y_s \, dA_s\right)$$

So, $\mathbb{E}\left(\int_{(0,t]} Y_s \, dA_s\right) = \mathbb{E}(Y_t A_t)$. Now, if A is natural then $\mathbb{E}\left(\int_{(0,t]} Y_s \, dA_s\right) = \mathbb{E}\left(\int_{(0,t]} Y_{s-} \, dA_s\right) = \mathbb{E}(Y_t A_t)$. On the other hand if $\mathbb{E}\left(\int_{(0,t]} Y_{s-} \, dA_s\right) = \mathbb{E}(Y_t A_t)$ then, as we have seen, we must have

$$\mathbb{E}\left(\int_{(0,t]} Y_{s-} \, dA_s\right) = \mathbb{E}\left(\int_{(0,t]} Y_s \, dA_s\right).$$

So A is natural.

(†) Since $A_{t_{k-1}} \in L^1(\mathcal{F}_{t_{k-1}})$ and $Y_{t_k} \in L^\infty(\mathcal{F}_{t_k})$ we cannot simply apply the theorem (M2) on expectations. Let $A^n_{t_{k-1}} = A_{t_{k-1}} I_{\{A_{t_{k-1}} \le n\}}$. Then $A^n_{t_{k-1}} \to A_{t_{k-1}}$ $\mathbb{P}$-almost surely. So $Y_{t_k}A^n_{t_{k-1}} \to Y_{t_k}A_{t_{k-1}}$, $\mathbb{P}$ a.s. and in L^1! (Dominated convergence). So $M_{t_{k-1}}(Y_{t_k}A^n_{t_{k-1}}) \to M_{t_{k-1}}(Y_{t_k}A_{t_{k-1}})$ in L^1. (M_{t_k} is an L^1 contraction) But by M2, $M_{t_{k-1}}(Y_{t_k}A^n_{t_{k-1}}) = M_{t_{k-1}}(Y_{t_k})A^n_{t_{k-1}} = Y_{t_{k-1}}A^n_{t_{k-1}} \to Y_{t_{k-1}}A_{t_{k-1}}$ $\mathbb{P}$ a.s. and in L^1. The two limits must agree, so, $Y_{t_{k-1}}A_{t_{k-1}} = M_{t_{k-1}}(Y_{t_k}A_{t_{k-1}})$.

DM6

Consider now the class, Υ, of stopping times, τ, of the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ for which $\mathbb{P}\{\tau < \infty) = 1$. A right continuous process, X, is said to be of <u>class D</u> if the family

$$\{X_\tau : \tau \in \Upsilon\}$$

is uniformly integrable. One variant this definition is;
"... Υ_α of stopping times, σ, of ... $(\mathcal{F}_t)_{t\in[0,\infty)}$ ... $\mathbb{P}\{\sigma < \alpha\} = 1$.
A right ... X ... is of class DL — if $\{X_\sigma : \sigma \in \Upsilon_\alpha\}$ is uniformly integrable, for every $0 < \alpha < \infty$.

<u>Theorem 1</u> With $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), [0,\infty))$ satisfying the usual conditions and X of class DL. Then X has a decomposition,

$$X_t = U_t + A_t$$

where (U_t) is a rt-cts martingale and (A_t) an increasing process. The process (A_t) can be taken to be predictable and in this case it is unique (up to indistinguishability).
Moreover, if X is of class D, then (U_t) is a uniformly integrable martingale and (A_t) is integrable.

<u>Pf</u>

<u>Uniqueness</u>

Suppose that $X_t = U_t + A_t = U_t' + A_t'$ are two decompositions, with (A_t) and (A_t') both natural. The process B,

$$B_t = A_t - A_t' = U_t' - U_t$$

is at once of bounded variation and a martingale. Now, B is also a <u>natural</u> process in the sense of Proposition 2.

DM7

To see this let π^n be an increasing sequence of partitions of $[0,t]$ with $\|\pi^n\| \to 0$ as $n \to \infty$. Then, for a bounded martingale, (ξ_s),

$$\mathbb{P}\text{ a.s. } \int_{(0,t]} \xi_{s-}\, dB_s = \lim_n \int_{(0,t]}^{\pi_n} \xi_s\, dB_s \triangleq \lim_n \sum_{\pi^n} \xi_{t^n_{i-1}} \Delta B_{t^n_i} \quad \binom{\text{BOUNDED}}{\text{CONV}}$$

So, by bounded convergence,

$$\mathbb{E}\left(\int_{(0,t]} \xi_{s-}\, dB_s \right) = \lim_n \mathbb{E}\left(\sum_{\pi^n} \xi_{t^n_{i-1}} \Delta B_{t^n_i} \right) = \mathbb{E}(\xi_t B_t)$$

as (A_t) and (A'_t) are natural.

Since $\Delta B_{t^n_i}$ is a martingale difference;

$$\mathbb{E}\left(\xi_{t^n_{i-1}} \Delta B_{t^n_i} \right) = \mathbb{E}\left(M_{t^n_{i-1}} \left(\xi_{t^n_{i-1}} \Delta B_{t^n_i} \right) \right)$$

$$= \mathbb{E}\left(\xi_{t^n_{i-1}} M_{t^n_{i-1}} (\Delta B_{t^n_i}) \right)$$

$$= 0.$$

So $\forall t$, $\mathbb{E}(B_t \xi_t) = 0$, for every bounded martingale, (ξ_t).

This shows that $B_t = 0$ $\mathbb{P}$ a.s. for every $t \in (0,\infty)$.

Finally, as (A_t) and (A'_t) are $\mathbb{P}$ a.s. right continuous, then for each $q \in \mathbb{Q}^+$ there is a null set, E_q, with $A_q = A'_q$ on $\Omega \setminus E_q$. Let $E = \bigcup_{q \in \mathbb{Q}^+} E_q$. It is a $\mathbb{P}$-null set. On $\Omega \setminus E$ we have $A_q = A'_q$ $\forall q \in \mathbb{Q}^+$. By right continuity, $A_t = A'_t$ $\forall t \in [0,\infty)^{(+)}$ on $\Omega \setminus E$. So they are indistinguishable.

$\boxed{\mathbb{Q}^+ = \mathbb{Q} \cap [0,\infty)}$ (+) They are already equal $\mathbb{P}$ a.s. for $t=0$, for $t>0$, look above.