Barriers

We got to the point where we had calculated that

$$\mathbb{P}\{ M_T^W < y , W_T < x \}$$

was given by

$$N\left(\frac{x}{\sqrt{T}}\right) - N\left(\frac{x-2y}{\sqrt{T}}\right).$$

When we want to do something similar for our stock, S, there is a difficulty:

$$S_t = S_0 e^{\sigma W_t + (r - \sigma^2/2)t}$$
$$= S_0 e^{\sigma (W_t + (r/\sigma - \sigma/2)t)}$$

So the condition $S_T < x$ translates into,

$$W_T + (r/\sigma - \sigma/2)T < \frac{1}{\sigma}\log\left(\frac{x}{S_0}\right)$$

and

$$M_T^S < y$$

becomes

$$M_T^{(W_t + (r/\sigma - \sigma/2)t)} < \frac{1}{\sigma}\log\left(\frac{y}{S_0}\right).$$

But $X_t \equiv W_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t \equiv W_t + \nu t$

(just introducing notation...) is not a $\mathbb{P}$-BM, it is a BM with drift.

To price our Barrier Option we need to compute,

$$\mathbb{E}^{\mathbb{P}}\left((S_T - K) I_{\{S_T > K,\, M_T^S < H\}} \right)$$

$$= \mathbb{E}^{\mathbb{P}}\left((S_0 e^{\sigma X_T} - K) I_{\left\{X_T > \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right),\, M_T^X < \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right)\right\}} \right)$$

Ideally, we would like to use the joint distribution of M_T^X and X_T to compute this expectation so, as a first target we are going to work out what

$$\mathbb{P}\{X_T < \hat{x},\, M_T^X < \hat{y}\}$$

is. We use a change of measure technique:

Let $\mathbb{Q}$ be the measure,

$$\mathbb{Q}(E) = \int_E e^{-\nu W_T - \frac{\nu^2}{2}T} \, d\mathbb{P}$$

remember, $\nu = \frac{r}{\sigma} - \frac{\sigma}{2}$.

Under this measure, $(X_t) = (W_t + \nu t)$ is a Brownian motion, (W_t^*), say. Under this measure,

$$S_t = S_0 e^{\sigma(W_t + \nu t)} \equiv S_0 e^{\sigma X_t} \equiv S_0 e^{\sigma W_t^*}.$$

Note in passing that,

$$\{ S_T < x, \; M_T^S < y \}$$

is exactly

$$\left\{ W_T^* < \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right), \; M_T^{W^*} < \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right) \right\}$$

$$= \left\{ X_T < \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right), \; M_T^X < \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right) \right\}$$

We can write down the $\mathbb{Q}$ probability of this event because $\mathbb{Q}$ sees X_T as a BM. In order to get at the $\mathbb{P}$-probability of the joint distribution of X_T and M_T^X!

would like to introduce a 'convenient fiction':

$$\mathbb{Q}\{M^X_T < \hat{y}, X_T = x\} \quad \text{— (1)}$$

by this I mean exactly

$$\left(\frac{\partial}{\partial x}\mathbb{Q}\{M^X_T < \hat{y}, X_T < x\}\right) dx$$

it amounts to the probability in (1) which, as it states, is the $\mathbb{Q}$ probability that $M^X_T < \hat{y}$ while $X_T = x$

If we introduce $\frac{d\mathbb{P}}{d\mathbb{Q}} = e^{\nu X_T - \frac{1}{2}\nu^2 T}$

(we could also write $W^*_T \ldots$)

then for a bounded Borel function, f,

$$\mathbb{E}^{\mathbb{P}}\left(I_{\{M^X_T < \hat{y}\}} f(X_T)\right) =$$

$$\mathbb{E}^{\mathbb{Q}}\left(I_{\{M^X_T < \hat{y}\}} f(X_T) \frac{d\mathbb{P}}{d\mathbb{Q}}\right) =$$

$$\mathbb{E}^{\mathbb{Q}}\left(I_{\{M^X_T < \hat{y}\}} f(X_T)\, e^{\nu X_T - \frac{\nu^2}{2}T}\right)$$

$$= \int_{-\infty}^{\hat{y}} f(x)\, \underbrace{\mathbb{Q}\{M^X_T < \hat{y}, X_T = x\}}_{\text{'convenient' fiction!}}\, e^{\nu x - \frac{\nu^2}{2}T}\, dx$$

$$= \int_{-\infty}^{\hat{y}} f(x) \frac{1}{\sqrt{T}} \left(\phi\left(\frac{x}{\sqrt{T}}\right) - \phi\left(\frac{x-2\hat{y}}{\sqrt{T}}\right) \right) e^{\nu x - \frac{\nu^2}{2} T} \, dx .$$

So $\mathbb{Q}\{M_T^X < \hat{y}, X_T = x\} = \frac{1}{\sqrt{T}} \left(\phi\left(\frac{x}{\sqrt{T}}\right) - \phi\left(\frac{x-2\hat{y}}{\sqrt{T}}\right) \right) dx$

where ϕ is the standard normal density.

Now observe that

$$\frac{1}{\sqrt{T}} \phi\left(\frac{x}{\sqrt{T}}\right) e^{\nu x - \frac{\nu^2}{2} T} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T} + \nu x - \frac{\nu^2}{2} T}$$

$$= \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x^2 - 2T\nu x + \nu^2 T^2)}{2T}}$$

$$= \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x - \nu T)^2}{2T}} = \frac{d}{dx} N\left(\frac{x - \nu T}{\sqrt{T}}\right).$$

Similarly

$$\frac{1}{\sqrt{T}} \phi\left(\frac{x-2\hat{y}}{\sqrt{T}}\right) e^{\nu x - \frac{\nu^2}{2} T} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{x-2\hat{y}}{\sqrt{T}}\right)^2 + \nu x - \frac{\nu^2}{2} T}$$

$$= \frac{e^{2\hat{y}\nu}}{\sqrt{T}} \phi\left(\frac{x - 2\hat{y} - \nu T}{\sqrt{T}}\right)$$

$$= \frac{d}{dx} \left(e^{2\hat{y}\nu} N\left(\frac{x - 2\hat{y} - \nu T}{\sqrt{T}}\right) \right).$$

If we now let $f(t) = I_{(-\infty, \hat{x}]}(t)$

So $f(x_T)\, I_{\{M^x_T < \hat{y}\}} = I_{\{M^x_T < \hat{y},\, x_T < \hat{x}\}}$

and

$$\mathbb{P}\{M^x_T < \hat{y},\, x_T < \hat{x}\} = \mathbb{E}^{\mathbb{Q}}\left(I_{\{M^x_T < \hat{y},\, x_T < \hat{x}\}} \times \frac{d\mathbb{P}}{d\mathbb{Q}} \right)$$

$$= \int_{-\infty}^{\hat{x}} \frac{1}{\sqrt{T}} \left(\phi\left(\frac{x}{\sqrt{T}}\right) e^{\nu x - \nu^2/2\, T} - \phi\left(\frac{x - 2\hat{y}}{\sqrt{T}}\right) e^{\nu x - \frac{\nu^2}{2} T} \right) dx$$

$$= \int_{-\infty}^{\hat{x}} \left[\frac{d}{dx}\left(N\left(\frac{x - \nu T}{\sqrt{T}}\right)\right) - e^{2\hat{y}\nu} \frac{d}{dx}\left(N\left(\frac{x - 2\hat{y} - \nu T}{\sqrt{T}}\right)\right) \right] dx$$

$$= N\left(\frac{\hat{x} - \nu T}{\sqrt{T}}\right) - e^{2\hat{y}\nu} N\left(\frac{\hat{x} - 2\hat{y} - \nu T}{\sqrt{T}}\right)$$

This is the joint distribution of X_T and M^x_T under $\mathbb{P}$.

The price of the option is,

$$\mathbb{E}^{\mathbb{P}}\left(e^{-rT} (S_0 e^{\sigma X_T} - K)\, I_{\{K < S_0 e^{\sigma X_T} < H,\, M^S_T < H\}} \right)$$

Now, $\{K < S_0 e^{\sigma X_T} < H,\ M^S_T < H\}$

$$= \left\{ \tfrac{1}{\sigma}\log\left(\tfrac{K}{S_0}\right) < X_T < \tfrac{1}{\sigma}\log\left(\tfrac{H}{S_0}\right),\ M_T^X < \tfrac{1}{\sigma}\log\left(\tfrac{H}{S_0}\right) \right\}$$

Which we can express via our 'convenient fiction' as

$$\int_{1/\sigma \log\left(\frac{K}{S_0}\right)}^{1/\sigma \log\left(\frac{H}{S_0}\right)} e^{-rT}\left(S_0 e^{\sigma x} - K\right) \mathbb{P}\left\{ M_T^X < \tfrac{1}{\sigma}\log\left(\tfrac{H}{S_0}\right),\ X_T = x \right\} dx$$

and we can get $\mathbb{P}\left\{M_T^X < \frac{1}{\sigma}\log\left(\frac{H}{S_0}\right), X_T = x\right\}$ from $\mathbb{P}\{M_T^X < \hat{y},\ X_T < \hat{x}\}$, just as before: it is

$$\left[\frac{1}{\sqrt{T}}\phi\left(\frac{x - \nu T}{\sqrt{T}}\right) - \frac{e^{\frac{2\nu}{\sigma}\log\left(\frac{H}{S_0}\right)}}{\sqrt{T}} \phi\left(\frac{x - \frac{2}{\sigma}\log\left(\frac{H}{S_0}\right) - \nu T}{\sqrt{T}}\right)\right] dx$$

You can now integrate up the term involving K in the integral above. It should agree with what we wrote down earlier today.

Learn: $\mathbb{P}\left(W_T < x,\ M_T^W < y\right)$ and

$\mathbb{P}\left(W_T + \nu T < x,\ M_T^{(W_t + \nu t)} < y\right)$.