We consider an up and in call option: Strike K, barrier L and expiry T. Assume $K \le L$, other-wise the payoff is identical with a normal call because it must "knock in" to be in the money.

Assume "$ud = 1$" and suppose the number of time steps, N, is even. Assume also that $L = Su^B L^{N-B}$ for $B \in \{0, 1, \dots, N\}$. Writing S_t for the value of the underlying at time t, we observe first of all that if $j \ge B$ then the probability that $S_T = Su^j d^{N-j}$ is,

$$\binom{N}{j} p^j (1-p)^{N-j}.$$

So now suppose $j < B$; a path $\omega = \{\omega_1, \dots, \omega_N\}$ with $S_T(\omega) = Su^j d^{N-j}$, with $S_t = L$ for some $t \in \{0, 1, 2, \dots, N\}$. Indeed suppose t is the first time that S achieves L along this path. We construct the reflected path of ω, denoted by ω'. We know that $S_t = L = Su^B d^{N-B}$ and we consider the case B is even(†). Now $t = k\Delta t$, ie. it takes k time steps up to time t; if we suppose that the path up to time t has r upward movements and (therefore) $k-r$ downward movements then it must be that $Su^r d^{k-r} = Su^B d^{N-B}$. Using the fact that "$ud = 1$";

$$1 = u^{B-r} d^{(N-B)-(k-r)}$$

ie.

$$B - r = (N-B) - (k-r)$$

$$B - r = N - B - k + r \quad \text{or} \quad 2r = 2B - N + k$$

(†) We have assumed $L = Su^B d^{N-B}$ ie. one of the final values of S, so perhaps B is necessarily even, if N is even, and $L > S$. [I've got an argument for this]

k. $r = B - (N-k)/2$ and $k - r = k - B + (N-k)/2$ $= (N+k)/2 - B$. Note that $k = N + 2(r-B)$, which is even. So up to time t our path ω' has $B-(N-k)/2$ up moves and $(N+k)/2 - B$ down moves. After time t the path ω has $j - B + (N-k)/2$ up moves and $N - j - (N+k)/2 + B$ down moves, this last number is $(N-k)/2 - j - B$. We swop the numbers of up and down moves to form the reflected path: So it will have $N-j-(N+k)/2+B$ up moves and $j-B+(N-k)/2$ down moves. This gives us a path with terminal value

$$S\, u^{B-(N-k)/2 + N-j-(N+k)/2+B}\, d^{(N+k)/2-B+j-B+(N-k)/2}$$

$$= S\, u^{2B-j} d^{N-(2B-j)}.$$

There is a 1-1 correspondance between paths hitting the barrier for the first time at time t and terminating at $Su^j d^{N-j}$ (for $j \le B$) and those hitting the barrier for the first time at time t and terminating at $Su^{2B-j} d^{N-(2B-j)}$. More generally, there is a 1-1 corres-pondance between paths which hit the barrier at some time before T and terminate at $Su^j d^{N-j}$ and paths which hit the barrier at some time before T and terminate at $Su^{2B-j} d^{N-(2B-j)}$ (†). However

$$Su^{2B-j} d^{N-(2B-j)} > Su^B d^{N-B}$$

and so any path terminating at $Su^{2B-j} d^{N-(2B-j)}$ must

(†) Any such path has a first hitting time for the barrier and therefore a reflected path corresponding to it. The first hitting times partition the paths terminating at $Su^j d^{N-j}$.

hit the barrier before T. So we have a way of counting all of the paths for the $j < B$ case. Summing over all of the possibilities gives us a value for the knock in call as

$$\pi(C) = \frac{1}{r^n}\left(\sum_{j=A}^{B-1} (Su^j d^{N-j} - K)\binom{N}{2B-j}^{(*)} p^j (1-p)^{N-j} + \sum_{j=B}^{N} (Su^j d^{N-j} - K)\binom{N}{j} p^j (1-p)^{N-j} \right)$$

We explain: Paths terminating at $Su^j d^{N-j}$, $j \leq B$, and hitting the barrier at some point are $\binom{N}{2B-j}$ in number. Each path has probability[(+)] $p^j (1-p)^{N-j}$. The payoff $(Su^j d^{N-j} - K)^+$ is only positive if j is greater than or equal to A where A is the least integer such that;

$$Su^A d^{N-A} > K$$

ie.
$$A \log u + (N-A)\log d > \log(K/S)$$

ie
$$A > \frac{\log(K/S) - N\log d}{\log u - \log d}.$$

The other term comprises the sum over all those paths which terminate at or above $Su^B d^{N-B}$, recall $K < Su^B d^{N-B}$ so the payoff is always positive for this case. Finally we have simply applied the risk-neutral valuation: The value is the discounted risk-neutral expectation of the payoff.

(+) Risk-neutral probability

(*) Of course this is zero if $2B-j < 0$ or $2B-j > N$

Now we simplify some terms: First of all recall that,

$$\binom{N}{\ell} = \frac{N!}{(N-\ell)!\,\ell!} = \binom{N}{N-\ell}$$

So that

$$\binom{N}{2B-j} = \binom{N}{N-2B+j} \qquad \text{(usual caveats about } 2B-j)$$

and as j runs from A to $B-1$, $\binom{N}{N-2B+j}$ runs from $\binom{N}{A-2B+N}$ to $\binom{N}{N-B-1}$, so if we write this as $\binom{N}{\ell}$ where ℓ runs from $A-2B+N$ to $N-B-1$ we have $\ell = N-2B+j$ so $j = \ell+2B-N$ (as ℓ runs from $A-2B+N$ to $N-B-1$) and

$$\binom{N}{2B-j} p^j (1-p)^{N-j} = \binom{N}{\ell} p^{\ell+2B-N} (1-p)^{N-(\ell+2B-N)}$$

$$= p^{2B-N} (1-p)^{N-2B} \binom{N}{\ell} p^{\ell} (1-p)^{N-\ell}$$

So that

$$\frac{K}{r^N} \sum_{j=A}^{B-1} \binom{N}{2B-j} p^j (1-p)^{N-j} = \frac{K}{r^N} \left(\sum_{\ell=N-2B+A}^{N-B-1} \binom{N}{\ell} p^{\ell} (1-p)^{N-\ell} \right) \left(\frac{p}{1-p} \right)^{2B-N}$$

$$= \frac{K}{r^N} \left(\frac{p}{1-p} \right)^{2B-N} \left(\Phi(N-2B+A, N, p) - \Phi(N-B, N, p) \right)$$

The term, $\dfrac{S}{r^N}\displaystyle\sum_{j=A}^{B-1} u^j d^{N-j}\binom{N}{2B-j}p^j(1-p)^{N-j}$ may

be similarly transformed to

$$\frac{S}{r^N}\sum_{\ell=N-2B+A}^{N-B-1} u^{\ell+2B-N} d^{N-(\ell+2B-N)}\binom{N}{\ell}p^{\ell+2B-N}(1-p)^{N-(\ell+2B-N)}$$

$$= S\sum_{\ell=N-2B+A}^{N-B-1}\left(\frac{up}{R}\right)^{\ell}\left(\frac{up}{R}\right)^{2B-N}\left(\frac{d(1-p)}{R}\right)^{N-\ell}\left(\frac{d(1-p)}{R}\right)^{N-2B}\binom{N}{\ell}$$

$$= \left(\frac{p'}{1-p'}\right)^{2B-N} S \sum_{N-2B+A}^{N-B-1}(p')^{\ell}(1-p')^{N-\ell}\binom{N}{\ell}$$

$$= \left(\frac{p'}{1-p'}\right)^{2B-N} S\left(\Phi(N-2B+A, N, p') - \Phi(N-B, N, p')\right)$$

Here $p' = \dfrac{up}{R}$ & $1-\dfrac{up}{R} = \dfrac{d(1-p)}{R}$ because p is the risk neutral measure (so $\mathbb{E}^p\left(\dfrac{S_t - S_{t-1}}{S_{t-1}}\right) = r$).

The term

$$\frac{1}{r^N}\sum_{j=B}^{N}\left(Su^j d^{N-j} - K\right)\binom{N}{j}p^j(1-p)^{N-j}$$

can — by incorporating r^N — be written as

$$= S\sum_{j=B}^{N}\binom{N}{j}(p')^j(1-p')^{N-j} - \frac{K}{r^N}\sum_{j=B}^{N}\binom{N}{j}p^j(1-p)^{N-j}$$

$$= S\,\Phi(B, N, p') - \frac{K}{r^N}\Phi(B, N, p),$$

Collecting terms gives ;

$$\pi(C) = S\,\Phi(B,N,p') + S\left(\frac{p'}{1-p'}\right)^{2B-N}\left\{\Phi(A-2B+N,N,p') - \Phi(N-B,N,p')\right\}$$

$$-Kr^{-N}\left\{\Phi(B,N,p) + \left(\frac{p}{1-p}\right)^{2B-N}\left\{\Phi(A-2B+N,N,p) - \Phi(N-B,N,p)\right\}\right\}$$

The next stage is to use non-standard methods to value a non-standard barrier option on a binary tree. We then take standard parts :