Abstract Integration

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function, g, is called an <u>indicator function</u> if it take the values 0 and 1 <u>only</u>. The set of points, $E = \{\omega : g(\omega) = 1\}$, is called "the set indicated by g" and "g is the indicator of E". For $E \in \mathcal{F}$ we write I_E for the indicator of E, so $I_E(\omega) = 1$ if $\omega \in E$ and is zero otherwise. We define the class of indicator functions, I_E, for $E \in \mathcal{F}$, to be the <u>elementary random variables</u> and we define,

$$\int_\Omega I_E \, d\mathbb{P} \triangleq \mathbb{P}(E).$$

A random variable is <u>simple</u> iff it is a finite linear combination of elementary random variables. It is clear that a simple random variable takes only finitely many distinct values. Let f be a simple random variable taking the distinct values, $\alpha_1, \dots, \alpha_n$. Then we may write

$$f(\omega) = \sum_{i=1}^{n} \alpha_i I_{E_i} \qquad \text{——} \qquad (1)$$

where $E_i = \{\omega : f(\omega) = \alpha_i\}$. Since the α_i's are different of one another, $E_i \cap E_j = \emptyset$ for $i \neq j$ and each E_i is non-empty. Any function which may be written as $\sum_{i=1}^{n}\alpha_i I_{E_i}$ where $E_i \in \mathcal{F}$, $1 \leq i \leq n$, $\alpha_i \in \mathbb{R}$ $1 \leq i \leq n$, is a simple random variable. Accordingly it may be written (uniquely) in the form $\sum_{i=1}^{k}\beta_i I_{F_i}$ where the β_i's are the distinct values that $\sum_{i=1}^{n}\alpha_i I_{E_i}$ takes, each $F_i \in \mathcal{F}$, $F_i \cap F_j = \emptyset$ for $i \neq j$. We define, for $f = \sum_{i=1}^{k}\beta_i I_{F_i}$,

$$\int_\Omega f \, d\mathbb{P} \triangleq \sum_{i=1}^{k}\beta_i \mathbb{P}(F_i) \quad \text{———} \quad (2)$$

A function of the form $\sum_{i=1}^{n}\alpha_i I_{E_i}$, with $E_i \in \mathcal{F}$, $1 \leq i \leq n$ may be written in many different ways as a linear combination of indicator functions. There is "a Lemma" which proves that such a function can be written as a linear combination of indicator functions of disjoint sets in $\mathcal{F}$ — non-uniquely — and that the integral defined <u>in the manner of</u> (2) above is <u>well defined</u> and the integrals are all equal to that defined in (2).

A simple random variable is positive if its range lies in $[0, \infty)$.

<u>Theorem</u>

Let $f : \Omega \to [0, \infty]$ be (measurable) a random variable. There exist simple random variables, (s_n), each of which are positive and

(1) $0 \leq s_1 \leq s_2 \leq s_3 \cdots \leq f$

(2) $s_n(\omega) \to f(\omega) \quad \forall \omega \in \Omega$

Pf For $n = 1, 2, \ldots,$ and $1 \leq i \leq n2^n$ let

$$E_{n,i} = f^{-1}\left(\left[\tfrac{i-1}{2^n}, \tfrac{i}{2^n}\right)\right)$$

$$F_n = f^{-1}([n, \infty])$$

set $\Delta_n = \sum_{i=1}^{n2^n} \left(\frac{i-1}{2^n}\right) I_{E_{n,i}} + nF_n$. By 'definition' $E_{n,i}$ and F_n are in $\mathcal{F}$ for each n and each i, and for fixed n these sets are disjoint. Property (1) follows because of the "halving" occuring in the contruction and the way the values of Δ_n are defined on $E_{n,i}$ and F_n. Property (2) follows because if $f(\omega) = \infty$ then $\Delta_n(\omega) = n$ and "$\Delta_n(\omega) \to \infty$" as $n \to \infty$. If $f(\omega) < \infty$ then, for large enough n, $f(\omega) \in E_{n,i}$ for some suitable choice of i, so $|f(\omega) - \Delta_n(\omega)| \leq 1/2^n$, enough.

Integrals of Simple Random Variables

Let $f = \sum_{i=1}^{n} \alpha_i I_{E_i}$, $E_i \in \mathcal{F}$, $E_i \cap E_j = \emptyset$, $i \neq j$, and $\alpha_1, \ldots \alpha_n$ are the distinct values of f. We define

$$\int_F f \, d\mathbb{P} = \int_\Omega f I_F \, d\mathbb{P} = \sum_{i=1}^{n} \alpha_i \, \mathbb{P}(E_i \cap F).$$

(convention $\infty \cdot 0 = 0$)

If g is also simple, $g = \sum_{j=1}^{m} \beta_j I_{F_j}$ ... the usual stuff being true for the F_j's and β_j's .... Then $f+g$ is clearly a simple function and, by adapting "that Lemma" alluded to earlier one can show that

$$\int_\Omega (f+g) \, d\mathbb{P} = \int_\Omega f \, d\mathbb{P} + \int_\Omega g \, d\mathbb{P}$$

Evidently, for $\lambda \in \mathbb{R}$, $\int_\Omega \lambda f \, d\mathbb{P} = \lambda \int_\Omega f \, d\mathbb{P}$ and for disjoint sets, $E, F \in \mathcal{F}$,

$$\int_{E \cup F} f \, d\mathbb{P} = \int_E f \, d\mathbb{P} + \int_F f \, d\mathbb{P}.$$

If $f \geq 0$, f simple then, obviously, $\int_\Omega f \, d\mathbb{P} \geq 0$ and $\int_\Omega f \, d\mathbb{P} = 0 \iff f = 0$ on a set of probability 1.

Integration of Non-Negative Random Variables

Let $f : \Omega \to [0, \infty]$ be a random variable. We define,

$$\int_\Omega f \, d\mathbb{P} = \sup \left\{ \int_\Omega s \, d\mathbb{P} : s \text{ is simple}, 0 \leq s \leq f \right\}$$

and for $E \in \mathcal{F}$,

$$\int_E f \, d\mathbb{P} = \sup \left\{ \int_E s \, d\mathbb{P} : s \text{ is simple}, 0 \leq s \leq f \right\}$$

Note that if f is <u>simple</u> then this is consistent with our previous definition because, f is itself simple and $\int_\Omega f \, d\mathbb{P} \geq \int_\Omega s \, d\mathbb{P}$ for every simple $s \leq f$ — (because the integral of +ve things is +ve)

<u>Theorem</u> Let f, g etc be random variables, $E, F, G, \ldots$ etc events in $\mathcal{F}$. Then,

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Remember, we deal with positive functions.

ⓐ If $f \leq g$ then $\int_\Omega f \, d\mathbb{P} \leq \int_\Omega g \, d\mathbb{P}$

ⓑ If $A, B \in \mathcal{F}$, $A \subseteq B$ then $\int_A f \, d\mathbb{P} \leq \int_B f \, d\mathbb{P}$

ⓒ If $0 \leq f$, $c \in [0, \infty)$ then $\int_E cf \, d\mathbb{P} = c \int_E f \, d\mathbb{P}$

ⓓ If $f(\omega) = 0$ on a set of probability 1, then $\int_E f \, d\mathbb{P} = 0$ for every $E \in \mathcal{F}$.

ⓔ If $\mathbb{P}(E) = 0$ then $\int_E f \, d\mathbb{P} = 0$ (even if $f = \infty$ on E)

ⓕ $\int_E f \, d\mathbb{P} = \int_\Omega f I_E \, d\mathbb{P}$

Pf ⓐ If s is simple, $s \leq f \leq g$ then $s \leq g$ so $\int_\Omega s \, d\mathbb{P} \leq \int_\Omega g \, d\mathbb{P}$. Taking the supremum over all such s leads to $\int_\Omega f \, d\mathbb{P} \leq \int_\Omega g \, d\mathbb{P}$.

ⓑ If $A \subseteq B$ and $f \geq 0$ then $f I_A \leq f I_B$ and by ⓐ, $\int_\Omega f I_A \, d\mathbb{P} \leq \int_\Omega f I_B \, d\mathbb{P}$. Now, if $s \leq f$ and s is simple then $s I_E$ is simple and $s I_E \leq f I_E$. So,

$$\int_E s \, d\mathbb{P} \leq \int_\Omega f I_E \, d\mathbb{P} .$$

Taking the supremum over all $s \leq f$ gives,

$$\int_E f \, d\mathbb{P} \leq \int_\Omega f I_E \, d\mathbb{P} .$$

On the other hand, if s is simple and $s \leq f I_E$ then $s \leq f$ and $s = s I_E$. If we choose s so that $\int_\Omega s \, d\mathbb{P} > \int_\Omega f I_E \, d\mathbb{P} - \epsilon$, (this assumes $\int_\Omega f I_E \, d\mathbb{P} < \infty$) then, as $s = s I_E$ and $s \leq f$,

$$\int_\Omega s \, d\mathbb{P} = \int_E s \, d\mathbb{P} \leq \int_E f \, d\mathbb{P}, \text{ so } \int_\Omega f I_E \, d\mathbb{P} < \int_E f \, d\mathbb{P} + \epsilon.$$

This shows $\int_E f \, d\mathbb{P} = \int_\Omega f I_E \, d\mathbb{P}$, proves (f) and concludes (b). If $\int_\Omega f I_E \, d\mathbb{P} = \infty$ then for each $n \in \mathbb{N}$ there is a simple random variable, s_n, $s_n \leq f I_E \leq f$, with $\int_\Omega s_n \, d\mathbb{P} \geq n$. But $s_n = s_n I_E$ and $\int_\Omega s_n \, d\mathbb{P} = \int_E s_n \, d\mathbb{P}$ showing $\int_E f \, d\mathbb{P} = \infty$ too.

(c) If $c = 0$ it's obvious. If $c > 0$ and s is simple $s \leq f$ then $cs \leq cf$, and cs is simple too as is $c s I_E$. Now,

$$\int_E cf \, d\mathbb{P} \geq \int_E cs \, d\mathbb{P} = c \int_E s \, d\mathbb{P}, \text{ so } \int_E cf \, d\mathbb{P} \geq c \int_E f \, d\mathbb{P}.$$

On the other hand, if t is simple, $t \leq cf$ then $\frac{t}{c} \leq f$. Now $\frac{t}{c}$ is simple and

$$\int_E f \, d\mathbb{P} \geq \int_E \frac{t}{c} \, d\mathbb{P} = \frac{1}{c} \int_E t \, d\mathbb{P},$$ taking the sup over t

gives $\int_E f \, d\mathbb{P} \geq \frac{1}{c} \int_E cf \, d\mathbb{P}$. This proves C.

(d) Let $E = \{\omega : f(\omega) = 0\}$ and $\alpha = \inf\{f(\omega) : \omega \in \Omega \setminus E\}$

then $s = \alpha I_{\Omega \setminus E}$ is a simple random variable and is the largest simple random variable less than f. So

$$\begin{aligned} \int_\Omega f \, d\mathbb{P} = \int_\Omega s \, d\mathbb{P} &= 0 \cdot \mathbb{P}(E) + \alpha \cdot \mathbb{P}(\Omega \setminus E) \\ &= 0 \cdot 1 + \alpha \cdot 0 \\ &= 0 \quad (\text{even if } \alpha = \infty) \end{aligned}$$

By (a),(b), $f I_G \leq f$ so $\int_\Omega f I_G \, d\mathbb{P} \leq \int_\Omega f \, d\mathbb{P} = 0$, for $G \in \mathcal{F}$.

ⓔ If $\mathbb{P}(E) = 0$ then fI_E is zero on a set of probability 1.
So $\int_\Omega f I_E \, d\mathbb{P} = 0$ by ⓓ, and the result follows from ⓕ.

The Monotone Convergence Theorem

Let (f_n) be a sequence of random variables, suppose

(i) $0 \leq f_1(\omega) \leq f_2(\omega) \leq \ldots \leq \infty \quad \forall \omega \in \Omega$,

(ii) $f_n(\omega) \longrightarrow f(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$.

Then f is a random variable and $\int_\Omega f_n \, d\mathbb{P} \longrightarrow \int_\Omega f \, d\mathbb{P}$.

Pf
We know $\left(\int_\Omega f_n \, d\mathbb{P}\right)$ is an increasing sequence. Let

$$\lim_n \int_\Omega f_n \, d\mathbb{P} = \alpha \leq \infty .$$

It is known that f is measurable (ask about this CB).
Since $f_n \leq f$ we have $\int_\Omega f_n \, d\mathbb{P} \leq \int_\Omega f \, d\mathbb{P} \;\; \forall n$, so,

$$\alpha \leq \int_\Omega f \, d\mathbb{P}.$$

Let s be simple, $0 \leq s \leq f$ and let $c \in (0,1)$.
Define $E_n = \{\omega : f_n(\omega) \geq c s(\omega)\}$, $n \in \mathbb{N}$. Each $E_n \in \mathcal{F}$. Also, $E_1 \subseteq E_2 \subseteq E_3$ and $\Omega = \bigcup_n E_n$ because the f_n's are increasing & converge to $f(\omega) \geq s(\omega)$.
⌈ If $f(\omega) = 0$ then $s(\omega) = 0$ and $f_1(\omega) = c s(\omega)$ so $\omega \in E_1$. If $f(\omega) > 0$ then $f(\omega) > c s(\omega)$ $(f \geq s)$, there will be an n such that $f_n(\omega) \geq c s(\omega)$, here $\omega \in E_n$ ⌋. Also,

$$\int_\Omega f_n \, d\mathbb{P} \geq \int_{E_n} f_n \, d\mathbb{P} \geq c \int_{E_n} s \, d\mathbb{P} .$$

Letting $n \to \infty$ we get

$$\alpha = \lim_n \int_\Omega f_n \, d\mathbb{P} \geq \lim_n c \int_{E_n} s \, d\mathbb{P}.$$

Fact: Do we need to prove this? The set function

$$\mathcal{F} \ni E \longmapsto \int_E s \, d\mathbb{P}$$

is a (finite) measure on $\mathcal{F}$.

Assuming this,

$$\alpha \geq c \int_\Omega s \, d\mathbb{P}$$

for every $c \in (0,1)$. This shows $\alpha \geq \int_\Omega s \, d\mathbb{P}$ and hence $\alpha \geq \int_\Omega f \, d\mathbb{P}$. So the result is proved.

<u>Corollary</u>

Let f and g be random variables. Then

$$\int_\Omega (f+g) \, d\mathbb{P} = \int_\Omega f \, d\mathbb{P} + \int_\Omega g \, d\mathbb{P}.$$

<u>Pf</u> From an earlier theorem we can find simple r.v.'s $s_n \uparrow f$ and $t_n \uparrow g$ with, according to the MCThm $\int_\Omega s_n \, d\mathbb{P} \to \int_\Omega f \, d\mathbb{P}$ and $\int_\Omega t_n \, d\mathbb{P} \to \int_\Omega g \, d\mathbb{P}$. Since $s_n + t_n \uparrow f+g$ the result follows because the integral is additive on simple r.v.'s.

Fatou's Lemma

If (f_n) is a sequence of r.v.'s

$$\int_\Omega \liminf_n f_n \, d\mathbb{P} \leq \liminf_n \int_\Omega f_n \, d\mathbb{P}$$

<u>Pf</u> Set $g_k = \bigwedge_{i \geq k} f_i$ then $g_k \leq f_k$ obviously and g_k is: $0 \leq g_1 \leq g_2 \leq \dots$. That g_k is a r.v. is "well known". Now $g_k \to \liminf_n f_n$ (by definition) so, by monotone convergence,

$$\int_\Omega g_k \, d\mathbb{P} \to \int_\Omega \liminf_n f_n \, d\mathbb{P} .$$

But $\int_\Omega g_k \, d\mathbb{P} \leq \int_\Omega f_k \, d\mathbb{P}$ and so

$\lim_k \int_\Omega g_k \, d\mathbb{P} = \liminf_k \int_\Omega g_k \, d\mathbb{P} \leq \liminf_k \int_\Omega f_k \, d\mathbb{P}$, showing the result is true.

The Dominated Convergence Theorem.

Suppose (f_n) is a sequence of r.v.'s such that;

(i) $\lim_n f_n(\Omega) = f(\omega)$ exists $\forall \omega \in \Omega$

(ii) There is a r.v., g, with $g \geq 0$ and $f_n \leq g \ \forall n$ <u>and</u> $\int_\Omega g \, d\mathbb{P} < \infty$.

Then, f is a r.v., $\int_\Omega f \, d\mathbb{P} < \infty$, $\lim_n \int_\Omega |f - f_n| \, d\mathbb{P} = 0$ and as a consequence, $\lim_n \int_\Omega f_n \, d\mathbb{P} = \int_\Omega f \, d\mathbb{P}$.

Pf Since $f \leq g$ then $\int_\Omega f \, d\mathbb{P} \leq \int_\Omega g \, d\mathbb{P} < \infty$, and that f is a r.v. is "well known". Now $|f_n - f| \leq 2g$ (†) and the sequence $2g - |f_n - f|$ converges to $2g$. By Fatou's Lemma

$$\int_\Omega \liminf_n (2g - |f_n - f|) \, d\mathbb{P} = \int_\Omega 2g \, d\mathbb{P} \leq \liminf_{n \to \infty} \int_\Omega 2g - |f - f_n|) \, d\mathbb{P}$$

$$= \int_\Omega 2g \, d\mathbb{P} + \liminf_{n \to \infty} \left(- \int_\Omega |f - f_n|) \, d\mathbb{P} \right)$$

$$= \int_\Omega 2g \, d\mathbb{P} - \limsup_{n \to \infty} \int_\Omega |f - f_n| \, d\mathbb{P}$$

Subtracting $\int_\Omega 2g \, d\mathbb{P}$ (which is finite!)

$$0 \geq \limsup_{n \to \infty} \int_\Omega |f - f_n| \, d\mathbb{P}$$

Since $\forall n$ $\int_\Omega |f - f_n| \, d\mathbb{P} \geq 0$ this forces the limsup and lim inf to be equal. So

$$\lim_n \int_\Omega |f_n - f| \, d\mathbb{P} = 0.$$

Since (again I've not proved this...)

$$\left| \int_\Omega f \, d\mathbb{P} - \int_\Omega f_n \, d\mathbb{P} \right| = \left| \int_\Omega (f_n - f) \, d\mathbb{P} \right| \leq \int_\Omega |f_n - f| \, d\mathbb{P},$$

this shows that $\lim_n \int_\Omega f_n \, d\mathbb{P} = \int_\Omega f \, d\mathbb{P}$

$\square$

(†) ok I'm cheating, we dont yet know that $|f_n - f|$ is a r.v.