# Stochastic and Belated Integrals for the Full Fock Space

C. Barnett & A. Voliotis
Department of Mathematics
Imperial College
London SW7 2BZ

August 30, 2002

**Abstract**

This paper forms a sequel to [4] where a family of conditional expectations $E_t$, for $t \in \mathcal{R}^+$, are constructed on the von Neumann algebra, $\mathcal{V}$, generated by the creation, annihilation and gauge operators acting on the full Fock space over $\mathcal{L}^2(\mathcal{R}_+)$. The aim here is to take the account a little further and to investigate some aspects of stochastic integration with respect to the 'basic' processes of creation, annihilation and gauge and to compare this with what one obtains if one adopts the 'abstract' Belated integral described in [3]. We were surprised at the outcome! Along the way we adapt the representation theorem, [9], due to I.F. Wilde to this context.

## 1 Introduction

Our aim was to write a complete account of our work on this subject, however it soon became clear that we needed a S.A.L.T. [1] in order to proceed. The problem is that even the proofs of the elementary results below are unavoidably long if the details are included. Accordingly, in an earlier version, we excised much routine material. We are grateful to Professor R. L. Hudson, who read this earlier version, he convinced us that we should go much further. The reader will have to decide if we have struck the correct balance between detail and brevity. In any event the details remain in [8]. We have also employed the font size "scriptstyle" in order to scale down some of the displayed mathematics, this makes the appearance of some arrays uneven, we hope not ugly.
We recall that the full Fock space, $\mathcal{F}$ over $\mathcal{L}^2(\mathcal{R}_+)$, is defined as follows

$$\mathcal{F} \equiv \mathcal{C} \oplus \oplus_{n=1}^{\infty} \mathcal{L}^2(\mathcal{R}_+)^{\otimes n}$$

[1] Strategic Arguments Limitation Theorem

it has the usual scalar product. Note that all scalar products are linear in the left argument. $\Omega$ will denote the vector $(1,0,0,...)$.

We define the annihilation operator, $l(f)$ and creation operator, $l^*(f)$, for $f \in \mathcal{L}^2(\mathcal{R}_+)$, as follows

$$\begin{aligned} l(f)f_1 \otimes ... \otimes f_n &= < f_1, f > f_2 \otimes ... \otimes f_n \\ l^*(f)f_1 \otimes ... \otimes f_n &= f \otimes f_1 \otimes ... \otimes f_n \\ l(f)\Omega &= 0 \\ l^*(f)\Omega &= f \end{aligned}$$

here $n \geq 1$ and $f_1, ..., f_n$ are in $\mathcal{L}^2(\mathcal{R}_+)$. The operators $l(f)$ and $l^*(f)$ are bounded and mutually adjoint. Furthermore,

$$\|l(f)\| = \|l^*(f)\| = \|f\|_2$$

Given any $\mathcal{T} \in \mathcal{B}(\mathcal{L}^2(\mathcal{R}_+))$ we define the operator $p(\mathcal{T})$ by;

$$\begin{aligned} p(\mathcal{T})f_1 \otimes ... \otimes f_n &= \mathcal{T}f_1 \otimes ... \otimes f_n \\ p(\mathcal{T})\Omega &= 0 \end{aligned}$$

for $f_i \in \mathcal{L}^2(\mathcal{R}_+), 1 \leq i \leq n$. The operator $p(\mathcal{T})$ is bounded with $\|p(\mathcal{T})\| = \|\mathcal{T}\|$ and $p(\mathcal{T})^* = p(\mathcal{T}^*)$. For $g \in \mathcal{L}^\infty(\mathcal{R}_+)$, $g$ will be considered to be the element of $\mathcal{B}[\mathcal{L}^2(\mathcal{R}_+)]$ obtained by letting $g$ act by multiplication on $\mathcal{L}^2(\mathcal{R}_+)$. This makes the meaning of $p(g)$ clear. Moreover the following identities hold:

$$\begin{aligned} l(g).l^*(f) &= < f, g > \mathcal{I} \\ p(\mathcal{T}_1).p(\mathcal{T}_2) &= p(\mathcal{T}_1.\mathcal{T}_2) \\ p(\mathcal{T})l^*(f) &= l^*(\mathcal{T}f) \\ l(g)p(\mathcal{T}) &= l(\mathcal{T}^*g) \end{aligned}$$

Let $D^o \subseteq \mathcal{F}$ be the set consisting of $\lambda\Omega$ with $\lambda \in \mathcal{C}$ and $|\lambda| \leq 1$ and vectors of the form $u_1 \otimes ... \otimes u_k$ with $k \in \mathcal{N}, u_j \in \mathcal{L}^2(\mathcal{R}_+) \cap \mathcal{L}^\infty(\mathcal{R}_+), \|u_j\|_2 \leq 1, \|u_j\|_\infty \leq 1$ for $1 \leq j \leq k$. For $k = 0, u_1 \otimes ... \otimes u_k = \Omega$. We use $D$ to denote the linear span of $D^o$ It is known that $D$ is dense in $\mathcal{F}$, and that $\mathcal{F}$ is separable. This leads to a useful fact about the strong topology, $\tau_s$, on $\mathcal{B}(\mathcal{F})$, the bounded operators on $\mathcal{F}$. The strong topology is metrisable on bounded subsets of $\mathcal{B}(\mathcal{F})$ and the metric is given by a norm. This follows because $\mathcal{F}$ is a separable Hilbert space and so there exists a countable base $(\varsigma_n)_{n=1}^\infty$. So for $x \in \mathcal{B}(\mathcal{F})$ we define

$$\|x\|_s = \left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\varsigma_n\|^2 \right\}^{\frac{1}{2}}$$

Then $\|.\|_s$ is a norm and $\|x - y\|_s$ provides a metric for the strong operator topology on bounded sets.

**Definition 1** *We define $\mathcal{A}$ to be the *-algebra generated by the annihilation and gauge operators $l(f), p(g)$ respectively and $\mathcal{I}$, where $f \in \mathcal{L}^2(\mathcal{R}_+), g \in \mathcal{L}^\infty(\mathcal{R}_+)$.*

*We shall denote by $\mathcal{V}$ the Von Neumann algebra, $\overline{\mathcal{A}}^{\tau_s}$ in $\mathcal{B}(\mathcal{F})$. Similarly, for any $t \in \mathcal{R}_+$, $\mathcal{A}_t$ is defined to be the *-subalgebra of $\mathcal{A}$ which is generated by $\mathcal{I}$ and the operators $l(f), p(g)$ with $g \in \mathcal{L}^\infty([0,t])$ and $f \in \mathcal{L}^2([0,t])$. $\mathcal{V}_t$ will denote the strong-operator closure of $\mathcal{A}_t$.*

We note that any element of $\mathcal{A}$ can be written as a sum of <u>basic elements</u> of the form $\lambda\mathcal{I}$ or

$$l^*(f_1)...l^*(f_r)p(g)l(h_1)...l(h_s)$$

or

$$l^*(f_1)...l^*(f_r)l(h_1)...l(h_s)$$

with the convention that $r = 0$ (respectively $s = 0$) denotes an element with no creation (respectively no annihilation) operators. Here $r, s \in \mathcal{N} \cup \{0\}$ and $f_i, h_j \in \mathcal{L}^2(\mathcal{R}_+)$ and $g \in \mathcal{L}^\infty(\mathcal{R}_+), 0 \leq i \leq r, 0 \leq j \leq s$. Furthermore if

$$\text{supp } f_i, \text{ supp } h_j, \text{ supp } g \subseteq [0,t] \quad \forall i, j$$

as above, then we get basic elements for $\mathcal{A}_t$. Indeed, for a basic element $x$, of $\mathcal{A}$, with, say,

$$x = l^*(f_1)....l^*(f_r)p(g)l(h_1)...l(h_s)$$

we can define

$$E_t(x) = l^*(f_1\chi_{[0,t)})....l^*(f_r\chi_{[0,t)})p(g\chi_{[0,t)})l(h_1\chi_{[0,t)})...l(h_s\chi_{[0,t)}).$$

A similar definition holds for the other basic elements. The map $E_t$ extends to a conditional expectation of $\mathcal{V}$ onto $\mathcal{V}_t$ with all of the usual properties, see Lemma 7 of [4]. We note in particular that $E_t$ is strongly continuous on bounded subsets of $\mathcal{V}$.
For $A \in \mathcal{A}$ we introduce the notations, $A_+$ and $A_-$. The first of these denotes the sum of those basic elements of $A$ containing only $I$ and creation operators while the second term denotes the sum of those basic elements containing only annihilation operators.

**Definition 2** *We define a process $F(t)$ to be a function*

$$F : \mathcal{R}^+ \to \{ \textit{ unbounded operators with domain containing } \mathrm{D} \ \}$$

*A $\mathcal{V}$-adapted process is a process such that $F(t) \in \mathcal{V}_t$ and similarly for $\mathcal{A}$-adapted processes. We shall call a process simple if it can be written in the form*

$$\sum_{j=1}^{n} F(t_j)\chi_{[t_j,t_{j+1})} \quad \text{with} \quad 0 = t_1 \leq ...t_j \leq t_{j+1} \leq ... \leq t_{n+1} = \infty,$$

$$1 \leq j \leq n \quad F(t_j) \in \mathcal{A}_{t_j}$$

## 2 Prerequisites for a stochastic integral

In this section we will develop some of the prerequisites for a further development of the stochastic integrals with respect to the basic processes. We begin with a summary of definitions and elementary results.

If $\{w_n\}_{n=1}^{\infty}$ is a countable base for $\mathcal{L}^2(\mathcal{R}_+)$ the set of vectors $\xi_n = w_1^{(i)} \otimes \ldots \otimes w_{k(i)}^{(i)}, i \in \mathcal{N}, k(i) \in \mathcal{N}$ and $w_j^{(i)} \in \{w_n\}_{n=1}^{\infty}$ for $1 \leq j \leq k(i)$, are orthonormal and can be chosen so that $\|w_n\|_2 = 1$ for each $n \in \mathcal{N}$ (we assume this hereafter). If we write

$$D^{o\prime} = \{\Omega\} \cup \{w_1^{(i)} \otimes \ldots \otimes w_{k(i)}^{(i)} : i \in \mathcal{N}, k(i) \in \mathcal{N}\}$$

Then $D^{o\prime}$ is a countable base for $\mathcal{F}$.

Let $d \in D^o$ and $\Phi_d : \mathcal{B}(\mathcal{F}) \to \mathcal{C}$ be defined by:

$$\Phi_d(x) =< xd, d >,$$

the vector state for d. Recalling the vectors $\xi_n$ identified above, define a state $\Phi : \mathcal{B}(\mathcal{F}) \to \mathcal{C}$ by

$$\Phi = \sum_{r=1}^{\infty} \frac{1}{2^n} \Phi_{\xi_n}.$$

Both $\Phi$ and $\Phi_d$ are normal faithful states on $\mathcal{V}$. We introduce an inner product on $\mathcal{B}(\mathcal{F})$ by:

$$< x, y >= \Phi(y^* x)$$

This inner product defines the norm:

$$\|x\|_s = \left\{ \sum_{n=1}^{\infty} \frac{1}{2^n} \|x\xi_n\|^2 \right\}^{\frac{1}{2}} \leq \|x\|$$

which, as we noted in the introduction, when restricted to $S$ gives the strong-operator topology on $S$.

With this norm $\mathcal{B}(\mathcal{F})$ becomes a pre-Hilbert space. In order to get a Hilbert space structure we shall consider those possibly unbounded operators $x$, with domain containing $D'$ which satisfy:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \|x\xi_n\|^2 < \infty$$

We identify two such operators, $x$ and $y$, if $y \supseteq x$, that is; $dom(x) \subseteq dom(y)$ and $xh = yh$ for $h \in dom(x)$. We shall denote this space by $\mathcal{L}_2\big\{\mathcal{B}(\mathcal{F})\big\}$. We have

**Lemma 1** $\mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big)$ *is a Hilbert space , with inner product:*

$$< x, y >= \sum_{n=1}^{\infty} \frac{1}{2^n} < x\xi_n, y\xi_n >$$

See [8] for the proof.

**Definition 3** *For a simple process, $F$, we define*

$$\int F dl = \sum_{j=1}^{n} F(t_j) l(\chi_{[t_j,t_{j+1})})$$

*and*

$$\int_0^T F dl = \sum_{j=1}^{n} F(t_j) \cdot l(\chi_{[t_j,t_{j+1})} \cdot \chi_{[0,T)}) \quad \forall T \in \mathcal{R}_+$$

Replacing $l$ with $l^*$ and $p$ above yields the corresponding integrals with respect to the creation and gauge processes. These integrals are linear functions from the space of simple processes into $\mathcal{V}$ and the *processes* $\int_0^t F dl, \int_0^t F dl^*, \int_0^t F dp$ are martingales with respect to the conditional expectation $E_t$. The following is easily proved.

**Lemma 2** *For any simple process $F$,*

$$\int F dl = \int_0^r F dl$$

*for $r \geq t_{n+1}$ and*

$$E_T \int F dl = \int_0^T F dl = \int \chi_{[0,T)} F dl \quad \forall T \in \mathcal{R}_+$$

*With identical results for $dl^*$ and $dp$.*

See [8] for the proof.

We remind ourselves of the definition of $a_+$ for $a \in \mathcal{A}$.

**Definition 4** *For an operator $\mathcal{A}$ with $a = \sum_{i \in I} a_i$ , and $a_i$ basic elements, we define*

$$a_+ = \sum_{i \in \mathcal{J}} a_i$$

*where*

$$\mathcal{J} = \left\{ i \in I : a_i = l^*(f_1^i) \dots l^*(f_{r_i}^i), r_i \in \mathcal{N} \text{ or } a_i = \lambda_i \mathcal{I},\, \lambda_i \in \mathcal{C} \right\}$$

Summarising Lemmas 3.10 and 3.11 of [8] we have;

**Lemma 3** *Let $a \in \mathcal{A}$ then*

$$\|a_+\| = \|a\Omega\| \leq \|a\|$$

*For basic elements, a and b with*

$$a = l^*(f_1)\dots l^*(f_r)p(g)l(h_1)\dots l(h_s)$$

*and*

$$b = l^*(f'_1)\dots l^*(f'_r)l(h'_1)\dots l(h'_s)$$

*with* $r \neq s$ *and each of the vectors* $\xi_n$ *we have:*

$$\Phi_{\xi_n}(a) = \Phi_{\xi_n}(b) = 0$$

See [8] for the proofs.

We can now prove some of the basic inequalities which will drive our integration theory. The philosophy here is that we arrive at an inequality, the left side of which is the integration theory we want to develop while the right side is something altogether more familiar. This allows us to 'bus' results across from the right to the left.

**Theorem 1** *Let* $F$ *a simple processes and* $d \in D^0$*. Recall the definitions of* $\Phi$*,* $\Phi_d$ *and* $\|\cdot\|$*. The following hold:*

$$\Phi_d\Big\{|\int F dl\ |^2\Big\} \leq \int \Big\|F(t)\Big\|^2 dt$$

*and*

$$\Big\|\int F dl\Big\|_s^2 \leq \int \Big\|F(t)\Big\|^2 dt$$

*For* $l^*$ *we get something very similar*

$$\Phi_d\{|\int F dl^*\ |^2\Big\} = \int \Phi_d\Big\{\|F_+(t)^*\|^2\}dt \leq \int \Big\|F(t)\Big\|^2 dt$$

*and*

$$\Big\|\int F dl^*\Big\|_s^2 = \int \Big\|F(t)\Big\|_s^2 dt \leq \int \Big\|F(t)\Big\|^2 dt.$$

*For gauge process we get*

$$\Phi_d\Big\{|\int F dp\ |^2\Big\} = \int \Phi\{|\ F_+(t)\ |^2\}d\mu_{(d)} \leq \int \Big\|F(t)\Big\|^2 dt$$

*where if* $d = u_1 \otimes \ldots \otimes u_k$ *then (recalling that* $\|u_i\|_\infty \leq 1$*)*

$$\mu_{(d)}(E) \equiv \|u_2\|_2^2 \ldots \|u_k\|_2^2 \int_E |u_1(t)|^2 dt \leq \int_E dt$$

*while* $\mu_{(\Omega)} \equiv 0$*. We also have*

$$\Big\|\int F dp\Big\|_s^2 = \int \Big\|F(t)\Big\|_s^2 d\mu \leq \int \Big\|F(t)\Big\|^2 dt$$

*where* $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_{\xi_n}(E)$. *The isometries that follow are nice.*

$$\Phi_d\Big\{\Big(| \int Fdl\Big)^* |^2\Big\} = \int \Phi_d\{| F(t)^* |^2\Big\}dt$$

*and*

$$\Big\|\Big(\int Fdl\Big)^*\Big\|_s^2 = \int \Big\|F(t)^*\Big\|_s^2 dt.$$

**Proof**

$$\Phi_d\Big\{\Big(\int Fdl\Big)^*\Big(\int Fdl\Big)\Big\} \le \Big\|\Big(\int Fdl\Big)^* \cdot \Big(\int Fdl\Big)\Big\| = \Big\|\Big(\int Fdl\Big)^*\Big\|^2.$$

Now since $\mathcal{V}$ is a $C^*$-algebra

$$\Big\|\Big(\int Fdl\Big)^*\Big\|^2 = \Big\|\Big(\int Fdl\Big) \cdot \Big(\int Fdl\Big)^*\Big\|$$

so

$$\begin{aligned} \Big\|\Big(\int Fdl\Big)^*\Big\|^2 &= \Big\|\sum_{j=1}^m \sum_{i=1}^m F(t_j) l(\chi_{[t_j,t_{j+1})}) l^*(\chi_{[t_i,t_{i+1})}) F(t_i)^*\Big\| \\ &= \Big\|\sum_{j=1}^m F(t_j)F(t_j)^*(t_{j+1} - t_j)\Big\| \end{aligned}$$

The last step follows from the relation for $ll^*$. It is now clear that

$$\Big\|\sum_{j=1}^m F(t_j)F(t_j)^*(t_{j+1} - t_j) \le \sum_{j=1}^m \|F(t_j)\|^2(t_{j+1} - t_j) = \int \|F(t)\|^2 dt.$$

Applying this to the expression for $\Big\|\int Fdl\Big\|_s^2$ gives the second inequality.

We turn to the integral with respect to $l^*$:

$$\begin{aligned} \int Fdl^* &= \sum_{j=1}^m F(t_j) l^*(\chi_{[t_j,t_{j+1})} \\ &= \sum_{j=1}^m F_+(t_j) l^*(\chi_{[t_j,t_{j+1})}) + \sum_{j=1}^m \Big\{F(t_j) - F_+(t_j)\Big\} \cdot l^*(\chi_{[t_j,t_{j+1})}) \end{aligned}$$

We now observe that any basic element of $F(t_j) - F_+(t_j)$ is a product of annihilation, gauge or creation operators. The last of these is either $l(h)$ or $p(g)$ with $h$ and $g$ supported in $[0, t_j)$.

Moreover

$$l(h) \cdot l^*(\chi_{[t_j,t_{j+1})}) = \int_{t_j}^{t_{j+1}} \overline{h}(t)dt = 0$$

and

$$p(g) \cdot l^*(\chi_{[t_j,t_{j+1})}) = l^*(\chi_{[t_j,t_{j+1})} \cdot g) = 0$$

Consequently:

$$\Big\{F(t_j) - F_+(t_j)\Big\} \cdot l^*(\chi_{[t_j,t_{j+1})}) = 0 \qquad 1 \leq j \leq m$$

Hence:

$$\int F dl^* = \sum_{j=1}^{m} F_+(t_j) l^*(\chi_{[t_j,t_{j+1})}) = \int F_+ dl^*.$$

At this point we introduce some notation to ease the presentation. We are going to adopt the (dangerous) practice of giving the same object different names according to the level of detail required in the proof. So, for example, a typical term in the expression for $F_+(t_j)$ will have the form;

$$l^*(f_1^{(j,k)}) \dots l^*(f_{r(j,k)}^{(j,k)})$$

and there will be $N(j)$, say, of these. We will write

$$L^*_{j,k} = L^*(f_{r(j,k)}^{(j,k)}) = l^*(f_1^{(j,k)}) \dots l^*(f_{r(j,k)}^{(j,k)}).$$

So we can write $F_+(t_j)$ in the following ways

$$F_+(t_j) = \sum_{k=1}^{N(j)} L^*_{j,k} + \lambda_j = \sum_{k=1}^{N(j)} L^*(f_{r(j,k)}^{(j,k)}) + \lambda_j \mathcal{I}.$$

Returning to the proof

$$\Phi_d\Big(\Big|\int F dl^*\Big|^2\Big) = \Phi_d\Big\{\sum_{i,j=1}^{m} l(\chi_{[t_j,t_{j+1})}) F_+^*(t_j) F_+(t_i) l^*(\chi_{[t_i,t_{i+1})})\Big\}$$

The right side above can be written

$$\sum_{j=1}^{m}\sum_{i=1}^{m} \Phi_d\Bigg\{ l(\chi_{[t_j,t_{j+1})}) \Big[\sum_{k=1}^{N(j)} (L^*_{k,j})^* + \overline{\lambda}_j \mathcal{I}\Big] \cdot \Big[\sum_{k'=1}^{N(i)} L^*_{k',i} + \lambda_i \mathcal{I}\Big] \cdot l^*(\chi_{[t_i,t_{i+1})})\Bigg\}.$$

Some of the terms are zero;

$$\begin{aligned}
&\Phi_d\Big\{\lambda_i l(\chi_{[t_j,t_{j+1})}) \cdot L^*(f_{r(j,k)}^{(j,k)})^* l^*(\chi_{[t_i,t_{i+1})})\Big\} = \\
&= \lambda_i < \chi_{[t_i,t_{i+1})}, f_1^{(j,k)} > \cdot \Phi_d\Big\{ l(\chi_{[t_j,t_{j+1})}) \cdot l(f_{r(j,k)}^{(j,k)}) \dots l(f_2^{(j,k)})\Big\} \\
&= 0
\end{aligned}$$

which follows from Lemma 3. In a similar fashion

$$\Phi_d\Big\{\overline{\lambda}_j l(\chi_{[t_j,t_{j+1})})\sum_{k'=1}^{N(i)} l^*(f_1^{(i,k')})\ldots l^*(f_{r(i,k')}^{(i,k')})l^*(\chi_{[t_i,t_{i+1})})\Big\}=0.$$

So $\Phi_d\Big(\big|\int Fdl^*\big|^2\Big)$ is equal to

$$\sum_{i,j=1}^{m}\Phi_d\Big\{\overline{\lambda}_j\lambda_i l(\chi_{[t_j,t_{j+1})})\cdot l^*(\chi_{[t_i,t_{i+1})})+\sum_{k=1}^{N(j)}\sum_{k'=1}^{N(i)} l(\chi_{[t_j,t_{j+1})})(L^*_{j,k})^*L^*_{i,k'}l^*(\chi_{[t_i,t_{i+1})})\Big\}.$$

and this in turn is equal to

$$\sum_{i,j=1}^{m}\Phi_d\Big\{\overline{\lambda}_j\lambda_i\delta_{ij}(t_{j+1}-t_j)\mathcal{I}+\sum_{k=1}^{N(j)}\sum_{k'=1}^{N(i)}\delta_{r(j,k)r(i,k')}\prod_{s=1}^{r(j,k)}<f_s^{(i,k')},f_s^{(j,k)}>\delta_{ij}(t_{j+1}-t_j)\Big\}.$$

Here we have used Lemma 3 on the terms for which $r(j,k)\neq r(i,k')$ (they are zero) and

$$l(\chi_{[t_j,t_{j+1})})\cdot l^*(\chi_{[t_i,t_{i+1})})=\int_{t_j}^{t_{j+1}}\chi_{[t_i,t_{i+1})}(t)dt=\delta_{ij}(t_{j+1}-t_j).$$

This leaves us with

$$\begin{aligned}
&\sum_{j=1}^{m}\Phi_d\Big\{\Big[\sum_{k=1}^{N(j)}L^*(f_{r(j,k)}^{(j,k)})^*+\overline{\lambda}_j\mathcal{I}\Big]\cdot\Big[\sum_{k'=1}^{N(j)}L^*(f_{r(j,k')}^{(j,k')})+\lambda_j\mathcal{I}\Big]\Big\}(t_{j+1}-t_j)\\
=&\sum_{j=1}^{m}\Phi_d\Big\{F_+(t_j)^*\cdot F_+(t_j)\Big\}\cdot(t_{j+1}-t_j)=\int\Phi_d\Big\{F_+(t)^*\cdot F_+(t)\Big\}dt\\
\leq&\int\|F_+(t)^*\cdot F_+(t)\|dt=\int\|F_+(t)\|^2dt\leq\int\|F(t)\|^2dt
\end{aligned}$$

We have used the fact that $\|d\|\leq 1$, that $\mathcal{V}$ is a $C^*$- algebra and the initial conclusions of Lemma 3. Using Monotone convergence and Lemma 3.

$$\begin{aligned}
\Big\|\int Fdl^*\Big\|_s^2 &= \Phi\Big(\Big|\int Fdl^*\Big|^2\Big)=\sum_{n=1}^{\infty}\frac{1}{2^n}\Phi_{\xi_n}\Big(\Big|\int Fdl^*\Big|^2\Big)\\
&= \sum_{n=1}^{\infty}\frac{1}{2^n}\int\Phi_{\xi_n}\Big\{|F_+(t)^*|^2\Big\}dt=\int\Phi\Big\{|F_+(t)^*|^2\Big\}dt\\
&= \int\|F_+(t)\|_s^2dt\leq\int\|F_+(t)\|^2dt\leq\int\|F(t)\|^2dt
\end{aligned}$$

Turning to the integral with respect to $p$, the proof follows what we have just done for $l^*$ initially, with the same justification. Following these steps we arrive at

$$\int Fdp=\sum_{j=1}^{m}F_+(t_j)p(\chi_{[t_j,t_{j+1})})=\int F_+dp.$$

Writing $F_+(t_j)$ as before we retrace the argument for $l^*$ noting that ,as before, some of the terms are zero. After some work we get that $\Phi_d\Big(\big|\int Fdp\big|^2\Big)$ is equal to

$$\sum_{j=1}^m \Phi_d\Big\{\Big[|\lambda_j|^2+\sum_{k=1}^{N(j)}\sum_{k'=1}^{N(i)}<f_1^{(i,k')},f_1^{(j,k)}>\ldots<f_{r(i,k')}^{(i,k')},f_{r(j,k)}^{(j,k)}>\Big]\mathcal{I}\Big\}\cdot\Phi_d\Big\{p(\chi_{[t_j,t_{j+1})})\Big\}$$

But now

$$\Phi_\Omega\Big\{p(\chi_{[t_j,t_{j+1})})\Big\}=0=\mu_{(\Omega)}\Big\{[t_j,t_{j+1})\Big\}$$

and for $d=u_1\otimes\ldots\otimes u_k$.

$$\begin{aligned}\Phi_d\Big\{p(\chi_{[t_j,t_{j+1})})\Big\} &= <(\chi_{[t_j,t_{j+1})}\cdot u_1)\otimes\ldots\otimes u_k, u_1\otimes\ldots u_k> \\ &= \int_{[t_j,t_{j+1})}|u_1(t)|^2dt\cdot\|u_2\|_2^2\ldots\|u_k\|_2^2 \\ &\equiv \mu_{(d)}\Big\{[t_j,t_{j+1})\Big\}\end{aligned}$$

And hence $\Phi_d\Big\{\Big(\int Fdp\Big)^*\cdot\Big(\int Fdp\Big)\Big\}$ is exactly

$$\begin{aligned} & \sum_{j=1}^m \Phi_d\Big\{|\lambda_j|^2+\sum_{k=1}^{N(j)}\sum_{k'=1}^{N(i)}<f_1^{(i,k')},f_1^{(j,k)}>\ldots<f_{r(i,k')}^{(i,k')},f_{r(j,k)}^{(j,k)}>\Big\}\cdot\mu_{(d)}\Big\{[t_j,t_{j+1})\Big\} \\ = & \sum_{j=1}^m \Phi_d\Big\{\Big[\overline{\lambda}_j\mathcal{I}+\sum_{k=1}^{N(j)}L^*(f_{r(j,k)}^{(j,k)})^*\Big]\cdot\Big[\lambda_j\mathcal{I}+\sum_{k'=1}^{N(j)}L^*(f_{r(j,k)}^{(j,k)})\Big]\Big\}\cdot\mu_{(d)}\Big\{[t_j,t_{j+1})\Big\} \\ = & \sum_{j=1}^m \Phi_d\Big\{F_+(t_j)^*\cdot F_+(t_j)\Big\}\cdot\mu_{(d)}\Big\{[t_j,t_{j+1})\Big\} \\ = & \int\Phi_d\Big\{F_+(t)^*\cdot F_+(t)\Big\}d\mu_{(d)} \\ = & \int\|F_+(t)\|_s^2d\mu_{(d)}\le\int\|F_+(t)\|^2d\mu_{(d)}\le\int\|F(t)\|^2d\mu_{(d)}\end{aligned}$$

Using what we have just proved above

$$\begin{aligned}\Big\|\int Fdp\Big\|_s^2 &= \Phi\Big(\big|\int Fdp\big|^2\Big)=\sum_{n=1}^\infty\frac{1}{2^n}\Phi_{\xi_n}\Big(\big|\int Fdp\big|^2\Big)=\sum_{n=1}^\infty\frac{1}{2^n}\int\|F_+(t)\|_s^2d\mu_{\xi_n} \\ &= \sum_{n=1}^\infty\frac{1}{2^n}\sum_{j=1}^m\|F_+(t_j)\|_s^2\mu_{\xi_n}\Big\{[t_j,t_{j+1})\Big\}=\sum_{j=1}^m\|F_+(t_j)\|_s^2\mu\Big\{[t_j,t_{j+1})\Big\} \\ &= \int\|F_+(t)\|_s^2d\mu\le\int\|F_+(t)\|^2d\mu\le\int\|F(t)\|^2d\mu\le\int\|F(t)\|^2dt.\end{aligned}$$

QED

The isometries which conclude the statement of the theorem are easily proved, the second requires the Monotone convergence theorem. We remark here that the restriction to adapted processes is unnecessary in this case, so our integrands can be non-adapted.

# 3 Integrals

Let $(N, \|\cdot\|)$ be a Banach space, $f : \mathcal{R}_+ \to N$ and $\nu$ a positive measure on $\mathcal{R}_+$. We define $f$ to be a step function if $f(s) = \sum_{j=1}^n f(s_j) \cdot \chi_{[s_j, s_{j+1})}(s)$ with $0 \le s_j < \ldots < s_{n+1} < \infty$.Define $P(\|\cdot\|, \nu)$ to be the space of ($\nu$-equivalence classes) of functions which are the limit in $\nu$-measure and $\nu$ almost everywhere of a sequence of step functions and which satisfy

$$\int \|f(s)\|^2 d\nu < \infty.$$

The norm on $P$ is the $L^2$-norm:

$$\| f \| = \{\int \| f(s) \|^2 d\nu\}^{\frac{1}{2}}.$$

One can show that $P(\|\cdot\|, \nu)$ is a Banach space. If $(N, \|\cdot\|)$ has a Hilbert space structure then $P(\|\cdot\|, \nu)$ is also a Hilbert space. We shall now define stochastic integrals for integrands belonging to a case of the space $P$. We take $N = \mathcal{V}$, the von Neumann algebra generated by the creation, annihilation and gauge operators equipped with the standard operator norm on $\mathcal{B}(\mathcal{F})$. The class of functions we shall be interested in are those which are adapted to the filtration, $(\mathcal{V}_t)$. So the step functions above are now the simple $(\mathcal{V}_t)$-adapted processes.

**Definition 5** *A process $F$ is integrable if $F \in P(\|\cdot\|, dt)$ and there exists a sequence $\big(F_n\big)_{n=1}^\infty$ of simple processes in $P(\|\cdot\|, dt)$ converging to $F$ in the $\|\cdot\|_p$ norm. We denote by $\mathcal{I}$ the set of integrable processes.*

Note: Since $F_n \to F$ as $n \to \infty$ in the $\|\cdot\|_p$ norm, then it is a routine argument to show that $F_n \to F$ in $dt$-measure, and that there is a subsequence $\big(F_{n_k}\big)_{k=1}^\infty$, of $\big(F_n\big)_{n=1}^\infty$, such that $F_{n_k} \to F$, as $k \to \infty$, $dt$-a.e.

**Lemma 4** *Let $F$ be an integrable process with $F_n$ a sequence of simple processes converging to $F$ in $P(\|\cdot\|, dt)$. Then each of the sequence of integrals*

$$\Big(\int F_n dl\Big)_{n=1}^\infty, \Big(\int F_n dl^*\Big)_{n=1}^\infty, \Big(\int F_n dp\Big)_{n=1}^\infty$$

*converge in $\mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big)$.*

So now we can make

**Definition 6** *For an integrable process $F$ we write:*

$$\begin{array}{llll} (i) & \int F dl & \equiv \lim_{n\to\infty} \int F_n dl & \text{in } \mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big) \\ (ii) & \int F dl^* & \equiv \lim_{n\to\infty} \int F_n dl^* & \text{in } \mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big) \\ (iii) & \int F dp & \equiv \lim_{n\to\infty} \int F_n dp & \text{in } \mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big) \end{array}$$

One can show that these integrals are well defined, and lie in $\mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big)$, see [8] for details.

**Lemma 5**

*(i)* *The set of integrable processes forms a vector subspace $\mathcal{J}$ in $P(\|\cdot\|, dt)$ and the maps: $F \mapsto \int Fdl, F \mapsto \int Fdp, F \mapsto \int Fdl^*$ are linear*

*(ii)* *Any integrable process is $\mathcal{V}$-adapted, $dt$-a.e.*

*(iii)* *If $F$ is integrable then so is $F^*$.*

*(iv)* *For an integrable process $F$ with simple processes $\Big(F_n\Big)_{n=1}^{\infty}$ converging to $F$ in $P(\|\cdot\|, dt)$: if*

$$\begin{aligned} \sup_n \Big\| \int F_n dl^* \Big\| &< \infty \\ \sup_n \Big\| \int F_n dl \Big\| &< \infty \\ \sup_n \Big\| \int F_n dp \Big\| &< \infty \end{aligned}$$

*then $\int Fdl^*, \int Fdl, \int Fdp$ are bounded operators, and*

$$\begin{aligned} \int F_n dl^* &\to \int Fdl^* \\ \int F_n dl &\to \int Fdl \\ \int F_n dp &\to \int Fdp \end{aligned}$$

*in the strong operator topology $\tau_s$.*

*(v)* *We can extend our definition of $\int Fdl$ to processes $F$ that are limits in $P(\|\cdot\|, dt)$ of not necessarily adapted step functions $f : \mathcal{R}^+ \to \mathcal{V}$, and $\int Fdl \in \mathcal{B}(\mathcal{F})$ for any such $F$.*

The proof of this Lemma is 'almost' routine. The details are in [8]. Recalling that $\mathcal{I}$ is the space of integrable processes we have,

**Lemma 6** *Suppose $F \in \mathcal{I}$. Letting $M$ denote any of the processes, $l$, $l^*$, $p$. Then*

$$\Big\| \int FdM \Big\|_s^2 \leq \int \|F(t)\|^2 dt$$

*Moreover $\mathcal{I}$ is a closed subspace of $P(\|\cdot\|, dt)$.*

Again we refer to [8] for the details. Once again denoting by $M$ any one of the basic processes, then for an integrable process $F$ we define

$$\int_0^T F dM = \int \chi_{[0,T)} \cdot F dM.$$

We observe that this definition agrees with our earlier one for simple processes and that each of the integrals exist. It is a short argument to show that $\chi_{[0,T)} \cdot F \in P(\|\cdot\|, dt)$. As one might expect this 'local' integrability for every $T > 0$ does not imply integrability.

We shall now prove martingale properties for our integrals; in order to do that we need to extend the definition of our conditional expectation $E_t$, as follows:

**Definition 7** *For $x$ a possibly unbounded operator with domain containing $D$, such that there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\mathcal{A}$ with $x_n d \to xd \ \forall d \in D^o$ then define:*

$$(E_t x)d = \lim_{n\to\infty} (E_t x_n)d, \quad \forall d \in D^o$$

The limit defining $(E_t x)d$ exists. This is proved in Lemma 5 of [4] which shows that $((E_t x_n)d)$ is Cauchy in $\mathcal{F}$. The same theorem may be used to show that $(E_t x)d)$ is well defined. We extend to $D$ by linearity. Finally, we note that for $x \in \mathcal{V}$ this definition agrees with definition 6 of [4] where it is shown that for a sequence $(a_n)_{n=1}^{\infty}$ in $\mathcal{A}$ such that $a_n \to x$ in the $\tau_s$-topology:

$$E_t x = \lim_{n\to\infty} E_t a_n$$

in the $\tau_s$-topology. Which of course implies

$$\Rightarrow (E_t x)d = \lim_{n\to\infty} (E_t a_n)d \qquad \forall d \in D^o$$

But $a_n \to x \ \ \tau_s$-on $D^o$, and so the two definitions agree.

**Lemma 7** *For an integrable process $F$ we can extend the operators*

$$\int F dl^*, \int F dl, \int F dp$$

*to have domain containing $D$.*

**Proof**

Suppose $(F_n)_{n=1}^{\infty}$ is a sequence of simple processes with $F_n \to F$ in $P(\|\cdot\|, dt)$ . Hence, $\forall \epsilon > 0 \ \ \exists N(\epsilon)$ such that:

$$n, m \geq N(\epsilon) \Rightarrow \|F_n - F_m\|_p < \epsilon$$

and using Theorem 1

$$\begin{aligned}
\Phi_d\Big\{\Big(\int (F_n - F_m)dl\Big)^* \cdot \Big(\int (F_n - F_m)dl\Big)\Big\} &\le \int \|F_n(t) - F_m(t)\|^2 dt < \epsilon^2 \\
\Phi_d\Big\{\Big(\int (F_n - F_m)dl^*\Big)^* \cdot \Big(\int (F_n - F_m)dl^*\Big)\Big\} &\le \int \|F_n(t) - F_m(t)\|^2 dt < \epsilon^2 \\
\Phi_d\Big\{\Big(\int (F_n - F_m)dp\Big)^* \cdot \Big(\int (F_n - F_m)dp\Big)\Big\} &\le \int \|F_n(t) - F_m(t)\|^2 dt < \epsilon^2
\end{aligned}$$

for $n, m \ge N(\epsilon)$. So $\Big[\Big(\int F_n dl\Big)d\Big]_{n=1}^{\infty}, \Big[\Big(\int F_n dl^*\Big)d\Big]_{n=1}^{\infty}, \Big[\Big(\int F_n dp\Big)d\Big]_{n=1}^{\infty}$ are Cauchy sequences in $\mathcal{F}$ and so we can define

$$\begin{aligned}
\Big(\int F dl\Big)d &\equiv \lim_{n\to\infty}\Big(\int F_n dl\Big)d \\
\Big(\int F dl^*\Big)d &\equiv \lim_{n\to\infty}\Big(\int F_n dl^*\Big)d \\
\Big(\int F dp\Big)d &\equiv \lim_{n\to\infty}\Big(\int F_n dp\Big)d
\end{aligned}$$

in $\mathcal{F}, \forall d \in D^o$.

This defines the operators $\int F dl, \int F dl^*, \int F dp$ on $D^o$ and by linearity, on $D$. One can prove that these operators are independent of the particular sequence, $(F_n)$, of simple processes. We can define the extension of the stochastic integrals $\int_0^T F dl, \int_0^T F dl^*, \int_0^T F dp$ above and these will now denote the operators $\int F \cdot \chi_{[0,T)} dl, \int F \cdot \chi_{[0,T)} dl^*, \int F \cdot \chi_{[0,T)} dp$ with domain containing $D$. Since span $D' \subseteq D^o$ we note that the above operators are extensions of those defined previously. QED

With this understood we have

**Theorem 2** *For $\int F dl, \int F dl^*, \int F dp$ described above the following holds;*

$$\begin{aligned}
E_t \int_0^T F dl &= \int_0^t F dl \qquad \text{for } t \le T < \infty \\
E_t \int_0^T F dl^* &= \int_0^t F dl^* \qquad \text{for } t \le T < \infty \\
E_t \int_0^T F dp &= \int_0^t F dp \qquad \text{for } t \le T < \infty
\end{aligned}$$

*and also;*

$$\begin{aligned}
E_t \int F dl &= \int_0^t F dl \\
E_t \int F dl^* &= \int_0^t F dl^* \\
E_t \int F dp &= \int_0^t F dp
\end{aligned}$$

*for $t \in \mathcal{R}^+$.*

**Proof**

From Lemma 5, for each integrable process $F$ there exists a sequence $\left(F_n\right)_{n=1}^{\infty}$ of simple processes for which:

$$\int F dl = \lim_{n\to\infty} \int F_n dl \quad \tau_s\text{- on } D$$

and

$$\begin{aligned} \int_0^T F dl &= \int F \cdot \chi_{[0,T)} dl \\ &= \lim_{n\to\infty} \int F_n \chi_{[0,T)} dl \quad \tau_s\text{- on } D \\ &\quad \text{by Lemma 5} \end{aligned}$$

Recalling how $E_t$ has been defined above, for $t \leq T < \infty$ we have:

$$\begin{aligned} E_t \textstyle\int F dl &= \lim_{n\to\infty} E_t \textstyle\int F_n dl & \tau_s\text{- on } D^o \\ \text{and} \quad E_t \textstyle\int_0^T F dl &= \lim_{n\to\infty} \textstyle\int F_n \cdot \chi_{[0,T)} dl & \tau_s\text{- on } D^o \end{aligned}$$

Hence:

$$\begin{aligned} E_t \int F dl &= \lim_{n\to\infty} \int \chi_{[0,t)} \cdot F_n dl \quad \tau_s\text{- on } D^o \\ \text{and } E_t \int_0^T F dl &= \lim_{n\to\infty} \int \chi_{[0,t)} \cdot F_n dl \quad \tau_s\text{- on } D^o \\ &\quad \text{by Lemmas 8 and 14} \\ \text{or } E_t \int F dl &= \int_0^t F dl \\ &= E_t \int_0^T F dl \end{aligned}$$

The other cases are similar. QED

We conclude this section by using the isometry properties of Theorem 1 to define three integrals $I_l, I_p, I_{l^*}$. In the next section we shall use these to represent certain operators as stochastic integrals.

First we shall denote by $\|\cdot\|_{s^*}$ the norm on $\mathcal{B}(\mathcal{F})$ given by $\|x\|_{s^*} = \|x^*\|_s$. with $\overline{\mathcal{V}}^{s^*}$ and $\mathcal{L}_2(\mathcal{V})$ the completions of $\mathcal{V}$ according to $\|\cdot\|_{s^*}$ and $\|\cdot\|_s$ respectively and, as described at the start of this section, we can define the Banach spaces $P(\|\cdot\|_{s^*}, dt)$ and $P(\|\cdot\|_s, d\mu)$ and $P(\|\cdot\|_s, dt)$.

**Definition 8**

*Let $F$ be a function $F : \mathcal{R}^+ \to \overline{\mathcal{V}}^s$ in $P(\|\cdot\|_{s^*}, dt)$. We call $F$ $l$-integrable iff there exists a sequence of step functions $\left(F_n\right)_{n=1}^{\infty} F_n : \mathcal{R}_+ \to \mathcal{V}$ such that: $F_n \to F$ in $P(\|\cdot\|_{s^*}, dt)$.*

*Let*

$$I_l(F) = \lim_{n\to\infty} \int F_n dl$$

*in* $\overline{\mathcal{B}(\mathcal{F})}^{s^*}$ *,the completion of* $\mathcal{B}(\mathcal{F})$ *with respect to the* $\|\cdot\|_{s^*}$*-norm.*

*(ii) Let* $F$ *a function* $F : \mathcal{R}_+ \to \mathcal{V}$ *in* $P(\|\cdot\|_s, d\mu)$*. We call* $F$ *p-integrable iff there exists a sequence of simple processes* $\left(F_n\right)_{n=1}^{\infty}$ *with :* $F_n \to F$ *in* $P(\|\cdot\|_s, d\mu)$*.*

*Let*

$$I_p(F) = \lim_{n\to\infty} \int F_n dp$$

*in* $\mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big)$*.*

*(iii) Let* $F$ *a function* $F : \mathcal{R}_+ \to \mathcal{V}$ *in* $P(\|\cdot\|_s, dt)$*. We call* $F$ $l^*$*-integrable iff there exists a sequence of simple processes* $\left(F_n\right)_{n=1}^{\infty}$ *with:* $F_n \to F$ *in* $P(\|\cdot\|_s, dt)$*.*

*Let*

$$I_{l^*}(F) = \lim_{n\to\infty} \int F_n dl^*$$

*in* $\mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big)$*.*

The above integrals are well defined by virtue of the isometries of Theorem 1.

These integrals are linear maps on the corresponding space of process and furthermore they are extensions of the stochastic integrals of Definition 6 since:

if $F$ is an integrable process with $\left(F_n\right)_{n=1}^{\infty}$ a sequence of simple processes such that $F_n \to F$ in $P(\|\cdot\|, dt)$ then,

$$\begin{array}{rccl} & F_n \to F & \text{in} & P(\|\cdot\|_s, d\mu) \\ & & \text{and in} & P(\|\cdot\|_s, dt) \\ \text{and also} & F_n \to F & \text{in} & P(\|\cdot\|_{s^*}, dt) \end{array}$$

since

$$\begin{aligned} \int \left\|F_n(t) - F(t)\right\|_{s^*}^2 dt &= \int \left\|F_n^*(t) - F(t)^*\right\|_s^2 dt \\ &\leq \int \left\|F_n^*(t) - F(t)^*\right\|^2 dt \\ &= \int \left\|F_n(t) - F(t)\right\|^2 dt \end{aligned}$$

Hence $F$ is $p$-integrable, $l^*$-integrable and $l$-integrable and:

$$\begin{aligned} \textstyle\int F dp &= \textstyle\lim_{n\to\infty} \int F_n dp \quad \text{in } \mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big) \\ &= I_p(F) \\ \textstyle\int F dl^* &= \textstyle\lim_{n\to\infty} \int F_n dl \quad \text{in } \mathcal{L}^2\Big(\mathcal{B}(\mathcal{F})\Big) \\ &= I_{l^*}(F) \end{aligned}$$

Finally,

$$\int F dl = \lim_{n\to\infty} \int F_n dl$$

in the $\tau_s$-topology of $\mathcal{B}(\mathcal{F})$.[by Lemma 5 (v)] and so

$$\Big(\int F dl\Big)^* = \lim_{n\to\infty} \Big(\int F_n dl\Big)^*$$

in the $\tau_w$-topology in $\mathcal{B}(\mathcal{F})$.

But

$$I_l(F)^* = \lim_{n\to\infty} \big(\int F_n dl\big)* \quad \text{in } \|\cdot\|_s\text{- norm}$$

$$\sup_n \|\big(\int F_n dl\big)^*\| < \infty \text{ by Lemma 5 (v)}$$

and so

$$\begin{aligned} I_l(F)^* &= \lim_{n\to\infty}\big(\textstyle\int F_n dl\big)^* \quad \text{in } \tau_s\text{-topology} \\ &\text{by Lemma 5} \\ \Rightarrow I_l(F)^* &= \lim_{n\to\infty}\big(\textstyle\int F_n dl\big)^* \quad \text{in } \tau_w\text{-topology in } \mathcal{B}(\mathcal{F}) \end{aligned}$$

Combining we get

$$I_l(F) = \int F dl.$$

# 4 A representation result

The result proved below was inspired by the proof of the representation result offered by I.F.Wilde in [9]. Indeed it is fair to say that our result is simply a reworking of his argument in this context.

**Definition 9** *Let $\mathcal{A}_l$ be the *-subalgebra of $\mathcal{A}$ generated by basic elements of the form:*

$$\begin{gathered} l^*(f_1)\dots l^*(f_r)p(g)l(h_1)\dots l(h_s) \\ l^*(f_1)\dots l^*(f_r)l(h_1)\dots l(h_s) \\ \lambda\mathcal{I} \end{gathered}$$

*where $f_i, h_j \in \mathcal{L}^2(\mathcal{R}_+), g \in \mathcal{L}^\infty(\mathcal{R}_+) \quad r, s \geq 1$ and $\lambda \in \mathcal{C}$.*

Now consider the strong closure of $\mathcal{A}_l$. Because the strong operator topology is in this case metrisable we see that for $x \in \overline{\mathcal{A}_l}^{\tau_s}$ there exists a sequence $(a_n)_{n=1}^\infty$ in $\mathcal{A}_l$ with;

$$a_n \to x \qquad \text{in } \tau_s\text{-topology.}$$

But for $x \in \overline{\mathcal{A}_l}^{\tau_s}, x^* \in \overline{\mathcal{A}_l}^{\tau_s}$ and so there exists a sequence $(b_n)_{n=1}^\infty$ in $\mathcal{A}_l$ such that:

$$\begin{aligned} b_n \to x^* \quad &\text{in} \quad \tau_s - \text{topology} \\ \Rightarrow b_n \to x^* \quad &\text{in} \quad \|\cdot\|_s - \text{topology} \\ &\qquad \text{because (Banach-Steinhaus): } \sup_n\|b_n\| < \infty \\ \Rightarrow b_n^* \to x \quad &\text{in} \quad \|\cdot\|_{s^*} - \text{topology} \end{aligned}$$

with $(b_n^*)_{n=1}^\infty$ in $\mathcal{A}_l$.

**Theorem 3** *For any* $x \in \overline{\mathcal{A}_l}^{\tau_s}$ $\exists \lambda \in \mathcal{C}$ *and an* $l$*-integrable process* $F$ *in* $P(\|\cdot\|_{s^*}, dt)$ *with:*

$$x = \lambda \mathcal{I} + I_l(F)$$

*This representation is unique.*

**Proof**

The set of step functions in $\mathcal{L}^2(\mathcal{R}_+)$ is dense in $\mathcal{L}^2(\mathcal{R}_+)$. Hence $\forall f \in \mathcal{L}^2(\mathcal{R}_+)$ there exists a sequence of step functions $\left(\ell^{(n)}\right)_{n=1}^\infty$ with

$$\|\ell^{(n)} - f\|_2 \to 0$$

so

$$\|l^*(\ell^{(n)}) - l^*(f)\| = \|\ell^{(n)} - f\|_2 \to 0$$

and

$$\|l(\ell^{(n)}) - l(f)\| = \|\ell^{(n)} - f\|_2 \to 0$$

so for

$$l^*(f_1)\dots l^*(f_r)p(g)l(h_1)\dots l(h_s)$$

or

$$l^*(f_1)\dots l^*(f_r)l(h_1)\dots l(h_s)$$

in $\mathcal{A}_l$, there are sequences $\left(\ell_i^{(n)}\right)_{n=1}^\infty$ and $\left(\eta_j^{(n)}\right)_{n=1}^\infty$ of step functions in $\mathcal{L}^2(\mathcal{R}_+)$, for $1 \le i \le r$, $1 \le j \le s$ such that: $l^*(\ell_i^{(n)}) \to l^*(f_i)$ and $l(\eta_j^{(n)}) \to l(h_j)$ in $\mathcal{B}(\mathcal{F})$.

Then

$$l^*(\ell_1^{(n)})\dots l^*(\ell_r^{(n)})p(g)l(\eta_1^{(n)})\dots l(\eta_s^{(n)}) \to l^*(f_1)\dots l^*(f_r)p(g)l(h_1)\dots l(h_s)$$

and

$$l^*(\ell_1^{(n)})\dots l^*(\ell_r^{(n)})l(\eta_1^{(n)})\dots l(\eta_s^{(n)}) \to l^*(f_1)\dots l^*(f_r)l(h_1)\dots l(h_s)$$

in $\mathcal{B}(\mathcal{F})$.

So any element in $\mathcal{A}_l$ can be written as a limit in $\mathcal{B}(\mathcal{F})$, of linear combinations of operators of the form:

$$\lambda \mathcal{I} \quad \lambda \in \mathcal{C}$$

$$\text{or} \quad l^*(\chi_{\mathcal{J}_1})\dots l^*(\chi_{\mathcal{J}_r})p(g)l(\chi_{\mathcal{J}'_1})\dots l(\chi_{\mathcal{J}'_s})$$
$$\text{or} \quad l^*(\chi_{\mathcal{J}_1})\dots l^*(\chi_{\mathcal{J}_r})l(\chi_{\mathcal{J}'_1})\dots l(\chi_{\mathcal{J}'_s})$$

with $\mathcal{J}_1, \dots, \mathcal{J}_r, \mathcal{J}'_1, \dots, \mathcal{J}'_s$ intervals in $\mathcal{R}_+, g \in \mathcal{L}^\infty(\mathcal{R}_+)$, and $r, s \ge 1$.

What this shows us is that if $x \in \overline{\mathcal{A}_l}^{\tau_s}$, there exists a sequence $(x_n)_{n=1}^{\infty}$ of linear combinations of elements of the form described immediately above and $(\lambda_n)_{n=1}^{\infty}$ in $\mathcal{C}$ with:

$$\lambda_n \mathcal{I} + x_n \to x \quad \text{in } \|\cdot\|_{s^*}\text{-topology}$$

Now a moments thought will convince one that

$$\begin{aligned} & l^*(\chi_{\mathcal{J}_1}) \dots l^*(\chi_{\mathcal{J}_r}) p(g) l(\chi_{\mathcal{J}'_1}) \dots l(\chi_{\mathcal{J}'_s}) = \\ &= \quad I_l \Big\{ l^*(\chi_{\mathcal{J}_1}) \dots l^*(\chi_{\mathcal{J}_r}) p(g) l(\chi_{\mathcal{J}'_1}) \dots l(\chi_{\mathcal{J}_{s-1}}) \cdot \chi_{\mathcal{J}'_s}(t) \Big\} \end{aligned}$$

with a similar relation for

$$l^*(\chi_{\mathcal{J}_1}) \dots l^*(\chi_{\mathcal{J}_r}) l(\chi_{\mathcal{J}'_1}) \dots l(\chi_{\mathcal{J}'_s}).$$

By linearity of $I_l$ we have step functions $F_n : \mathcal{R}^+ \to \mathcal{V}$ with

$$x_n = \int F_n dl$$

Since $\Big(\lambda_n \mathcal{I} + x_n\Big)_{n=1}^{\infty}$ is $\|\cdot\|_{s^*}$ Cauchy we get that

$$\begin{aligned} & \Big\{ \big(\lambda_n \mathcal{I} + x_n\big)^* \Omega \Big\}_{n=1}^{\infty} && \text{is Cauchy} \\ \Rightarrow \quad & \Big\{ < (\lambda_n \mathcal{I} + x_n)\Omega, \Omega > \Big\}_{n=1}^{\infty} && \text{is Cauchy} \\ \Rightarrow \quad & (\lambda_n)_{n=1}^{\infty} && \text{is Cauchy} \\ & && \text{since } x_n \Omega = 0 \ \ \forall n \in \mathcal{N} \end{aligned}$$

Let $\lambda = \lim_{n\to\infty} \lambda_n$. Then $\lambda_n \mathcal{I} \to \lambda \mathcal{I}$ in $\|\cdot\|_{s^*}$-norm and so :

$$x_n \to x - \lambda \mathcal{I} \qquad \text{in } \|\cdot\|_{s^*}\text{-norm}$$

Hence

$$I_l(F_n) \to x - \lambda \mathcal{I} \qquad \text{in } \|\cdot\|_{s^*}\text{-norm}$$

By the isometry property (iv) of Theorem 1, $(F_n)_{n=1}^{\infty}$ is Cauchy in $P(\|\cdot\|_{s^*}, dt)$ and so there exists a function $F : \mathcal{R}_+ \to \overline{\mathcal{V}}^{s^*}$ in $P(\|\cdot\|_{s^*}, dt)$ such that: $F_n \to F$ in $P(\|\cdot\|_{s^*}, dt)$. Hence $F$ is $l$-integrable and $I_l(F) = \lim_n I_l(F_n)$ in $\|\cdot\|_{s^*}$-norm.

Hence

$$x = \lambda \mathcal{I} + I_l(F)$$

and the existence part of this result is proved.

If $\exists \lambda' \in \rfloor, F' \in P(\|\cdot\|_{s^*}, dt)$ such that $x = \lambda' \mathcal{I} + I_l(F')$ then

$$\begin{aligned} (\lambda - \lambda')\mathcal{I} + I_l(F - F') &= 0 \\ \Rightarrow (\lambda - \lambda')\mathcal{I}\Omega &= 0 \\ \Rightarrow \lambda &= \lambda' \end{aligned}$$

since $I_l(F)\Omega = 0 \ \ \forall F$ which are $l$-integrable

Furthermore

$$\begin{aligned} 0 &= \|I_l(F-F')\|_{s^*}^2 \\ &= \lim_{n\to\infty} \| \int G_n dl\|_{s^*}^2 \\ &\quad \text{with } G_n \text{ simple } G_n \to F-F' \text{ in } P(\|\cdot\|_{s^*}, dt) \\ &= \lim_{n\to\infty} \|G_n\|_p^2 \\ &= \|F-F'\|_p \end{aligned}$$

Therefore,

$$F = F'$$

and the representation is unique. QED

We make a few observations.

1. Similar results can be obtained for representations of certain classes of operators, as $I_{l^*}$ and $I_p$ integrals of $l^*$ and $p$-integrable processes respectively.

2. The *-operator $\mathcal{V} \to \mathcal{V}$ is an antilinear isometry that can be extended to a surjective antilinear isometry $\Psi : \mathcal{L}^2(\mathcal{V}) \to \overline{\mathcal{V}}^{s^*}$ thus providing us with a concrete representation of the abstract completion $\overline{\mathcal{V}}^{s^*}$.

# 5 Belated Integrals

We compare our results of the earlier sections with the outcome of applying the theory of *Belated Integrals* to our situation. Belated integrals were developed by Barnett and Wilde in [3]. It is a vector integration theory derived from the general bilinear vector integral of R. G. Bartle, [5]. In Bartles' theory both the integrand and the integrator are 'vector' valued. In order to arrive at a numerical assay for subsets of a measurable space one employs the *semivariation* of the vector measure. This 'measure' of the size of sets allows one to develop a general integral. In the Belated theory one uses the *belated semivariation* of the vector measure in place of the usual semivariation. The reason for this modification is that the usual semivariation is infinite in many cases where the belated semivariation is finite. We apply this theory to the present situation.

Fix a real number $T > 0$. Let $\Phi$ denote the field of subsets of $[0, T]$ generated by the intervals. We note that each element of $\Phi$ can be written as a finite union of disjoint intervals. For $E \in \Phi$ we define

$$l(E) = l(I_E), l^*(E) = l^*(I_E), p(E) = p(I_E).$$

Each of these functions is a finitely additive set function on $\Phi$. The belated semivariation of $E \in \Phi$ with respect to the set function $l$ is

$$\|E\|^l = sup \| \sum_i \alpha_i l(E_i)\|_s$$

where the supremum is taken over all partitions of $E$ in $\Phi$, and choices $\alpha_i \in \mathcal{A}_{infE_i}$ with $\|\alpha_i\| \leq 1$. The semivariations with respect to $l^*$ and $p$ are defined in exactly the same fashion. We observe here that

$$\sum_i \alpha_i l(E_i) = \int \phi dl$$

where

$$\phi = \sum_i \alpha_i I_{E_i}.$$

and $\phi$ is a simple process; the simple process associated with the partition $E_i$ and choice of $\alpha_i$'s. Indeed we can assume that the partition of $E$ comprises *intervals* $E_i$. With this in mind we have

**Lemma 8** *For $E \in \Phi$ we have*

$$\|E\|^l \leq \lambda(E)^{\frac{1}{2}}$$

*where $\lambda$ denotes Lebesgue measure. We also have*

$$\|E\|^{l^*} = \lambda(E)^{\frac{1}{2}}$$

*and*

$$\|E\|^p = \mu(E)^{\frac{1}{2}}$$

*where $\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_{\xi_n}(E)$ and for $\xi_n = w_1^{(n)} \otimes \ldots \otimes w_{k(n)}^{(n)}$*

$$\mu_{\xi_n}(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \int \mid w_1^n(t) \mid^2 I_E dt.$$

**Proof**
Let $(E_i)$ be a partition of $E$ and $\alpha_i$ a choice of elements from the unit ball(s) of $\mathcal{A}_{infE_i}$. Denote by $\phi$ the associated simple process. By Theorem 1 (i)

$$(\|E\|^l)^2 = sup \| \int \phi dl\|_s^2 \leq sup \int \|\phi(t)\|^2 dt$$

the last supremum is clearly dominated by $\lambda(E)$. Now for the gauge semivariation we have

$$\|E\|^p = sup\| \int \phi dp\|_s^2 = sup \int \|\phi_+(t)\|_s^2 d\mu.$$

Now

$$\int \|\phi_+(t)\|_s^2 dt = \int \sum_{n=1}^{\infty} \frac{1}{2^n} \|\phi(t)_+(\xi_n)\|^2 dt.$$

Since $\phi(t)_+$ contains sums of basic elements with creation operators

$$\|\phi(t)_+(\xi_n)\|^2 = \|\phi(t)_+(\Omega) \otimes \xi_n\|^n = \|\phi(t)_+(\Omega)\|^2.$$

It now follows that if we select $\phi(t) = \phi(t)_+ = \mathcal{I}$ then the supremum is attained at exactly $\mu(E)$. A very similar calculation proves the result for the $l^*$ semivariation. QED

So each of the semivariations associated with the basic processes is finite on bounded subsets of $\mathcal{R}$. What is more, there is in each case a *control measure* and so the results of section 2 of [3] apply here. With the semivariations playing the role usually occupied by measures we can define the notions of convergence in $dl$-measure (respectively $dl^*$ and $dp$ measure) and convergence $dl$-almost everywhere (respectively $dl^*$ and $dp$ almost everywhere). For a $dl$ (respectively $dl^*$, $dp$)-measurable function $f$ with $(f_n)$ a sequence of simple processes converging to $f$ in $dl$ (resp' $dl^*$, $dp$) measure we say that $f$ is $dl$ (resp' $dl^*$, $dp$) integrable if $(f_n)$ can be chosen so that

$$\forall \epsilon > 0 \, \exists \, \delta > 0 \, : \, E \in \Phi \;\; \|E\|^l < \delta \; then \; \forall n \; \| \int_E f_n dl \|_s < \epsilon$$

with the appropriate modifications for the other cases. Note that the second condition, (b), of definition 3.1 of [3] is automatically satisfied because the semivariation of $[0, T]$ is finite (in every case). We say that $f$ is belated integrable with respect to $dl$ (resp' $dl^*$, $dp$ ). It is natural to ask if a process which is integrable in the sense of definiton 7 above with respect to one of the basic processes is belated integrable with respect to that process. The answer is

**Theorem 4** *The set of processes integrable with respect to dl in the sense of definition 7 is a subset of the set of processes which are belated integrable with respect to dl and the two integrals agree.*

**Proof**

Let $f \in P(\| \cdot \|, dt)$ and let $(f_n)$ be a sequence of simple processes converging to $f$ in $P(\| \cdot \|, dt)$ we can assume that the convergence is in Lebesgue measure and Lebesgue almost everywhere. In view of the last lemma this means that $(f_n)$ converges to $f$ in $\| \cdot \|^l$ measure and $\| \cdot \|^l$ almost everywhere. So we must prove

$$\forall \epsilon > 0 \exists \delta > 0 : E \in \Phi \; and \; \|E\|^l < \delta \; then \; sup_n \| \int_E f_n dl \|_s < \epsilon.$$

We argue by contradiction. Suppose that there is $\epsilon > 0$ and a sequence of sets, $(E_m)$ in $\Phi$ with $\|E_m\|^l < \frac{1}{2^m}$ and

$$sup \| \int_{E_m} f_n dl \|_s^2 > \epsilon$$

Define $B_n = \cup_{m=n}^{\infty} E_m$ then $\cap_{n=1}^{\infty} B_n = limsup E_m$. The sets, $B_n$, are outer sets with their semivariation defined by (2.11 of [3])

$$\|B_n\|^l lim_{M \longrightarrow \infty} \| \cup_{m=n}^M E_m \|^l$$

and because the semivariation is countably subadditive (2.10 of [3]) then

$$\|B_n\| \leq \sum_{m=n}^{\infty} \|E_m\| < \sum_{m=n} (\frac{1}{2^m}) = \frac{1}{2^{n+1}}.$$

Now it is easy to see that

$$\| \cup_{m=n}^{M} E_m \|^l \geq \| l(I_{\cup_{m=n}^{M} E_m}) \|_s$$

and so

$$\| l(I_{\cup_{m=n}^{M} E_m}) \|_s^2 = \sum_n \frac{1}{2^n} \| l(I_{\cup_{m=n}^{M} E_m}) \xi_n \|^2 = \sum_n \frac{1}{2^n} \mid \int w_1^{(n)} I_{\cup_{m=n}^{M} E_m} dt \mid^2 .$$

And by the dominated convergence theorem

$$\int w_1^{(n)} I_{\cup_{m=n}^{M} E_m} dt \longrightarrow \int w_1^{(n)} I_{B_n} dt$$

so that

$$\sum_n \frac{1}{2^n} \mid \int w_1^{(n)} I_{\cup_{m=n}^{M} E_m} dt \mid^2 \longrightarrow \sum_n \frac{1}{2^n} \mid \int w_1^{(n)} I_{\cup B_n} dt \mid^2$$

as $M \longrightarrow \infty$ by the monotone convergence theorem. Therefore

$$\| l(I_{\cup_{m=n} E_m}) \|_s \longrightarrow \| l(I_{B_n}) \|_s$$

and by taking a limit $\|B_n\|^l \geq \|l(B_n)\|_s$. Using much the same argument as above we have

$$\| l(I_{limsupE_n}) \|_s = lim \| l(I_{B_n}) \|_s = 0.$$

From this we deduce that $\lambda(limsupE_n) = 0$. However, from Theorem 1 and the assumptions concerning the sequence $(E_n)$ we have

$$sup_n \int_{E_m} \|f_n\|^2 dt > \epsilon$$

which, since $B_m \supseteq E_m$, tells us that

$$sup_n \int_{B_m} \|f_n\|^2 dt > \epsilon.$$

But $\lambda(B_m) \longrightarrow \lambda(limsupE_m) = 0$ and so

$$\int_{B_m} \|f_n\|^2 dt \leq \int_{[0,T]} \|f - f_n\|^2 dt + \int_{B_m} \|f\|^2 dt + \{\int_{B_m} \| f \|^2 dt\}^{\frac{1}{2}} \cdot \{\int \|f - f_n\|^2 dt\}^{\frac{1}{2}} .$$

Now the first of the right hand terms tends to 0 by the hypothesis of the theorem while the remaining terms are both small once $m \in \mathcal{N}$ is chosen large enough, $m \geq M$, say. So there is $N \in \mathcal{N}$ such that for $n \geq N$ and $m \geq M$ we have

$$\int_{B_m} \|f_n\|^2 dt < \epsilon.$$

For $n \in \{1, 2, 3, \ldots, N-1\}$ we use the dominated convergence theorem, $N-1$ times, to obtain $M_1$,$M_2$,....,$M_{N-1}$, such that for $m > max\{M_1, M_2, ... M_{N-1}, m\}$ we have

$$sup_n \int_{B_m} \|f_n\|^2 dt \leq \epsilon.$$

This is contradiction. So $f$ is belated integrable and it is clear that the two kinds of integral coincide. QED

We turn to the other integrals now.

**Theorem 5** *The set of processes which are integrable with respect to $l^*$ in the sense of definition 5 are a subset of the the belated integrable processes and the two integrals agree.*

**Proof**

If $f$ is integrable in the manner of definition 5 then there are simple processes, $(f_n)$ converging to $f$ in $P(\|\cdot\|, dt)$, and Lebesgue almost everywhere (and therefore in measure since $T < \infty$). The following inequalities hold,

$$\begin{aligned} \|\int_E f_n dL^*\|_s^2 &\leq \int_E \|f_n\|^2 dt \\ &\leq \int \|f-f_n\|^2 dt + \int_E \|f\|^2 dt + 2\{\int_E \|f-f_n\|^2 dt\}^{\frac{1}{2}}\{\int_E \|f\|^2 dt\}^{\frac{1}{2}} \end{aligned}$$

the first follows directly from theorem 1 the second from the triangle inequality. Now the first term can be made small by choosing $n \in \mathcal{N}$ sufficiently large, greater than $N$ say, and the second term can be made small by choosing $\lambda(E)$ small enough. This takes care of the third term also. Now as $\|E\|^{l^*} = \lambda(E)^{\frac{1}{2}}$ we see that by taking the $l^*$ - semivariation small enough then

$$\|\int_E f_n dl^*\|_s^2 < \epsilon$$

for $n \geq N$. For $1 \leq N-1$ one chooses the $l^*$ semivariation appropriately to make the $s-norm$ of the integrals small, again using theorem 1. It is then clear the $f$ is belated integrable and the the two integrals coincide. QED

One can prove a corresponding result for the gauge integral employing arguments parallel to those in the last theorem. We leave the details for the reader and state

**Corollary 1** *The collection of p-integrable processes (in the sense of definition 7) is a subset of the set of p-belated integrable processes and the two integrals agree.*

We conclude our discussion with an example. Let $[0, T]$ be the interval $[0, 1]$. Consider the function

$$f(s) = \frac{1}{s} l(I_{[0,s]})$$

this is an adapted process. Let $I(k, n)$ be the indicator function of the interval $[\frac{k}{2^n}, \frac{k+1}{2^n})$ and $I_0(k, n)$ the indicator function of the interval $[0, \frac{k}{2^n})$. Define

$$fn(s) = \sum_{k=0}^{2^n-1} I(k, n)(\frac{2^n}{k}) l(I_0(k, n)).$$

We note that $f_n$ is an adapted process. We consider the $l^*$ integral of $f$. First we note that

$$\int f_n dl^* = \sum_k (\frac{2^n}{k}) l(I_0(k,n) l^*(I(k,n))$$

and the sum on the right is zero because the annihilation and creation act on functions with disjoint supports. Therefore the integrals of the sequence, $(f_n)$, will be uniformly small on sets of small $l^*$ semivariation. Moreover, it is not difficult to see that $(f_n)$ converges $\| \cdot \|^{l^*}$ almost everywhere on $[0,1]$. So $f$ is *integrable* with respect to $l^*$ in the belated sense and has zero integral. But a computation shows

$$\int_{[0,1]} \|f\|^2 dt = \infty.$$

So $f$ is not in $P(\| \cdot \|, dt)$. This means that the inclusion refered to in theorem 5 is strict!

**References**

[1] C. Barnett, R.F. Streater and I.F. Wilde, "The Ito-Clifford Integral", J. Funct. Anal. ,48, 1982, pp 142-212

[2] C. Barnett, R.F. Streater and I.F. Wilde, "Quasi-Free Quantum Stochastic Integrals for the C.A.R. and C.C.R.", J. Funct. Anal., 52,1983, pp 17-47

[3] C.Barnett, I.F. Wilde, : "Belated Integrals", J. Funct. Analysis, (1985) Vol 66 No. 3 pp 283-307

[4] C. Barnett, A. Voliotis : "A Conditional Expectation for the Full Fock Space". J. Operator Theory, 44(2000), 3-23

[5] R.G.Bartle : "A general bilinear vector integral". Studia Math 15 (1950) 337-352.

[6] F.Fagnola, "On Quantum Stochastic Integration with respect to free noise", Quantum Probability and related topics, World Sci. Publishing, 1991, pp. 285-304.

[7] R.Speicher, "Survey on the Stochastic Integration on the Full-Fock space", Quantum Probability and related topics, World Sci. Publishing (1991) pp 421-435

[8] A. Voliotis :"Martingales and Stochastic Integration on the Full Fock Space". PhD Thesis, Imperial College London (1994).

[9] I.F.Wilde ,"Quantum Martingales and Stochastic Integrals", Oberwolfach Proceedings, 1987