Eigenvarieties

Kevin Buzzard

August 2, 2006

Abstract

We axiomatise and generalise the "Hecke algebra" construction of the Coleman-Mazur Eigencurve. In particular we extend the construction to general primes and levels. Furthermore we show how to use these ideas to construct "eigenvarieties" parametrising automorphic forms on totally definite quaternion algebras over totally real fields.

1 Introduction

In a series of papers in the 1980s, Hida showed that classical ordinary eigenforms form p-adic families as the weight of the form varies. In the non-ordinary finite slope case, the same turns out to be true, as was established by Coleman in 1995. Extending this work, Coleman and Mazur construct a geometric object, the eigencurve, parametrising such modular forms (at least for forms of level 1 and in the case $p > 2$). On the other hand, Hida has gone on to extend his work in the ordinary case to automorphic forms on a wide class of reductive groups. One might optimistically expect the existence of non-ordinary families, and even an "eigenvariety", in some of these more general cases.

Anticipating this, we present in Part I of this paper (sections 2–5) an axiomatisation and generalisation of the Coleman-Mazur construction. In his original work on families of modular forms, Coleman in [10] developed Riesz theory for orthonormalizable Banach modules over a large class of base rings, and, in the case where the base ring was 1-dimensional, constructed the local pieces of a parameter space for normalised eigenforms. There are two places where we have extended Coleman's work. Firstly, we set up Coleman's Fredholm theory and Riesz theory (in sections 2 and 3 respectively) in a slightly more general situation, so that they can be applied to spaces such as direct summands of orthonormalizable Banach modules; the motivation for this is that at times in the theory we meet Banach modules which are invariants of orthonormalizable Banach modules under the action of a finite group; such modules are not necessarily orthonormalizable, but we want to use Fredholm theory anyway. And secondly we show in sections 4–5 that given a projective Banach module and a collection of commuting operators, one of which is compact, one can glue the local pieces constructed by Coleman to form an eigenvariety, in the case where the base ring is an arbitrary reduced affinoid. At one stage we are forced to use Raynaud's theory of formal models; in particular this generalisation is not an elementary extension of Coleman's ideas.

The resulting machine can be viewed as a construction of a geometric object from a family of Banach spaces equipped with certain commuting linear maps. Once one has this machine, one can attempt to feed in Banach spaces of "overconvergent automorphic forms" into the machine, and get "eigenvarieties" out. We extend the results of [9] in Part II of this paper (sections 6 and 7), constructing an eigencurve using families of overconvergent modular forms, and hence removing some of the assumptions on p and N in the main theorems of [9]. Note that here we do not need the results of section 4, as weight space is 1-dimensional and Coleman's constructions are enough.

There are still technical geometric problems to be resolved before one can give a definition of an overconvergent automorphic form on a general reductive group, but one could certainly hope for an elementary definition if the group in question is compact mod centre at infinity, as the geometry then becomes essentially non-existent. As a concrete example of this, we propose in Part III (sections 8–13) a definition of an overconvergent automorphic form in the case when the reductive group is a compact form of GL_2 over a totally real field, and apply our theory to this situation to construct higher-dimensional eigenvarieties.

Chenevier has constructed Banach spaces of overconvergent automorphic forms for compact forms of GL_n over $\mathbf{Q}$ and one can feed his spaces into the machine also to get eigenvarieties for these unitary groups.

This work began in 2001 during a visit to Paris-Nord, and the author would like to thank Jacques Tilouine for the invitation and Ahmed Abbes for several useful conversations. In fact the author believes that he was the first to coin the phrase "eigenvarieties", in 2001. Part I of this paper was written at that time, as well as some of Part III. The paper then remained in this state for three years, and the author most sincerely thanks Gaetan Chenevier for encouraging him to finish it off. In fact Theorem 4.6 of this paper is assumed both by Chenevier in [8], and Yamagami in [17], who independently announced results very similar to those in Part III of this paper, the main difference being that Yamagami works with the U operator at only one prime above p and fixes weights at the other places, hence his eigenvarieties can have smaller dimension than ours, but they see more forms (they are only assumed to have finite slope at one place above p). My apologies to both Chenevier and Yamagami for the delay in writing up this construction; I would also like to thank both of them for several helpful comments.

A lot has happened in this subject since 2001. Matthew Emerton has recently developed a general theory of eigenvarieties which in many cases produces cohomological eigenvarieties associated to a large class of reductive algebraic groups. As well as Coleman and Mazur, many other people (including Emerton, Ash and Stevens, Skinner and Urban, Mazur and Calegari, Kassaei, Kisin and Lai, Chenevier, and Yamagami), have made contributions to the area, all developing constructions of eigenvarieties in other situations. We finish this introduction with an explanation of the relationship between Emerton's work and ours. Emerton's approach to eigenvarieties is more automorphic and more conceptual than ours. His machine currently needs a certain spectral sequence to degenerate, but this degeneration occurs in the case of the Coleman-Mazur eigencurve and hence Emerton has independently given a construction of this eigencurve for arbitrary N and p as in Part II of this paper. However, Emerton's construction is less "concrete" and in particular the results in [6] and [7] rely on the construction of the 2-adic eigencurve presented in this paper. On the other hand Emerton's ideas give essentially the same construction of the eigenvariety associated to a totally definite quaternion algebra over a totally real field, in the sense that one can check that his more conceptual approach, when translated down, actually becomes equivalent to ours.

We would like to thank Peter Schneider for pointing out an error in an earlier version of Lemma 5.6, Elmar Grosse-Kloenne for pointing out a simplification in the definition of our admissible cover, and the referee for several helpful remarks, in particular for pointing out that flatness was necessary in Lemma 5.5.

PART I: The eigenvariety machine.

2 Compact operators on K-Banach modules

In this section we collect together the results we need from the theory of commutative Banach algebras. A comprehensive source for the terminology we use is [1]. Throughout this section, K will be a field complete with respect to a non-trivial non-archimedean valuation $|.|_K$, and A will be a commutative

Noetherian K-Banach algebra. That is, A is a commutative Noetherian K-algebra equipped with a function $|.| : A \to \mathbf{R}_{\geq 0}$, and satisfying

- $|1| \leq 1$, and $|a| = 0$ iff $a = 0$,
- $|a+b| \leq \max\{|a|, |b|\}$,
- $|ab| \leq |a||b|$,
- $|\lambda a| = |\lambda|_K |a|$ for $\lambda \in K$,

and such that A is complete with respect to the metric induced by $|.|$. For elementary properties of such algebras we refer the reader to [1], §3.7 and thereafter. Such algebras are Banach algebras in the sense of Coleman [10]. Later on we shall assume (mostly for simplicity) that A is a reduced K-affinoid algebra with its supremum norm, but this stronger assumption does not make the arguments of this section or the next any easier.

From the axioms one sees that either $|1| = 0$ and hence $A = 0$, or $|1| = 1$, in which case the map $K \to A$ is injective and the norm on A extends the norm on K. Fix once and for all $\rho \in K^\times$ with $|\rho|_K < 1$. Such ρ exists as we are assuming the valuation on K is non-trivial. We use ρ to "normalise" vectors in several proofs. If A^0 denotes the subring $\{a \in A : |a| \leq 1\}$ then one easily checks that the ideals of A^0 generated by ρ^n, $n = 1, 2, \ldots$, form a basis of open neighbourhoods of zero in A. Note that A^0 may not be Noetherian (for example if $A = K = \mathbf{C}_p$).

Let A be a commutative Noetherian K-Banach algebra. A *Banach A-module* is an A-module M equipped with $|.| : M \to \mathbf{R}_{\geq 0}$ satisfying

- $|m| = 0$ iff $m = 0$,
- $|m+n| \leq \max\{|m|, |n|\}$,
- $|am| \leq |a||m|$ for $a \in A$ and $m \in M$,

and such that M is complete with respect to the metric induced by $|.|$. Note that A itself is naturally a Banach A-module, as is any closed ideal of A. In fact all ideals of A are closed, by Proposition 3.7.2/2 of [1].

If M and N are Banach A-modules, then we define a norm on $M \oplus N$ by $|m \oplus n| = \text{Max}\{|m|, |n|\}$. This way $M \oplus N$ becomes a Banach A-module. In particular we can give A^r the structure of a Banach A-module in a natural way.

By a finite Banach A-module we mean a Banach A-module which is finitely-generated as an abstract A-module. We use the following facts several times in what follows:

Proposition 2.1. *(a) (Open Mapping Theorem) A continuous surjective K-linear map between Banach K-modules is open.*

(b) The category of finite Banach A-modules, with continuous A-linear maps as morphisms, is equivalent to the category of finite A-modules. In particular, any A-module homomorphism between finite Banach A-modules is automatically continuous, and if M is any finite A-module then there is a unique (up to equivalence) complete norm on M making it into a Banach A-module.

Proof. (a) is Théorème 1 in Chapter I, §3.3 of [3] (but note that "homomorphisme" here has the meaning assigned to it in §2.7 of Chapter III of [2], and in particular is translated as "strict morphism" rather than "homomorphism").

(b) is proved in Propositions 3.7.3/2 and 3.7.3/3 of [1].

□

Note that by (b), a finite A-module M has a canonical topology, induced by any norm that makes M into a Banach A-module. We call this topology the *Banach topology* on M.

As an application of these results, we prove the following useful lemma:

Lemma 2.2. *If M is a Banach A-module, and P is a finite Banach A-module, then any abstract A-module homomorphism $\phi : P \to M$ is continuous.*

Proof. Let $\pi : A^r \to P$ be a surjection of A-modules, and give A^r its usual Banach A-module norm. Then π is open by the Open Mapping Theorem, and $\phi\pi$ is bounded and hence continuous. So ϕ is also continuous. $\square$

If I is a set, and for every $i \in I$ we have a_i, an element of A, then by the statement $\lim_{i\to\infty} a_i = 0$, we simply mean that for all $\epsilon > 0$ there are only finitely many $i \in I$ with $|a_i| > \epsilon$. This is no condition if I is finite, and is the usual condition if $I = \mathbf{Z}_{\geq 0}$. For general I, if $\lim_{i\to\infty} a_i = 0$ then only countably many of the a_i can be non-zero. We also mention here a useful convention: occasionally we will take a max or a supremum over a set (typically a set of norms) which can be empty in degenerate cases (e.g., if a certain module or ring is zero). In these cases we will define the max or the supremum to be zero. In other words, throughout the paper we are implicitly taking suprema in the set of non-negative reals rather than the set of all reals.

Let A be a non-zero commutative Noetherian K-Banach algebra, let M be a Banach A-module, and consider a subset $\{e_i : i \in I\}$ of M such that $|e_i| = 1$ for all $i \in I$. Then for any sequence $(a_i)_{i\in I}$ of elements of A with $\lim_{i\to\infty} a_i = 0$, the sum $\sum_i a_i e_i$ converges. We say that a Banach A-module M is *orthonormalizable*, or *ONable* for short, if there exists such a subset $\{e_i : i \in I\}$ of M with the following two properties:

- Every element m of M can be written uniquely as $\sum_{i\in I} a_i e_i$ with $\lim_{i\to\infty} a_i = 0$, and
- If $m = \sum_i a_i e_i$ then $|m| = \max_{i\in I} |a_i|$.

Such a set of elements $\{e_i\}$ is called an *orthonormal basis*, or an ON basis, for M. Note that the second condition implies that $|e_i| = 1$ for all $i \in I$.

Again assume $A \neq 0$. If I is a set, we define $c_A(I)$ to be A-module of functions $f : I \to A$ such that $\lim_{i\to\infty} f(i) = 0$. Addition and the A-action are defined pointwise. We define $|f|$ to be $\mathrm{Max}\{|a_i| : i \in I\}$. With respect to this norm, $c_A(I)$ becomes a Banach A-module. If $i \in I$ and we define e_i to be the function sending $j \in I$ to 0 if $i \neq j$, and to 1 if $i = j$, and if furthermore $A \neq 0$, then it is easily checked that the e_i are an ON basis for $c_A(I)$, and we call $\{e_i : i \in I\}$ the *canonical ON basis* for $c_A(I)$. If M is any ONable Banach A-module, then to give an ON basis $\{e_i : i \in I\}$ for M is to give an isometric (that is, metric-preserving) isomorphism $M \cong c_A(I)$. Note also that $c_A(I)$ has the following universal property: if M is any Banach A-module then there is a natural bijection between $\mathrm{Hom}_A(c_A(I), M)$ and the set of bounded maps $I \to M$, given by sending $\phi : c_A(I) \to M$ to the map $i \mapsto \phi(e_i)$.

If $A = 0$ then the only A-module is $M = 0$, and we regard this module as being ONable of arbitrary rank. We have chosen to ignore this case in the definitions above because if we had included it then we should have to define an ONable Banach module as being a collection of e_i as above but with $|e_i| = |1|$ and so on; however this just clutters notation. There is no other problem with the zero ring in this situation. We will occasionally assume $A \neq 0$ in proofs, and leave the interested reader to fill in the trivial details in the case $A = 0$.

We recall some basic results on "matrices" associated to endomorphisms of Banach A-modules. The proofs are elementary exercises in analysis. Let M and N be Banach A-modules, and let $\phi : M \to N$ be an A-module homomorphism. Then a standard argument (see Corollary 2.1.8/3 of [1]) using the fact that one can use ρ to renormalise elements of M, shows that ϕ is continuous iff it is bounded, and in this case we define $|\phi| = \sup_{0 \neq m \in M} \frac{|\phi(m)|}{|m|}$ (this set of reals is bounded above if ϕ is continuous). Now assume that M is ONable, with ON basis $\{e_i : i \in I\}$. One easily checks that if ϕ is continuous and $\phi(e_i) = n_i$, then the n_i are a bounded collection of elements of N which uniquely determine ϕ. Furthermore, if n_i are an arbitrary bounded collection of elements of N there is a unique continuous map $\phi : M \to N$ such that $\phi(e_i) = n_i$ for all i, and $|\phi| = \sup_{i \in I} |n_i|$.

Now assume that N is also ONable, with basis $\{f_j : j \in J\}$. If $\phi : M \to N$ is a continuous A-module homomorphism, we can define its associated matrix coefficients $(a_{i,j})_{i\in I, j\in J}$ by[1]

$$\phi(e_i) = \sum_{j\in J} a_{i,j} f_j.$$

One checks easily from the arguments above that the collection $(a_{i,j})$ has the following two properties:

- For all i, $\lim_{j\to\infty} a_{i,j} = 0$.
- There exists a constant $C \in \mathbf{R}$ such that $|a_{i,j}| \leq C$ for all i, j.

In fact C can be taken to be $|\phi|$, and furthermore we have $|\phi| = \sup_{i,j} |a_{i,j}|$.

Conversely, given a collection $(a_{i,j})_{i\in I, j\in J}$ of elements of A, satisfying the two conditions above, there is a unique continuous $\phi : M \to N$ with norm $\sup_{i,j} |a_{i,j}|$ whose associated matrix is $(a_{i,j})$. As a useful consequence of this, we see that if ϕ and $\psi : M \to N$ are continuous, with associated matrices $(a_{i,j})$ and $(b_{i,j})$, then $|\phi - \psi| \leq \epsilon$ iff $|a_{i,j} - b_{i,j}| \leq \epsilon$ for all i and j.

Let A be a commutative Noetherian K-Banach algebra and let M, N be Banach A-modules. The A-module $\mathrm{Hom}(M, N)$ of continuous A-linear homomorphisms from M to N is then also a Banach A-module: completeness follows because if ϕ_n is a Cauchy sequence in $\mathrm{Hom}(M, N)$ then for all $m \in M$, $\phi_n(m)$ is a Cauchy sequence in N, and one can define $\phi(m)$ as its limit; then ϕ is the limit of the ϕ_n.

A continuous A-module homomorphism $M \to N$ is said to be of *finite rank* if its image is contained in a finitely-generated A-submodule of N. The closure in $\mathrm{Hom}(M, N)$ of the finite rank homomorphisms is the set of *compact* homomorphisms (many

[1] Here we follow Serre's conventions in [15], rather than writing $a_{j,i}$ for $a_{i,j}$.

authors use the term "completely continuous"). Let M and N be ONable Banach modules, and let $\phi : M \to N$ be a continuous homomorphism, with associated matrix $(a_{i,j})$. We wish to give a simple condition which is expressible only in terms of the $a_{i,j}$, and which is equivalent to compactness. Such a result is announced in Lemma A1.6 of [10] for a more general class of rings A, but the proof seems to be incomplete. This is not a problem with the theory however, as the proof can be completed in all cases of interest without too much trouble. We complete the proof here in the case of commutative Noetherian K-Banach algebras. We start with some preliminary results. As ever, A is a Noetherian K-Banach algebra. If M is an ONable Banach A-module, with ON basis $\{e_i : i \in I\}$, and if $S \subseteq I$ is a finite subset, then we define A^S to be the submodule $\oplus_{i \in S} A e_i$, and we define the projection $\pi_S : M \to A^S$ to be the map sending $\sum_{i \in I} a_i e_i$ to $\sum_{i \in S} a_i e_i$. Note that this projection is norm-decreasing onto a closed subspace of M.

Lemma 2.3. *Let M be an ONable Banach A-module, with basis $\{e_i : i \in I\}$, and let P a finite submodule of M.*

(a) There is a finite set $S \subseteq I$ such that $\pi_S : M \to A^S$ is injective on P.

(b) P is a closed subset of M, and hence is complete.

(c) For all $\epsilon > 0$, there is a finite set $T \subseteq I$ such that for all $p \in P$ we have $|\pi_T(p) - p| \leq \epsilon |p|$.

Proof. Say P is generated by $m_1, \ldots, m_r$ and for $1 \leq \alpha \leq r$ we have $m_\alpha = \sum_i a_{\alpha,i} e_i$.

(a) For $i \in I$ let v_i be the element $(a_{\alpha,i})_{1 \leq \alpha \leq r}$ of A^r. The A-submodule of A^r generated by the v_i is finitely-generated, as A is Noetherian, and hence there is a finite set $S \subseteq I$ such that this module is generated by $\{v_i : i \in S\}$. It is now an easy exercise to check that this S works, because if $\pi_S(\sum_\alpha b_\alpha m_\alpha)$ is zero, then $\sum_\alpha b_\alpha a_{\alpha,i}$ is zero for all $i \in S$ and hence for all $i \in I$.

(b) P is a finite A-module, and hence there is, up to equivalence, a unique complete A-module norm on P. Let Q denote

P equipped with this norm. The algebraic isomorphism $Q \to P$ induces a map $Q \to M$ which is continuous by Lemma 2.2, and hence the algebraic isomorphism $Q \to P$ is continuous. On the other hand, if S is chosen as in part (a), then the injection $P \to A^S$ induces a continuous injection from P onto a submodule of A^S which is closed by Proposition 3.7.3/1 of [1], and this submodule is algebraically isomorphic to Q, and hence isomorphic to Q as a Banach A-module. We hence have continuous maps $Q \to P \to Q$, which are algebraic isomorphisms, and hence the norms on P and Q are equivalent. So the maps are also homeomorphisms, and P is complete with respect to the metric induced from M, and is hence a closed submodule of M.

(c) By (b), P is complete and hence a K-Banach space. The map $A^r \to P$ sending (a_α) to $\sum_\alpha a_\alpha m_\alpha$ is thus a continuous surjection between K-Banach spaces, and hence by the open mapping theorem there exists $\delta > 0$ such that if $p \in P$ with $|p| \leq \delta$ then $p = \sum_{\alpha=1}^r a_\alpha m_\alpha$ with $|a_\alpha| \leq 1$ for all α. Choose T such that for all m_α we have $|\pi_T(m_\alpha) - m_\alpha| \leq \epsilon\delta|\rho|$. This T works: if $p \in P$ is arbitrary, then either $p = 0$ and hence the condition we are checking is automatic, or $p \neq 0$. In this case, there exists some $n \in \mathbf{Z}$ such that $|\rho|\delta < |\rho|^n|p| \leq \delta$, and then $\rho^n p = \sum_\alpha a_\alpha m_\alpha$ with $|a_\alpha| \leq 1$ for all α. Then

$$\begin{aligned}
|\pi_T(p) - p| &= \left|\sum_\alpha \rho^{-n} a_\alpha(\pi_T(m_\alpha) - m_\alpha)\right| \\
&\leq |\rho|^{-n}\epsilon\delta|\rho| \\
&\leq \epsilon|p|
\end{aligned}$$

and we are done. □

Proposition 2.4. *Let M, N be ONable Banach A-modules, with ON bases $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$. Let $\phi : M \to N$ be a continuous A-module homomorphism, with basis $(a_{i,j})$. Then ϕ is compact if and only if $\lim_{j\to\infty} \sup_{i\in I} |a_{i,j}| = 0$.*

Proof. If the matrix of ϕ satisfies $\lim_{j\to\infty}\sup_{i\in I}|a_{i,j}| = 0$, then ϕ is easily seen to be compact: for any $\epsilon > 0$ there is a finite subset $S \subseteq J$ such that $|\phi - \pi_S\phi| \leq \epsilon$.

The other implication is somewhat more delicate. It suffices to prove the result when ϕ has finite rank. If $\phi = 0$ then the result is trivial, so assume $0 \neq \phi$ and $\phi(M) \subseteq P$, where $P \subseteq N$ is finite. By part (c) of Lemma 2.3, for any $\epsilon > 0$ we may choose T such that $|\pi_T(p) - p| \leq \epsilon|p|/|\phi|$, and hence $|\pi_T\phi - \phi| \leq \epsilon$. Hence $|a_{i,j}| \leq \epsilon$ if $j \notin T$, and we are home because ϵ was arbitrary.

□

Remark. If we allow A to be non-Noetherian then we do not know whether the preceding proposition remains true.

From this result, it easily follows that a compact operator $\phi : M \to M$, where M is an ONable Banach A-module, has a characteristic power series $\det(1 - X\phi) = \sum_{n\geq 0} c_n X^n \in A[[X]]$, defined in terms of the matrix coefficients of ϕ using the usual formulae, which we recall from §5 of [15] for convenience: firstly choose an ON basis $\{e_i : i \in I\}$ for M, and say ϕ has matrix $(a_{i,j})$ with respect to this basis. If S is any finite subset of I, then define $c_S = \sum_{\sigma:S\to S}\operatorname{sgn}(\sigma)\prod_{i\in S} a_{i,\sigma(i)}$, where the sum ranges over all permutations of S, and for $n \geq 0$ define $c_n = (-1)^n\sum_S c_S$, where the sum is over all finite subsets of I of size n. One easily checks that this sum converges, using Proposition 2.4. Furthermore, again using Proposition 2.4 and following Proposition 7 of [15], one sees that the resulting power series $\det(1 - X\phi) = \sum_n c_n X^n$ converges for all $X \in A$. However, from our definition it is not clear to what extent the power series depends on the choice of ON basis for M. We now investigate to what extent this is the case. We begin with some observations in the finite-dimensional case.

If P is any finite free ONable Banach A-module, with ON basis (e_i), and $\phi : P \to P$ is any A-module homomorphism, then $\det(1 - X\phi)$, defined as above with respect to the e_i, is the usual algebraically-defined $\det(1 - X\phi)$, because the definition above coincides with the usual classical definition, which is independent of choice of basis. Next we recall the well-known fact that if P and Q are both free A-modules of finite rank, and $u : P \to Q$ and $v : Q \to P$ are A-module homomorphisms, then $\det(1 - Xuv) = \det(1 - Xvu)$. Now let M be an ONable Banach A-module, with ON basis $\{e_i : i \in I\}$. We use this fixed basis for computing characteristic power series in the lemma below.

Lemma 2.5. *(a) If $\phi_n : M \to M$, $n = 1, 2, \ldots$ are a sequence of compact operators that tend to a compact operator ϕ, then* $\lim_n \det(1 - X\phi_n) = \det(1 - X\phi)$*, uniformly in the coefficients.*

(b) If $\phi : M \to M$ is compact, and furthermore if the image of ϕ is contained in $P := \oplus_{i \in S} Ae_i$, for S a finite subset of I, then $\det(1 - X\phi) = \det(1 - X\phi|_P)$*, the right hand determinant being the usual algebraically-defined one.*

(c) (strengthening of (b)) If $\phi : M \to M$ is compact, and if the image of ϕ is contained within an arbitrary submodule Q of M which is free of finite rank, then $\det(1 - X\phi) = \det(1 - X\phi|_Q)$*, where again the right hand side is the usual algebraically-defined determinant.*

Proof. (a) This follows *mutatis mutandis* from [15], Proposition 8.

(b) If $(a_{i,j})$ is the matrix of ϕ then $a_{i,j} = 0$ for $j \notin S$ and the result follows immediately from the definition of the characteristic power series.

(c) Choose $\epsilon > 0$. By Lemma 2.3(c), there is a finite set $T \subseteq I$ such that $\pi_T : Q \to P := A^T$ has the property that $|\pi_T - i| \leq \epsilon$, where $i : Q \to M$ is the inclusion. Define $\phi^T = \pi_T \phi : M \to P \subseteq M$. By (b) we see that $\det(1 - X\phi^T)$ equals the algebraically-defined polynomial $\det(1 - X\phi^T|_P)$. Furthermore, by consideration of the maps $\phi : P \to Q$ and $\pi_T : Q \to P$, we see that this polynomial also equals the algebraically-defined $\det(1 - X\phi_T)$, where $\phi_T = \phi\pi_T : Q \to Q$. One can compute this latter determinant with respect to an arbitrary algebraic A-basis of Q. By Lemma 2.3(b), Q with its subspace topology is complete, and hence the topology on Q is the Banach topology. Now as ϵ tends to zero, $\phi_T : Q \to Q$ tends to $\phi : Q \to Q$ and $\phi^T : M \to M$ tends to $\phi : M \to M$, and the result follows by part (a).

□

Corollary 2.6. *Let M be an A-module, and let $|.|_1$ and $|.|_2$ be norms on M both making M into an ONable Banach A-module, and both inducing the same topology on M. Then an A-linear map $\phi : M \to M$ is compact with respect to $|.|_1$ iff it is compact with respect to $|.|_2$, and furthermore if $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$ are ON bases for $(M, |.|_1)$ and $(M, |.|_2)$ respectively, then the definitions of* $\det(1 - X\phi)$ *with respect to these bases coincide.*

Proof. All one has to do is to check that ϕ can be written as the limit as maps ϕ_n which have image contained in free modules of finite rank, and then the result follows from parts (a) and (c) of the Lemma. To do this, one can simply use Lemma 2.3 to construct ϕ as a limit of $\pi_{T_n}\phi$, for T_n running through appropriate finite subsets of I. Note that ϕ_n will then tend to ϕ with respect to both norms (recall that two norms on M are equivalent iff they induce the same topology, because the valuation on K is non-trivial) and the result follows. □

The corollary enables us to conclude that the notion of a characteristic power series only depends on the topology on M, when A is a commutative Noetherian K-Banach algebra. In particular it does not depend on the choice of an orthonormal basis for M. Coleman proves in corollary A2.6.1 of [10] that the definition of the characteristic power series only depends on the topology on M when A is semi-simple; on the other hand neither of these conditions on A implies the other.

Next we show that the analogue of Corollaire 2 to Proposition 7 of [15] is true in this setting. Coleman announces such an analogue in Proposition A2.3 of [10] but again we have not been able to complete the proof in the generality in which Coleman is working. We write down a complete proof when A is a commutative Noetherian K-Banach algebra and remark that it is actually slightly delicate. We remark also that in the case where A is a reduced affinoid, which will be true in the applications, one can give an easier proof by using Corollary 2.10 to reduce to the case treated by Serre.

Lemma 2.7. *If M and N are ONable Banach A-modules, if $u : M \to N$ is compact and $v : N \to M$ is continuous, then uv and vu are compact, and* $\det(1 - Xuv) = \det(1 - Xvu)$.

Proof. If there exist finite *free* sub-A-modules F of M and G of N such that $u(M)$ is contained in G and $v(G)$ is contained in F, then $u : F \to G$ and $v : G \to F$, and by Lemma 2.5(c) it suffices to check that the algebraically-defined characteristic polynomials of $uv : G \to G$ and $vu : F \to F$ are the same, which is a standard result. We reduce the general case to this case by several applications of Lemma 2.3, the catch being that it is not clear (to the author at least) whether any finite submodule of an ONable Banach module is contained within a finite free submodule.

We return to the general case. Write u as a limit of finite rank operators u_n. Then $u_n v$ and vu_n are finite rank, so uv and vu are both the limit of finite rank operators and are hence compact. By Lemma 2.5(a), it suffices to prove that $\det(1 - Xu_n v) = \det(1 - Xvu_n)$ for all n, and hence we may assume that u is finite rank. Let $Q \subseteq N$ denote a finite A-module containing the image of u.

Choose ON bases $\{e_i : i \in I\}$ for M and $\{f_j : j \in J\}$ for N. Now for any positive integer n we may, by Lemma 2.3(c), choose a finite subset $T_n \subseteq J$ such that $|\pi_{T_n} q - q| \leq \frac{1}{n}|q|$ for all $q \in Q$. It follows easily that $|\pi_{T_n} u - u| \leq |u|/n$ and hence $\lim_{n\to\infty} \pi_{T_n} u = u$. Hence $v\pi_{T_n} u \to vu$ and $\pi_{T_n} uv \to uv$ and again by Lemma 2.5(a) we may replace u by $\pi_{T_n} u$ and in particular we may assume that the image of u is contained in a finite *free* A-submodule of N. Let G denote this submodule. Now

$P = v(G)$ is a finite submodule of M, and for any positive integer n we may, as above, choose a finite subset $S_n \subseteq I$ such that $|\pi_{S_n} p - p| \leq \frac{1}{n}|p|$ for all $p \in P$.

It is unfortunately not the case that $\pi_{S_n} v \to v$ as $n \to \infty$, as v is not in general compact. However we do have $\pi_{S_n} vu \to vu$ and hence the characteristic power series of $\pi_{S_n} vu$ tends (uniformly in the coefficients) to the characteristic power series of vu. Also, the image of $uv : N \to N$ and $u\pi_{S_n} v$ are both contained within G and hence the characteristic power series of uv (resp. $u\pi_{S_n} v$) is equal to the algebraically-defined characteristic power series of $uv : G \to G$ (resp. $u\pi_{S_n} v : G \to G$). Once one has restricted to G, one *does* have $u\pi_{S_n} v \to uv$, and hence the characteristic power series of $u\pi_{S_n} v$ tends to the characteristic power series of uv. We may hence replace v by $\pi_{S_n} v$ and in particular may assume that the image of v is contained within a finite free A-submodule F of M. We have now reduced to the algebraic case dealt with at the beginning of the proof. $\square$

Corollary 2.6 also enables us to slightly extend the domain of definition of a characteristic power series: if M is a Banach A-module, then we say that M is *potentially ONable* if there exists a norm on M equivalent to the given norm, for which M becomes an ONable Banach A-module. Equivalently, M is potentially ONable if there is a bounded collection $\{e_i : i \in I\}$ of elements of M with the following two properties: firstly, every element m of M can be uniquely written as $\sum_i a_i e_i$ with $\lim_{i \to \infty} a_i = 0$, and secondly there exist positive constants c_1 and c_2 such that for all $m = \sum_i a_i e_i$ in M, we have $c_1 \sup_i |a_i| \leq |m| \leq c_2 \sup_i |a_i|$. We call the collection $\{e_i : i \in I\}$ a *potentially ON basis* for M. Being potentially ONable is probably a more natural notion than being ONable, because it is useful to be able to work with norms only up to equivalence, whereas ONability of a module really depends on the precise norm on the module. Note that to say a module is ONable is equivalent to saying that it is isometric to some $c_A(I)$, and to say that it is potentially ONable is just to say that it is isomorphic to some $c_A(I)$ (in the category of Banach modules, with continuous maps as morphisms).

If M is potentially ONable then one still has the notion of the characteristic power series of a compact operator on M, defined by choosing an equivalent ONable norm and using this norm to define the characteristic power series. By Corollary 2.6, this is independent of all choices. We note that certainly there can exist Banach A-modules which are potentially ONable but not ONable, for example if $A = K = \mathbf{Q}_p$ and $M = \mathbf{Q}_p(\sqrt{p})$ with its usual norm, then $|M| \neq |A|$ and so M is not ONable, but is potentially ONable.

A useful result is

Lemma 2.8. *If $h : A \to B$ is a continuous morphism of Noetherian K-Banach algebras, and M is a potentially ONable Banach A-module, then $M\widehat{\otimes}_A B$ is a potentially ONable Banach B-module, and furthermore if $\{e_i : i \in I\}$ is a potentially ON basis for M, then $\{e_i \otimes 1 : i \in I\}$ is a potentially ON basis for $M\widehat{\otimes}_A B$.*

Proof. Set $N = c_B(I)$, and let $\{f_i : i \in I\}$ be its canonical ON basis. Then there is a natural A-bilinear bounded map $M \times B \to N$ sending $(\sum_i a_i e_i, b)$ to $\sum_i b h(a_i) f_i$, which induces a continuous map $M\widehat{\otimes}_A B \to N$. On the other hand, if $n \in N$, one can write n as a limit of elements of the form $\sum_{i \in S} b_i f_i$, where S is a finite subset of I. The element $\sum_{i \in S} e_i \otimes b_i$ of $M \otimes_A B$ has norm bounded above by a constant multiple of $\max_{i \in S} |b_i|$ and hence as S increases, the resulting sequence $\sum_{i \in S} e_i \otimes b_i$ is Cauchy and so its image in $M\widehat{\otimes}_A B$ tends to a limit. This construction gives a well-defined continuous A-module homomorphism $N \to M\widehat{\otimes}_A B$ which is easily checked to be an inverse to the natural map $M\widehat{\otimes}_A B \to N$, and now everything follows. $\square$

Note that because we are only working in the "potential" world, we do not need to assume the map $A \to B$ is contractive, although in the applications we have in mind it usually will be.

Corollary 2.9. *If $h : A \to B$ is a continuous morphism of commutative Noetherian K-Banach algebras, M and N are potentially ONable Banach A-modules with potentially ON bases (e_i) and (f_j), and $\phi : M \to N$ is compact, with matrix $(a_{i,j})$, then $\phi \otimes 1 : M\widehat{\otimes}_A B \to N\widehat{\otimes}_A B$ is also compact and if $(b_{i,j})$ is the matrix of $\phi \otimes 1$ with respect to the bases $(e_i \otimes 1)$ and $(f_j \otimes 1)$ then $b_{i,j} = h(a_{i,j})$.*

Proof. Compactness of $\phi \otimes 1$ follows from Proposition 2.4 and the rest is easy. □

Corollary 2.10. *With notation as above, if* $\det(1 - X\phi) = \sum_n c_n X^n$ *then* $\det(1 - X(\phi \otimes 1)) = \sum_n h(c_n) X^n$.

Proof. Immediate. □

In practice we need to extend the notion of the characteristic power series of a compact operator still further, to the natural analogue of projective modules in this setting. Let us say that a Banach A-module P *satisfies property* (Pr) if there is a Banach A-module Q such that $P \oplus Q$, equipped with its usual norm, is potentially ONable. I am grateful to the referee for pointing out the following universal property for such modules: P has property (Pr) if and only if for every surjection $f : M \to N$ of Banach A-modules and for every continuous map $\alpha : P \to N$, α lifts to a map $\beta : P \to M$ such that $f\beta = \alpha$. The proof is an elementary application of the Open Mapping Theorem; the key point is that if $P = c_A(I)$ for some set I, then to give $\alpha : P \to N$ is to give a bounded map $I \to N$, and such a map lifts to a bounded map $I \to M$ by the Open Mapping Theorem. Note however that it would be perhaps slightly disingenuous to call such modules "projective", as there are epimorphisms in the category of Banach A-modules whose underlying module map is not surjective.

One can easily check that if P is a finite Banach A-module which is projective as an A-module, then P has property (Pr). The converse is also true:

Lemma 2.11. *If P is a finite Banach A-module with property (Pr) then P is projective as an A-module.*

Proof. Choose a surjection $A^n \to P$ for some n and then use the universal property above. □

Note that potentially ONable Banach A-modules have property (Pr), but in general the converse is false—for example if there are finite A-modules which are projective but not free then such modules, equipped with any complete Banach A-module norm, will satisfy (Pr) but will not be potentially ONable.

Say P satisfies property (Pr) and $\phi : P \to P$ is a compact morphism. Define $\det(1 - X\phi)$ thus: firstly choose Q such that $P \oplus Q$ is potentially ONable, and define $\det(1 - X\phi) = \det(1 - X(\phi \oplus 0))$; note that $\phi \oplus 0 : P \oplus Q \to P \oplus Q$ is easily seen to be compact. This definition may *a priori* depend on the choice of Q, but if R is another Banach A-module such that $P \oplus R$ is also potentially ONable, then so is $P \oplus Q \oplus P \oplus R$, and the maps $\phi \oplus 0 \oplus 0 \oplus 0$ and $0 \oplus 0 \oplus \phi \oplus 0$ are conjugate via an isometric A-module isomorphism, and hence have the same characteristic power series. Now the fact that $\det(1 - X\phi)$ is well-defined independent of choice of Q follows easily from the fact that if M and N are ONable A-modules, and $\phi : M \to M$ is compact, then the characteristic power series of ϕ and $\phi \oplus 0 : M \oplus N \to M \oplus N$ coincide.

Many results that we have already proved for potentially ONable Banach A-modules are also true for modules with property (Pr), and the proofs are typically easy, because one can reduce to the potentially ONable case without too much difficulty. Indeed the trick used in the example above is typically the only idea one needs. One sometimes has to also use the following standard ingredients: Firstly, if R is any commutative ring, P is a finite projective R-module, and $\phi : P \to P$ is an R-module homomorphism, then there is an algebraically-defined $\det(1 - X\phi)$, defined either by localising and reducing to the free case, or by choosing a finite projective R-module Q such that $P \oplus Q$ is free, and defining $\det(1-X\phi)$ to be $\det(1-X(\phi\oplus 0))$. And secondly, if M and N both have property (Pr) and $\phi : M \to M$ and $\psi : N \to N$ are compact, then $\det(1-X(\phi\oplus\psi)) = \det(1-X\phi)\det(1-X\psi)$. Finally, we leave it as an exercise for the reader to check the following generalisations of Lemma 2.7 and Lemma 2.8–Corollary 2.10.

Lemma 2.12. *If M and N are Banach A-modules with property (Pr), if $u : M \to N$ is compact and $v : N \to M$ is continuous, then uv and vu are compact, and* $\det(1-Xuv) = \det(1-Xvu)$.

Lemma 2.13. *If M is a Banach A-module with property (Pr), $\phi : M \to M$ is compact, and $h : A \to B$ is a continuous morphism of commutative Noetherian K-Banach algebras, then $M\widehat{\otimes}_A B$ has property (Pr) as a B-module, $\phi\otimes 1$ is compact, and* $\det(1-X(\phi\otimes 1))$ *is the image of* $\det(1-X\phi)$ *under h.*

3 Resultants and Riesz theory

We wish now to mildly extend the results in sections A3 and A4 of [10] to the case where A is a Noetherian K-Banach algebra and M is a Banach A-module satisfying property (Pr). Fortunately much of what Coleman proves already applies to our situation, or can easily be modified to do so. We make what are hopefully some helpful comments in case the reader wants to check the details. This section is not self-contained, and anyone wishing to check the details should read it in conjunction with §A3 and §A4 of [10].

Section A3 of [10] applies to commutative Noetherian K-Banach algebras already (apart from the comments relating to semi-simple algebras, because in general a commutative Noetherian K-Banach algebra may contain nilpotents). We give some hints for following the proofs in this section of [10]. We define the ring $A\{\{T\}\}$ to be the subring of $A[[T]]$ consisting of power series $\sum_{n\geq 0} c_n T^n$ with the property that for all $R \in \mathbf{R}_{>0}$, we have $|c_n|R^n \to 0$ as $n \to \infty$. One could put a norm on $A\{\{T\}\}$, for example $|\sum_n c_n T^n| = \mathrm{Max}_n |c_n|$, but $A\{\{T\}\}$ is not in general complete with respect to this norm. One very useful result about this ring is that if $H(T) \in A\{\{T\}\}$ and $D(T)$ is a monic polynomial of degree $d \geq 0$ then $H(T) = Q(T)D(T)+R(T)$ with $Q(T) \in A\{\{T\}\}$ and $R(T)$ a polynomial of degree less than d. Furthermore, $Q(T)$ and $R(T)$ are uniquely determined. A word on the proof: uniqueness uses the kind of trick in Lemma A3.1 of [10]. For existence one reduces to the case where all the coefficients of D have norm at most 1 and proves the result for polynomials first, and then takes a limit.

If $Q \in A[T]$ is a monic polynomial, and $P \in A\{\{T\}\}$, then Coleman defines the *resultant* $\mathrm{Res}(Q, P)$ on the top of p434 of [10]. Many of the formulae that Coleman needs are classical when P is a polynomial, and can be extended to the power series case using the following trick: straight from the definition it follows that if $u \in A^\times$ then $\mathrm{Res}(Q, P) = \mathrm{Res}(u^{-n}Q(uT), P(uT))$. This normalisation can be used to renormalise either Q or P into $A^0\langle T\rangle$, where $A^0 := \{a \in A : |a| \leq 1\}$. If S_n denotes the symmetric group acting naturally on $A^0\langle T_1, \ldots, T_n\rangle$ then the subring left invariant by the action is $A^0\langle e_1, \ldots, e_n\rangle$, where the e_i are the elementary symmetric functions of the T_i. Hence if $P, Q \in A^0\langle T\rangle$ then $\mathrm{Res}(Q, P) \in A^0$. If $Q \in A^0[T]$ is monic then one can check that the definition of a resultant makes sense for $P \in A\langle T\rangle$, and furthermore that $\mathrm{Res}(Q, -)$ is locally uniformly continuous in the second variable (in the sense that for all $M \in \mathbf{R}$, $\mathrm{Res}(Q, -)$ is a uniformly continuous function from $\{P \in A\langle T\rangle : |P| \leq M\}$ to A).

Coleman defines a function D sending a pair $B, P \in A[X]$ to an element $D(B, P) \in A[T]$. In fact if P has degree n and we define $P^*(X) = X^n P(X^{-1})$, then $D(B, P) = \mathrm{Res}(P^*(X), 1 - TB(X))$ where the resultant is computed in $A\langle T\rangle\{\{X\}\}$. One can check that $D(B(uX), P(u^{-1}X)) = D(B, P)$ if $u \in A^\times$, and that D is locally uniformly continuous in the B variable. It is also locally uniformly continuous in the P variable, because $\mathrm{Res}(X, C(X)) = 1$ if $C(0) = 1$. This is enough to check that

Coleman's definition of $D(B,P)$ makes sense when B and P are in $A\{\{T\}\}$. In fact we shall only need it when B is a polynomial. Another useful formula is that $D(uB,P)(T) = D(B,P)(uT)$ for $u \in A^\times$.

In §A4 of [10] Coleman assumes his hypothesis (M), which tends not to be true for affinoids over K if K is not algebraically closed. Coleman also assumes that he is working with an ON-able Banach A-module. We work in our more general situation. Hence let A denote a commutative Noetherian K-Banach algebra, let M be a Banach A-module satisfying property (Pr) and let $\phi : M \to M$ be a compact morphism, with characteristic power series $P(X) = \det(1 - X\phi)$. We define the *Fredholm resolvant* of ϕ to be $P(X)/(1 - X\phi) \in A[\phi][[X]]$. Exactly as in Proposition 10 of [15], one can prove that if $F(X) = \sum_{n\geq 0} v_m X^m$ then for all $R \in \mathbf{R}_{>0}$ the sequence $|v_m|R^m$ tends to zero, where v_m is thought of as being an element of $\mathrm{Hom}(M,M)$. Lemma A4.1 of [10] goes through unchanged, and we recall it here (Notation: if $Q(X)$ is a polynomial of degree n then $Q^*(X)$ denotes $X^n Q(X^{-1})$):

Lemma 3.1. *With A, M, ϕ and P as above, if $Q(X) \in A[X]$ is monic then Q and P generate the unit ideal in $A\{\{X\}\}$ if and only if $Q^*(\phi)$ is an invertible operator on M.*

□

Before we continue, let us make some remarks on zeroes of power series. If $f = \sum_{n\geq 0} a_n T^n$ is in $A[[T]]$ and $s \in \mathbf{Z}_{\geq 0}$ then we define $\Delta^s f = \sum_{n\geq 0} \binom{n+s}{s} a_{n+s} T^n \in A[[T]]$. If $f, g \in A[[T]]$ then it is possible to check that $\Delta^s(fg) = \sum_{i=0}^{s} \Delta^i(f)\Delta^{s-i}(g)$. One also easily checks that if A is a Noetherian K-Banach algebra then Δ^s sends $A\{\{T\}\}$ to itself. We say that $a \in A$ is a *zero of order* h of $H \in A\{\{T\}\}$ if $(\Delta^s H)(a) = 0$ for $s < h$ and $(\Delta^h H)(a)$ is a unit. If $h \geq 1$ and $H = 1 + a_1 T + \ldots$ then this implies that $-1 = a(a_1 + a_2 a + \ldots)$ and hence that a is a unit. One now checks by induction on h that $H(T) = (1 - a^{-1}T)^h G(T)$, where $G \in A\{\{T\}\}$, and then that $G(a)$ is a unit.

Again let M be a Banach A-module with property (Pr) and let $\phi : M \to M$ be a compact morphism, with characteristic power series $P(T)$. Say $a \in A$ is a zero of $P(T)$ of order h.

Proposition 3.2. *There is a unique decomposition $M = N \oplus F$ into closed ϕ-stable submodules such that $1-a\phi$ is invertible on F and $(1-a\phi)^h = 0$ on N. The submodules N and F are defined as the kernel and the image of a projector which is in the closure in* $\mathrm{Hom}(M, M)$ *of $A[\phi]$. Moreover, N is projective of rank h, and assuming $h > 0$ then a is a unit and the characteristic power series of ϕ on N is $(1 - a^{-1}T)^h$.*

Proof. We start by following Proposition 12 of [15], much of which goes through unchanged in our setting. We find that there are maps $p, q \in \mathrm{Hom}(M, M)$, both in the closure of $A[\phi]$, such that $p^2 = p$, $q^2 = q$ and $p + q = 1$, and if we consider the decomposition $M = N \oplus F$ corresponding to these projections, $N = \ker(p)$, then $(1 - a\phi)^h = 0$ on N, and $(1 - a\phi)$ is invertible on F. The decomposition is visibly unique, as if $\psi = (1 - a\phi)^h$ then $N = \ker(\psi)$ and $F = \mathrm{Im}(\psi)$. We now diverge from Proposition 12 of [15].

It is clear that N satisfies (Pr), but furthermore we have $(1 - a\phi)^h = 0$ on N which implies that the identity is compact on N. An elementary argument (change the metric on N to an equivalent one if necessary and reduce to a computation of matrices) shows that if $\beta \in \mathrm{Hom}(N, N)$ has sufficiently small norm, then $|\beta^n| \to 0$ and hence $1 - \beta$ is invertible. Because 1 is compact, we can choose $\alpha : N \to N$ of finite rank such that $1 - \alpha$ is sufficiently small, and hence α is invertible and so N is finitely-generated. By Lemma 2.11, N is projective.

If $h = 0$ then $N = 0$ and $F = M$, as can be seen from Lemma 3.1, and we are home. So assume for the rest of the proof that $h > 0$. Then $P(a) = 0$ and this implies that a must be a unit. If P_N and P_F denote the characteristic power series of ϕ on N and F respectively, then $P = P_N P_F$ and by Lemma 3.1 we see that $(T - a)^h$ and P_F generate the unit ideal in $A\{\{T\}\}$. Hence

$(1-a^{-1}T)^h$ divides P_N in $A\{\{T\}\}$. Moreover, P_N is a polynomial because N is finitely-generated, and hence $(1-a^{-1}T)^h$ divides P_N in $A[T]$. Moreover, $(1-a^{-1}T)^h$ has constant term 1 and is hence not a zero-divisor in $A\{\{T\}\}$, hence if $P_N(T) = D(T)(1-a^{-1}T)^h$ and $P(T) = (1-a^{-1}T)^h G(T)$ then $D(T)$ divides $G(T)$ in $A\{\{T\}\}$ and so $D(a)$ is a unit.

We know that $D(T)$ is a polynomial. Furthermore, because $(1-a\phi)^h = 0$ on N we see that ϕ has an inverse on N and hence that the determinant of ϕ is in $A^\times$. Hence the leading term of D is a unit. Reducing the situation modulo a maximal ideal of A we see that the reduction of P_N must be a power of the reduction of $(1-a^{-1}T)$ and this is enough to conclude that $D = 1$. Hence the characteristic power series of ϕ on N is $(1-a^{-1}T)^h$. Finally, the fact that ϕ is invertible on N means that the rank of N at any maximal ideal must equal the degree of P_N modulo this ideal, and hence the rank is h everywhere. □

Keep the notation: M has (Pr) and $\phi : M \to M$ is compact, with characteristic power series $P(T)$.

Theorem 3.3. *Suppose $P(T) = Q(T)S(T)$, where $S = 1+\ldots \in A\{\{T\}\}$ and $Q = 1+\ldots$ is a polynomial of degree n whose leading coefficient is a unit, and which is relatively prime to S. Then there is a unique direct sum decomposition $M = N \oplus F$ of M into closed ϕ-invariant submodules such that $Q^*(\phi)$ is zero on N and invertible on F. The projectors $M \to N$ and $M \to F$ are elements of $\mathrm{Hom}(M, M)$ which are in the closure of $A[\phi]$. Furthermore, N is projective of rank n and the characteristic power series of ϕ on N is $Q(T)$.*

Proof. We follow Theorem A4.3 of [10]. The operator $v = 1 - Q^*(\phi)/Q^*(0)$ has a characteristic power series which has a zero at $T = 1$ of order n. Applying the previous proposition to v, we see $M = N \oplus F$, where N and F are defined as the kernel and the image of a projector in the closure of $A[v]$ and hence in the closure of $A[\phi]$. Hence both N and F are ϕ-stable. Unfortunately, by the end of the proof of Theorem A4.3 of [10] one can only deduce that $Q^*(\phi)^n$ is zero on N and invertible on F, so we are not quite home yet. However, by Proposition 3.2, N is projective of rank n. Moreover, the characteristic power series of ϕ on F is coprime to Q, by Lemma 3.1. Hence if $G(T)$ is the characteristic power series of ϕ on N, we see that Q divides G. But G and Q have degree n and the same constant term, and furthermore the leading coefficient of Q is a unit. This is enough to prove that $G = Q$.

□

4 An admissible covering

The key aim in this section is to generalise some of the results of section A5 of [10] (especially Proposition A5.8) to the case where the base is an arbitrary reduced affinoid. In fact almost all of Coleman's results go through unchanged, but there are some differences, which we summarise here. Firstly it is not true in general that the image of an affinoid under a quasi-finite map is still affinoid. However if one works with finite unions of affinoids then one can deal with the problems that this causes. Secondly Coleman uses the notion of a strict neighbourhood of a subspace of the unit disc. We slightly modify this notion to one which suits our purpose. Lastly we need some kind of criterion for when a quasi-finite map of rigid spaces of constant degree is finite. We use a theorem of Conrad whose proof invokes Raynaud's theory of admissible formal models of rigid spaces.

We set up some notation. Let K be a field with a complete non-trivial non-Archimedean valuation. Let R denote a reduced K-affinoid algebra, and let $B = \mathrm{Max}(R)$ be the associated affinoid variety. We equip R with its supremum semi-norm, which is a norm in this case. Let $R\{\{T\}\}$ denote the ring of power series $\sum_{n\geq 0} a_n T^n$ with $a_n \in R$ such that for all real $r > 0$ we have $|a_n|r^n \to 0$ as $n \to \infty$. Then $R\{\{T\}\}$ is just the ring of functions on $B \times_K \mathbf{A}^{1,\mathrm{an}}$, where here $\mathbf{A}^{1,\mathrm{an}}$ denotes the analytification of affine 1-space over K.

Let $P(T) = \sum_{n\geq 0} r_n T^n \in R\{\{T\}\}$ be a function with $r_0 = 1$. Our main object of study is the rigid space cut out by $P(T)$, that is, the space $Z \subseteq B \times_K \mathbf{A}^{1,\mathrm{an}}$ defined by the zero locus of $P(T)$. In practice, $P(T)$ will be the characteristic power series of a compact endomorphism of a Banach R-module.

Certainly Z is a rigid analytic variety, equipped with projection maps $f : Z \to B$ and $g : Z \to \mathbf{A}^{1,\mathrm{an}}$. We frequently make use of the following cover of Z: If $r \in \sqrt{|K^\times|}$ (that is, some power of r is the norm of a non-zero element of K) then let $B[0,r]$ denote the closed affinoid disc over K of radius r, considered as an admissible open subspace of $\mathbf{A}^{1,\mathrm{an}}$. Let Z_r denote the zero locus of $P(T)$ on the space $B \times_K B[0,r]$. Then Z_r is an affinoid, and the Z_r admissibly cover Z. Let $f_r : Z_r \to B$ denote the canonical projection. Note that any affinoid subdomain of Z will be admissibly covered by its intersections with the Z_r, which are affinoids, and hence will be contained within some Z_r.

Now let $\mathcal{C}$ denote the set of affinoid subdomains Y of Z with the following property: there is an affinoid subdomain X of B (depending on Y) with the property that $Y \subseteq Z_X := f^{-1}(X)$, the induced map $f : Y \to X$ is finite and surjective, and Y is disconnected from its complement[2] in Z_X, that is, there is a function $e \in \mathcal{O}(Z_X)$ such that $e^2 = e$ and Y is the locus of Z_X defined by $e = 1$.

Our goal is (c.f. Proposition A5.8 of [10])

Theorem. *$\mathcal{C}$ is an admissible cover of Z.*

The reason we want this result is that in later applications Z will be a "spectral variety", and the $Y \in \mathcal{C}$ are exactly the affinoid subdomains of Z over which one can construct a Hecke algebra, and hence an eigenvariety, without any technical difficulties.

[2]Elmar Grosse-Kloenne has pointed out that this condition in fact follows from the others; one can use 9.6.3/3 and 9.5.3/5 of [1] to check that $Y \to Z_X$ is both an open and a closed immersion.

We prove the theorem after establishing some preliminary results.

Lemma 4.1. *$f_r : Z_r \to B$ is quasi-finite and flat.*

Proof. By increasing r if necessary, we may assume $r \in |K^\times|$ and hence, by rescaling, that $r = 1$. The situation we are now in is as follows: R is an affinoid and $P(T) = \sum_{n\geq 0} r_n T^n \in R\langle T\rangle$ with $r_0 = 1$, and we must show that $R \to R\langle T\rangle/(P(T))$ is quasi-finite and flat. Quasi-finiteness is immediate from the Weierstrass preparation theorem. For flatness observe that $R\langle T\rangle$ is flat[3] over R and that if P is any maximal ideal of $R\langle T\rangle$ then $\mathcal{P} := P \cap R$ is a maximal ideal of R. Hence $R\langle T\rangle/(\mathcal{P}R\langle T\rangle) = (R/\mathcal{P}R)\langle T\rangle$ is an integral domain and the image of $P(T)$ in $(R/\mathcal{P}R)\langle T\rangle$ is non-zero, as its constant term is non-zero. Hence the image of $P(T)$ is not a zero-divisor and flatness now follows from Theorem 22.6 of [14]. □

Corollary 4.2. *If $Y \subseteq Z$ is an affinoid then $f : Y \to B$ is quasi-finite and flat.*

Proof. Y is affinoid and hence $Y \subseteq Z_r$ for some $r \in \sqrt{|K^\times|}$, so the result follows from the previous lemma. □

Corollary 4.3. *If $Y \subseteq Z$ is an affinoid, and $X \subseteq B$ is an admissible open such that $Y \subseteq f^{-1}(X)$, and if there is an integer $d \geq 0$ such that all fibres of the induced map $f : Y \to X$ have degree d, then $f : Y \to X$ is finite and flat.*

Proof. $f : Y \to X$ is flat by Corollary 4.2. It is also quasi-compact and separated, so finiteness follows from Theorem A.1.2 of [11]. □

Note that this latter result uses the full force of Raynaud's theory of formal models.

[3] One can prove flatness by using the Open Mapping Theorem and mimicking the proof of the result stated in Exercise 7.4 of [14], noting that the solution to the exercise is on p289 of loc. cit.

Lemma 4.4. *If $r \in \sqrt{|K^\times|}$ and $f_r : Z_r \to B$, then for $i \geq 0$ define $U_i := \{x \in B : \deg(f_r^{-1}(x)) \geq i\}$. Then each U_i is a finite union of affinoid subdomains of B, and U_i is empty for i sufficiently large.*

Proof. The sequence $|r_n|r^n$ tends to zero as $n \to \infty$, and hence for any $x \in B$, the set $\{|r_n(x)|r^n : n \geq 0\}$ has a maximum, denoted M_x, which is attained. Note that $|r_0(x)| = 1$ and hence $M_x \geq 1$, and in particular if N is an integer such that $|r_n|r^n < 1$ for all $n \geq N$ then $M_x = \mathrm{Max}\{|r_n(x)|r^n : 0 \leq n < N\}$ and $M_x > |r_n(x)|r^n$ for all $n \geq N$. For $i \geq 0$ let S_i denote the affinoid subdomain of B defined by $\{x \in B : |r_i(x)|r^i = M_x\}$. Then S_i is empty for $i \geq N$. A calculation on the Newton polygon shows that

$$U_i = \cup_{j \geq i} S_j$$

and the result follows. □

Definition. *If S and T are admissible open subsets of the affinoid B, such that both S and T are finite unions of affinoid subdomains of B, then we say that T is a* strict neighbourhood *of S (in B) if $S \subseteq T$ and there is an admissible open subset U of B with the following properties:*

- *U is a finite union of affinoid subdomains of B,*
- *$U \cap S$ is empty,*
- *$U \cup T = B$.*

The intersection of two affinoid subdomains of B is an affinoid subdomain of B. Hence if U and V are admissible open subsets of B which are both the union of finitely many affinoid subdomains, then so is $U \cap V$. As a consequence, we see that if T_α is a strict neighbourhood of S_α for $1 \leq \alpha \leq n$ then $\cup_\alpha T_\alpha$ is a strict neighbourhood of $\cup_\alpha S_\alpha$. We now prove the key technical lemma that we need.

Lemma 4.5. *Suppose $r \in \sqrt{|K^\times|}$ and $V \subseteq B$ is an affinoid subdomain with the property that $f_r : f_r^{-1}(V) \to V$ is finite of constant degree $d > 0$. Then there is an affinoid subdomain X of B which is a strict neighbourhood of V in B, and $s \in \sqrt{|K^\times|}$ with $s > r$ such that the affinoid $Y = f_s^{-1}(X)$ contains $f_r^{-1}(V)$, lies in $\mathcal{C}$, and is finite flat of degree d over X.*

Proof. (c.f. Lemma A5.9 of [10]). If $x \in V$ then let $P_x(T) = \sum_{n\geq 0} r_n(x)T^n$ denote the specialisation of $P(T)$ to $K(x)\{\{T\}\}$. The statement that the degree of $f_r^{-1}(x)$ is d translates by the theory of the Newton polygon to the statement that for all $x \in V$ we have $|r_d(x)| \neq 0$ (so r_d is a unit in $\mathcal{O}^{\mathrm{an}}(V)$) and furthermore that for all integers $n \geq 0$ we have $-\log(|r_n(x)|) \geq (n-d)\log(r) - \log(|r_d(x)|)$, with strict inequality when $n > d$. Here log is the usual logarithm, with the usual convention that $-\log(0) = +\infty$. Because $P(T)$ is entire, there exists an integer $N > d$ such that for $n \geq N$ we have $-\log|r_n(x)| > n\log(r+1)$ for all $x \in B$ and hence $-\log(|r_n(x)|) \geq (n-d)\log(r+1) - \log(|r_d(x)|)$ for all $n \geq N$. For $d < n < N$ we have $-\log(|r_n(x)|) > (n-d)\log(r) - \log(|r_d(x)|)$ and hence $|r_n(x)/r_d(x)| < r^{d-n}$ for all $x \in V$. Because functions on affinoids attain their bounds, there is some $t \in \sqrt{|K^\times|}$ with $r < t < r+1$ and $|r_n(x)/r_d(x)| < t^{d-n}$ for $d < n < N$, and hence for all $x \in V$ we have $-\log(|r_n(x)|) \geq (n-d)\log(t) - \log(|r_d(x)|)$ for all $n \geq 0$, with equality iff $n = d$. Now choose γ_1, γ_2, δ_1, $\delta_2 \in \log\left(\sqrt{|K^\times|}\right)$ such that $\delta_2 > -\log|r_d(x)| > \delta_1$ for all $x \in V$ and $\log(r) < \gamma_1 < \gamma_2 < \log(t)$. Let X be the affinoid subdomain of B defined by the N equations

$$\delta_1 \leq -\log|r_d(x)| \leq \delta_2,$$

$$-\log|r_n(x)| + \log|r_d(x)| \geq (n-d)\gamma_1 \text{ for } 0 < n < d,$$

and

$$-\log|r_n(x)| + \log|r_d(x)| \geq (n-d)t \text{ for } d < n < N.$$

These equations define X as a Laurent subdomain of B, and if $x \in V$ then not only are these equations satisfied, but strict inequality holds in every case. Hence $V \subset X$ and moreover if we consider the N affinoids, each defined by one of the N equations

$$\delta_1 \geq -\log|r_d(x)|,$$

$$-\log|r_d(x)| \geq \delta_2,$$

$$-\log|r_n(x)| + \log|r_d(x)| \leq (n-d)\gamma_1,\ 0 < n < d,$$

$$-\log|r_n(x)| + \log|r_d(x)| \leq (n-d)t,\ d < n < N,$$

and let W be the union of these N affinoids, then $X \cap W$ is empty and $X \cup V = B$. Hence X is a strict neighbourhood of V in B, in the sense we defined above. Let $s = \exp(\gamma_2)$ and set $Y = f_s^{-1}(X)$. Then Y is an affinoid in Z and by the previous corollary and the way we have arranged the Newton polygons, $f : Y \to X$ is finite of degree d (note that by our choice of t we have

$$-\log|r_n(x)| + \log|r_d(x)| \geq (n-d)t \text{ for all } n \geq N,$$

with strict inequality for $x \in V$). Furthermore if $x \in X$ then no slope of the Newton polygon of $P_x(T)$ can equal s, and hence the projection from $f^{-1}(X)$ to $\mathbf{A}^{1,\mathrm{an}}$ contains no elements of norm s. Hence Y is disconnected from its complement in $f^{-1}(X)$, and in particular is an affinoid subdomain of Z, so $Y \in \mathcal{C}$. □

We are now ready to prove the theorem.

Theorem 4.6. *$\mathcal{C}$ is an admissible cover of Z.*

Proof. Again we follow Coleman. We know that Z is admissibly covered by the Z_r, $r \in \sqrt{|K^\times|}$, and hence it suffices to prove that for every Z_r, there is a finite collection of affinoids in $\mathcal{C}$ whose union contains Z_r. Recall that for $i \geq 0$, U_i is the subset of points in B such that $\deg(f_r^{-1}(a)) \geq i$, and that U_i is a finite union of affinoids. Furthermore, clearly $U_{i+1} \subseteq U_i$. If U_1 is empty there is nothing to prove, so let us assume that it is not. Let d denote the largest i such that U_i is non-empty. For $1 \leq i \leq d$ let $H(i)$ denote the following statement:

$H(i)$: "There is a finite set $Y_1, Y_2 \dots Y_{n(i)}$ of affinoid subdomains of Z, and a finite set $X_1, X_2 \dots X_{n(i)}$ of affinoid subdomains of B, such that for $1 \leq \alpha \leq n(i)$ we have $f : Y_\alpha \to X_\alpha$ is finite flat and surjective, $Y_\alpha \in \mathcal{C}$, $f_r^{-1}(U_i) \subseteq \cup_{\alpha=1}^{n(i)} Y_\alpha$, and $\cup_{\alpha=1}^{n(i)} X_\alpha$ is a strict neighbourhood of U_i in B."

If we can establish $H(1)$ then we are home because $f_r^{-1}(U_1) = Z_r$. We firstly establish $H(d)$, and then show that $H(i)$ implies $H(i-1)$ for $i \geq 2$, and this will be enough. For $H(d)$ we cover U_d by finitely many affinoid subdomains $V_1, V_2, \dots V_{n(1)}$ of B (in fact it is not difficult to show that U_d is itself an affinoid, but we shall not need this). By Corollary 4.3 we know that $f_r^{-1}(V_\alpha) \to V_\alpha$ is finite and flat. Now applying Lemma 4.5 to V_α we get a strict affinoid neighbourhood X_α of V_α, and if $Y_\alpha = f_s^{-1}(X_\alpha)$ (s as in the Lemma) then $H(d)$ follows immediately.

Now let us assume $H(i)$, $i \geq 2$. Then choose a finite union of affinoid subdomains $W \subset B$ such that $W \cap U_i$ is empty and $W \cup \bigcup_{\alpha=1}^{n(i)} X_\alpha = B$. Then $W \cap U_{i-1}$ is a finite union $V_{n(i)+1}$, $V_{n(i)+2} \dots V_{n(i)+m}$ of affinoid subdomains. Set $n(i-1) = n(i)+m$. Note that $f_r : f_r^{-1}(V_\alpha) \to V_\alpha$ is finite of degree $i-1$ for $n(i) < \alpha \leq n(i-1)$, hence one is in a position to apply Lemma 4.5 to get $Y_\alpha \to X_\alpha$ finite flat of degree $i-1$, $Y_\alpha \in \mathcal{C}$, $f_r^{-1}(V_\alpha) \subseteq Y_\alpha$, and X_α a strict neighbourhood of V_α, for $n(i) < \alpha \leq n(i-1)$. We now show that $\bigcup_{\alpha=1}^{n(i-1)} X_\alpha$ is a strict neighbourhood of U_{i-1}. We know that $\bigcup_{\alpha=n(i)+1}^{n(i-1)} X_\alpha$ is a strict neighbourhood of $\bigcup_{\alpha=n(i)+1}^{n(i-1)} V_\alpha$, so choose a finite union of affinoid subdomains W' such that $W' \cap (\bigcup_{\alpha=n(i)+1}^{n(i-1)} V_\alpha)$ is empty and $W' \cup (\bigcup_{\alpha=n(i)+1}^{n(i-1)} X_\alpha) = B$. Now set $W'' = W \cap W'$. Then W'' is a finite union of affinoids, $W'' \cap U_{i-1} = W' \cap W \cap U_{i-1} = W' \cap (\bigcup_{\alpha=n(i)+1}^{n(i-1)} V_\alpha)$ is empty, and $W'' \cup (\bigcup_{\alpha=1}^{n(i-1)} X_\alpha) = B$, and we are done. □

5 Spectral varieties and eigenvarieties

Let R be a reduced affinoid K-algebra equipped with its supremum norm, let M be a Banach R-module satisfying (Pr), and let $\mathbf{T}$ be a commutative R-algebra equipped with an R-algebra homomorphism to $\mathrm{End}_R(M)$, the continuous R-module endomorphisms of M. In practice $\mathbf{T}$ will be a polynomial R-algebra generated by (typically infinitely many) Hecke operators. We frequently identify $t \in \mathbf{T}$ with the endomorphism of M associated to it. Fix once and for all an element $\phi \in \mathbf{T}$, and assume that the induced endomorphism $\phi : M \to M$ is compact. Let $F(T) = 1 + \sum_{n\geq 1} c_n T^n$ be the characteristic power series of ϕ. We define the *spectral variety* Z_ϕ associated to ϕ to be the closed subspace of the rigid space $\mathrm{Max}(R) \times \mathbf{A}^1$ cut out by F. The spectral variety is a geometric object parametrising, in some sense, the reciprocals of the non-zero eigenvalues of ϕ. Its formulation is compatible with base change, by Lemma 2.13. Our main goal in this section is to write down a finite cover of this spectral variety, the *eigenvariety* associated to the data $(R, M, \mathbf{T}, \phi)$. Points on the eigenvariety will correspond to systems of eigenvalues for all the operators in $\mathbf{T}$, such that the eigenvalue for ϕ is non-zero. The construction is just an axiomatisation of Chapter 7 of [9] and is really not deep (in fact by far the deepest part of the entire construction is the fact that the cover $\mathcal{C}$ of Section 4 is admissible, as this appealed to the theory of formal models at one point). Unfortunately the construction does involve a lot of bookkeeping.

We begin with a finite-dimensional example, where ϕ is invertible and hence where we may avoid the technicalities of §4. Let R be a reduced affinoid K-algebra, and let M denote a finitely-generated projective R-module of rank d. Let $\mathbf{T}$ be an arbitrary R-algebra equipped with an R-algebra homomorphism to $\mathrm{End}_R(M)$, and let ϕ be an element of $\mathbf{T}$. Assume furthermore that $\phi : M \to M$ has an inverse, that is, there is an R-linear $\phi^{-1} : M \to M$ such that $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi$ is the identity on M. Define $P(T) = \det(1 - T\phi) = 1 + \ldots \in R[T]$; then the leading term of P, that is, the coefficient of T^d, is a unit. Let Z_ϕ denote the zero locus of $P(T)$ regarded as a function on $\mathrm{Max}(R) \times \mathbf{A}^1$. Then $R[T]/(P(T))$ is a finite R-algebra and hence an affinoid algebra, and Z_ϕ is the affinoid rigid space associated to this affinoid algebra. Let $\mathbf{T}(Z_\phi)$ denote the image of $\mathbf{T}$ in $\mathrm{End}_R(M)$; then $\mathbf{T}(Z_\phi)$ is a finite R-algebra and hence an affinoid algebra. By the Cayley-Hamilton theorem we have $\phi^{-1} \in \mathbf{T}(Z_\phi)$, and furthermore there is a natural map $R[T]/P(T) \to \mathbf{T}(Z_\phi)$ sending T to ϕ^{-1}. Set $D_\phi = \mathrm{Max}(\mathbf{T}(Z_\phi))$. Then the maps $R \to R[T]/P(T) \to \mathbf{T}(Z_\phi)$ of affinoids give maps $D_\phi \to Z_\phi \to \mathrm{Max}(R)$. We call Z_ϕ the *spectral variety* and D_ϕ the *eigenvariety* associated to this data. As a concrete example, consider the case where $R = K\langle X, Y\rangle$, $M = R^2$, $\mathbf{T} = R[\phi, t]$, where ϕ acts on M as the matrix $\left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$, and t acts as $\left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right)$. Then $\det(1 - T\phi) = (1 - T)^2$ so Z_ϕ is non-reduced, and $\mathbf{T}(Z_\phi)$ is the ring $R \oplus I\epsilon$, with $I = (X, Y)$ and $\epsilon^2 = 0$. Note that in this case the maps $D_\phi \to Z_\phi$ and $D_\phi \to \mathrm{Max}(R)$ are not flat, and D_ϕ is not reduced either.

It would not be unreasonable to say that what follows in this section is just a natural generalisation of this set-up, the main complication being that M is not necessarily finitely-generated, the purpose of the admissible cover $\mathcal{C}$ being to remedy this. See also Chenevier's thesis, where he develops essentially the same theory in essentially the same way (assuming Theorem 4.6 of this paper). The example above shows that in this generality one cannot expect D_ϕ and Z_ϕ to have too many "good" geometric properties; however one can hope that the examples of spectral and eigenvarieties arising "in nature" are better behaved.

Let us now go back to our more general situation, where M is a Banach R-module satisfying property (Pr) and $\mathbf{T}$ is a commutative R-algebra equipped with an R-algebra map $\mathbf{T} \to \mathrm{End}_R(M)$, such that the endomorphism of M induced by $\phi \in \mathbf{T}$ is compact. Let Z_ϕ be the closed subspace of $\mathrm{Max}(R) \times \mathbf{A}^1$ defined by the zero locus of the characteristic power series of ϕ.

Let $\mathcal{C}$ be the admissible cover of Z_ϕ constructed in section 4. Let Y be an element of this admissible cover, with image $X \subseteq \mathrm{Max}(R)$. By definition, X is an affinoid subdomain of $\mathrm{Max}(R)$, so set $A = \mathcal{O}(X)$. Then A is reduced by Corollary 7.3.2/10 of [1]. Define $M_A := M \widehat{\otimes}_R A$ and, for $t \in \mathbf{T}$, let t_A denote the A-linear continuous endomorphism of M_A induced by $t : M \to M$. Note that $\phi_A : M_A \to M_A$ is still compact, by Lemma 2.13. Let $F_A(T)$ be the characteristic power series of ϕ_A on M_A. Again by Lemma 2.13, F_A is just the image of F in $A\{\{T\}\}$.

Let us assume first that X is connected. Then we wish to associate to Y a factor of $F_A(T)$ so that we are in a position to apply Theorem 3.3. We do this as follows. We know that $\mathcal{O}(Y)$ is a finite flat A-module, and hence it is projective of some rank d. The element T of $\mathcal{O}(Y)$ is a root of its characteristic polynomial Q', which is monic of degree d, and hence gives us a map $A[T]/(Q'(T)) \to \mathcal{O}(Y)$. In fact, Y is a closed subspace of $X \times B[0, r]$ for some r, and hence if S is some appropriate K-multiple of T then the natural map $A\langle S\rangle \to \mathcal{O}(Y)$ is surjective. By Proposition 3.7.4/1 of [1] and its proof, any residue norm on $\mathcal{O}(Y)$ will be equivalent to any of the Banach norms that $\mathcal{O}(Y)$ inherits from being a finite complete A-module. One can deduce from this that the map $A[T]/(Q'(T)) \to \mathcal{O}(Y)$ is surjective. Hence $A[T]/(Q'(T)) \to \mathcal{O}(Y)$ is an isomorphism, because both sides are locally free A-modules of rank d. This means that the image of $F_A(T)$ in $A\{\{T\}\}/(Q'(T))$ is zero, and hence that Q' divides F_A in $A\{\{T\}\}$. Comparing constant terms, we see that $Q' = a_0 + a_1 T + \ldots$ with a_0 a unit, and hence we can define $Q = a_0^{-1} Q'$ and we are in a position to invoke Theorem 3.3 to give a decomposition $M_A = N \oplus F$ where N is projective of rank d over A. Note that in general N will not be free. Because the projector $M_A \to N$ is in the closure of $A[\phi]$, it commutes with all the endomorphisms of M_A induced by elements of $\mathbf{T}$, and hence N is t-invariant for all $t \in \mathbf{T}$. Define $\mathbf{T}(Y)$ to be the A-sub-algebra of $\mathrm{End}_A(N)$ generated by all the elements of $\mathbf{T}$. Now $\mathrm{End}_A(N)$ is a finite A-module, and hence $\mathbf{T}(Y)$ is a finite

A-algebra and hence an affinoid. Let $D(Y)$ denote the associated affinoid variety. We know that $Q^*(\phi)$ is zero on N, and hence $\mathbf{T}(Y)$ is naturally a finite $A[S]/(Q^*(S))$-algebra, via the map sending S to ϕ. Because the constant term of Q^* is a unit, there is a canonical isomorphism $A[S]/(Q^*(S)) = \mathcal{O}(Y)$ sending S to T^{-1}. Hence $\mathbf{T}(Y)$ is a finite $\mathcal{O}(Y)$-algebra, and thus there is a natural finite map $D(Y) \to Y$.

For general $Y \in \mathcal{C}$, the image of Y in $\mathrm{Max}(R)$ may not be connected, but Y can be written as a finite disjoint union $Y = \cup Y_i$ corresponding to the connected components of the image of Y in $\mathrm{Max}(R)$. We define $D(Y)$ as the disjoint union of the $D(Y_i)$. This construction gives us, for each $Y \in \mathcal{C}$, a finite cover $D(Y)$ of Y. We wish to glue together the $D(Y)$, as Y ranges through all elements of $\mathcal{C}$, and the resulting curve D, which will be a finite cover of Z_ϕ, will be the *eigenvariety* associated to the data $(R, M, \mathbf{T}, \phi)$. We firstly establish a few lemmas.

Lemma 5.1. *If $Y \in \mathcal{C}$ with image $X \subseteq \mathrm{Max}(R)$, and X' is an affinoid subdomain of X then Y', the pre-image of X' under the map $Y \to X$, is in $\mathcal{C}$, and is an affinoid subdomain of Y. Furthermore, $D(Y')$ is canonically isomorphic to the pre-image of Y' under the map $D(Y) \to Y$.*

Proof. Y' is the pre-image of X' under the map $Y \to X$ and is hence an affinoid subdomain of Y by Proposition 7.2.2/4 of [1]. The map $Y' \to X'$ is finite and surjective, and if e is the idempotent in $\mathcal{O}(Z_X)$ showing that Y is disconnected from its complement (that is, $e|_Y = 1$ and $e|_{Z_X \setminus Y} = 0$), then the restriction of e to $\mathcal{O}(Z_{X'})$ will do the same for Y'. Hence $Y' \in \mathcal{C}$. It is now elementary to check that $\mathbf{T}(Y') = \mathbf{T}(Y) \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(X')$ and hence $D(Y')$ is the pre-image of Y' under the map $D(Y) \to Y$. $\square$

Lemma 5.2. *If $Y_1, Y_2 \in \mathcal{C}$ then $Y := Y_1 \cap Y_2 \in \mathcal{C}$. Furthermore for $1 \leq i \leq 2$, Y is an affinoid subdomain of Y_i, and $D(Y)$ is canonically isomorphic to the pre-image of Y under the map $D(Y_i) \to Y_i$.*

Proof. Let X_i denote the image of Y_i in $\mathrm{Max}(R)$. Then the X_i are affinoid subdomains of $\mathrm{Max}(R)$, and hence so is their intersection. Let X denote a component of $X_1 \cap X_2$. It suffices to prove the assertions of the lemma with Y replaced by $Y \cap Z_X$, so let us re-define Y to be $Y \cap Z_X$.

Let Y_i' denote the pre-image of X under the map $Y_i \to X_i$. Then Y_i' is an affinoid subdomain of Y_i containing Y and by Lemma 5.1 we have $Y_i' \in \mathcal{C}$ with $D(Y_i')$ the pre-image of Y_i' under the map $D(Y_i) \to Y_i$. Now $Y = Y_1' \cap Y_2'$ is finite and flat over Y_1' and hence finite and flat over X. Let $e_i \in \mathcal{O}(Z_X)$ be the idempotent associated to Y_i', and set $e = e_1 e_2$. Then Y is the subset of Z_X defined by $e = 1$, and hence Y is locally free of finite rank over X. One easily checks that Y is a union of components of Y_i' for $1 \leq i \leq 2$, in fact. If Y is empty then the rest of the lemma is clear. If not then the map $Y \to X$ is surjective, and $Y \in \mathcal{C}$. Finally, for $1 \leq i \leq 2$, the idempotents e and $e_i(1 - e_{3-i})$ sum to 1 on Y_i' showing that $D(Y)$ is actually a union of connected components of $D(Y_i')$, pulling back the inclusion $Y \subseteq Y_i'$. □

Now by Proposition 9.3.2/1 of [1] we can glue the $D(Y)$ for $Y \in \mathcal{C}$ to get a rigid space D_ϕ (the cocycle conditions are satisfied because they are satisfied for the cover $\mathcal{C}$ of Z_ϕ), and by Proposition 9.3.3/1 of [1] we can glue the maps $D(Y) \to Y$ to get a map $D_\phi \to Z_\phi$. We say that the rigid space D_ϕ is the *eigenvariety* associated to the data $(R, M, \mathbf{T}, \phi)$. We have already seen in the finite-dimensional case that the map $D_\phi \to Z_\phi$ might not be flat, and that Z_ϕ and D_ϕ may be non-reduced. We summarise the obvious positive results about Z_ϕ and D_ϕ that come out of their construction:

Lemma 5.3. *D_ϕ and Z_ϕ are separated, and the map $D_\phi \to Z_\phi$ is finite.*

Proof. Z_ϕ is separated by, for example, Proposition 9.6/7 of [1] (applied to the admissible covering $\{Z_r\}$ of Z_ϕ defined in the previous section). The construction of D_ϕ over Z_ϕ shows that $D_\phi \to Z_\phi$ is finite; hence $D_\phi \to Z_\phi$ is separated, which implies that D_ϕ is separated. □

We need to establish further functorial properties of this construction. As before, let R be a reduced K-affinoid algebra equipped with its supremum norm, let M be a Banach R-module satisfying (Pr), and let $\mathbf{T}$ be a commutative R-algebra equipped with a distinguished element ϕ and an R-algebra homomorphism $\mathbf{T} \to \mathrm{Hom}_R(M, M)$, such that the image of ϕ is a compact endomorphism. We now consider what happens when we change R. More specifically, let R' denote another reduced K-affinoid algebra equipped with a map $R \to R'$, and let $M', \mathbf{T}', \phi'$ denote the obvious base extensions. The constructions above give us maps $D_\phi \to Z_\phi \to \mathrm{Max}(R)$ and $D_{\phi'} \to Z_{\phi'} \to \mathrm{Max}(R')$. The map $R \to R'$ gives us a map $\mathrm{Max}(R') \to \mathrm{Max}(R)$.

Lemma 5.4. *$Z_{\phi'} \to \mathrm{Max}(R')$ is canonically isomorphic to the pullback of $Z_\phi \to \mathrm{Max}(R)$ to $\mathrm{Max}(R')$.*

Proof. This is an immediate consequence of Lemma 2.13. □

In particular there is a natural map $Z_{\phi'} \to Z_\phi$.

Lemma 5.5. *If $R \to R'$ is flat then $D_{\phi'} \to Z_{\phi'}$ is canonically isomorphic to the pullback of $D_\phi \to Z_\phi$ under the map $Z_{\phi'} \to Z_\phi$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ denote usual the admissible covers of Z_ϕ and $Z_{\phi'}$. If $Y \in \mathcal{C}$ and Y' is the pullback of Y to $Z_{\phi'}$ then one checks without too much difficulty that $Y' \in \mathcal{C}'$. It is not immediately clear whether every element of $\mathcal{C}'$ arises in this way (although this is the case if $\mathrm{Max}(R') \subseteq \mathrm{Max}(R)$ is an affinoid subdomain, which will be the only case we are interested in in practice). However, this does not matter because the elements of $\mathcal{C}'$ which arise in this way still form an admissible covering of $Z_{\phi'}$, as they cover the separated space $Z_{\phi'}$, and all the elements of $\mathcal{C}'$ are affinoids so one can use Proposition 9.1.4/2 of [1]. Hence we may construct $D_{\phi'}$ by gluing the $D(Y')$ for all Y' which arise in this way. One checks that $D(Y')$ is the pullback of $D(Y)$ under the map $Y' \to Y$ (this is where we use flatness, the point being that without flatness one cannot deduce that the natural map from $D(Y')$ to the pullback of $D(Y)$ is an isomorphism) and that everything is compatible with gluing, and after this somewhat tedious procedure one deduces that the maps $D(Y') \to D(Y)$ identify $D_{\phi'}$ with the pullback of D_ϕ as indicated. $\square$

We will only be applying the above lemma in the case where $\mathrm{Max}(R')$ is an affinoid subdomain of $\mathrm{Max}(R)$ and in particular $R \to R'$ is flat in this case. I am grateful to the referee for pointing out that flatness is necessary for this lemma to be true. Indeed, one checks that in the example at the beginning of this section (with $R = K\langle X, Y\rangle$), if $R' = K$ and the map $R \to R'$ sends X and Y to zero, then the pullback of the eigenvariety is not isomorphic to the eigenvariety associated to the pullback. On the other hand, one can use the q-expansion principle to check that the construction of the cuspidal Coleman-Mazur eigencurve (see part II of this paper) will commute with any base change of reduced affinoids. Had we set up the theory for non-reduced bases one could no doubt even check that the construction commutes with arbitrary base change. The same arguments do not work for the eigenvarieties associated to totally definite quaternion algebras over totally real fields (see part III of this paper) and for Chenevier's unitary group eigenvarieties [8]. One does not have a q-expansion principle in these cases, and whether construction of these eigenvarieties commutes with all base changes (even those coming from the inclusion of a point into weight space) seems to be an open question, related to multiplicity one issues for overconvergent automorphic eigenforms in these settings. See Lemma 5.9 for a partial result. Another way of setting up the foundations of the theory of eigenvarieties might be to construct the eigenvariety as a limit of spectral varieties such as those in sublemma 6.2.3 of [9]; such a construction might well commute with all base changes, but would not see subtleties such as non-semisimplicity of eigenspaces at a fixed weight.

We now analyse how eigenvarieties change under a specific type of change of module. As always, let R be a reduced affinoid, let M and M' denote Banach modules satisfying property (Pr), let $\mathbf{T}$ be a commutative R-algebra equipped with maps $\mathbf{T} \to \mathrm{End}_R(M)$ and $\mathbf{T} \to \mathrm{End}_R(M')$, such that our chosen element $\phi \in \mathbf{T}$ acts compactly on both M and M'.

In practice we are interested only in modules M and M' which are related in a specific way, which we now axiomatise. We say that a continuous R-module and $\mathbf{T}$-module homomorphism $\alpha : M' \to M$ is a "primitive link" if there is a compact R-linear and $\mathbf{T}$-linear map $c : M \to M'$ such that $\phi : M \to M$ is $\alpha \circ c$ and $\phi : M' \to M'$ is $c \circ \alpha$. Note that these assumptions force the characteristic power series of ϕ on M and M' to coincide, by Lemma 2.12. Note also that the identity map $M \to M$ is a primitive link (take $c = \phi$). We say that a continuous R-module and $\mathbf{T}$-module homomorphism $\alpha : M' \to M$ is a "link" if one can find a sequence $M' = M_0, M_1, M_2, \ldots, M_n = M$ of Banach R-modules satisfying property (Pr) with $\mathbf{T}$-actions, and continuous R-module and $\mathbf{T}$-module maps $\alpha_i : M_i \to M_{i+1}$ such that each α_i is a primitive link, and α is the compositum of the α_i. We apologise for this terrible notation but the underlying notion is what occurs in applications; our motivation is the study of r-overconvergent modular forms as r changes. More precisely, with notation as in Part II of this manuscript, if $0 < r \leq r' < p/(p+1)$ and α is the inclusion from r'-overconvergent forms to r-overconvergent forms, then α will be a primitive link if $r' \leq pr$, but if $r' > pr$ then α may only be a link. Perhaps all of this can be avoided if one sets up the theory with a slightly more general class of topological modules.

Lemma 5.6. *Let R, M, M', $\mathbf{T}$, ϕ be as above, and assume that we are given a link $\alpha : M' \to M$ in the sense above. Let D_ϕ denote the eigenvariety associated to $(R, M, \mathbf{T}, \phi)$ and let D'_ϕ denote the eigenvariety associated to $(R, M', \mathbf{T}, \phi)$. Then D_ϕ and D'_ϕ are isomorphic.*

Proof. This is clear if α is an isomorphism, so we may assume that α is a primitive link, and thus that there is a compact $c : M \to M'$ such that αc and $c\alpha$ are equal to the endomorphisms of M and M' induced by ϕ. We use a dash to indicate the analogue of one of our standard constructions, applied to M' (for example Z'_ϕ, $\mathcal{C}'$ and so on). By Lemma 2.13, Z_ϕ and Z'_ϕ are equal, as are $\mathcal{C}$ and $\mathcal{C}'$ (as their construction does not depend on the underlying Banach module). Choose $Y \in \mathcal{C}$ with connected image $X \subseteq \mathrm{Max}(R)$. It will suffice to prove that α induces an isomorphism $D'(Y) = D(Y)$ that commutes with all the glueing data on both sides, and this will follow if we can show that, after base extension to $A = \mathcal{O}(X)$, α induces an isomorphism between the finite flat sub-R-modules N' and N of M' and M corresponding to Y, and hence that α (which recall is $\mathbf{T}$-linear) induces an isomorphism $\mathbf{T}(Y) = \mathbf{T}'(Y)$.

Recall that there is a polynomial $Q = 1 + \ldots$ associated to Y as in the definition of $D(Y)$, such that the leading term of Q is a unit, and such that N' and N are the kernels of $Q^*(\phi)$ on M' and M respectively. From this one can conclude that α maps N' to N, that c maps N to N', and that there is an element ψ of $R[\phi] \subseteq \mathbf{T}$, such that ψ is an inverse to ϕ on both N and N'. We now see that $\alpha : N' \to N$ must be an R-module isomorphism, because it is elementary to check that $\psi \circ c$ is a two-sided inverse (recall that ψ is a polynomial in ϕ and hence commutes with c). Now everything else follows without too much trouble. □

I thank Peter Schneider for pointing out a problem with the proof of the above lemma in the initial version of this manuscript.

We now have enough for our eigenvariety machine. The data we are given is the following: we have a reduced rigid space $\mathcal{W}$, a commutative R-algebra $\mathbf{T}$, and an element $\phi \in \mathbf{T}$. For any admissible affinoid open $X \subseteq \mathcal{W}$, with $\mathcal{O}(X) = R_X$ (equipped with its supremum norm), we have a Banach R_X-module M_X satisfying (Pr), and an R-module homomorphism $\mathbf{T} \to \mathrm{Hom}_{R_X}(M_X, M_X)$, denoted $t \mapsto t_X$, such that ϕ_X is compact. Finally, if $Y \subseteq X \subseteq \mathcal{W}$ are two admissible affinoid opens, then we have a continuous $\mathcal{O}(Y)$-module homomorphism $\alpha : M_Y \to M_X \widehat{\otimes}_{R_X} R_Y$ which is a "link" in the above sense, and such that if $X_1 \subseteq X_2 \subseteq X_3 \subseteq \mathcal{W}$ are all affinoid subdomains then $\alpha_{13} = \alpha_{23}\alpha_{12}$ where α_{ij} denotes the map $M_{X_i} \to M_{X_i} \widehat{\otimes}_{\mathcal{O}(X_i)} \mathcal{O}(X_j)$.

Construction 5.7 (eigenvariety machine). *To the above data we may canonically associate the* eigenvariety D_ϕ, *a rigid space equipped with a map to* $\mathcal{W}$, *with the property that for any affinoid open* $X \subseteq \mathcal{W}$, *the pullback of* D_ϕ *to* X *is canonically isomorphic to the eigenvariety associated to the data* $(R_X, M_X, \mathbf{T}, \phi_X)$.

There is very little left to check in this construction. If $Y \subseteq X$ are affinoid subdomains of $\mathcal{W}$ then by Lemma 5.6 the eigenvarieties associated to $(R_Y, M_Y, \mathbf{T}, \phi_Y)$ and $(R_Y, M_X \widehat{\otimes}_{R_X} R_Y, \mathbf{T}, \phi_X)$ are isomorphic. By Lemma 5.5 the eigenvariety associated to $(R_Y, M_X \widehat{\otimes}_{R_X} R_Y, \mathbf{T}, \phi_X)$ is isomorphic to the pullback to Y of the eigenvariety associated to $(R_X, M_X, \mathbf{T}, \phi_X)$. The assumption on compatibility of the α ensures that the cocycle condition is satisfied, and hence the D_{ϕ_i} glue together to give an eigenvariety D_ϕ over $\mathcal{W}$ whose restriction to X_i is D_{ϕ_i}.

As we have seen earlier, one cannot expect D_ϕ to be reduced or flat over $\mathcal{W}$ in this generality. However, here are some positive results.

Lemma 5.8. *Assume* $\mathcal{W}$ *is equidimensional of dimension* n. *Then* D_ϕ *is also equidimensional of dimension* n. *The finite map* $D_\phi \to Z_\phi$ *has the property that each irreducible component of* D_ϕ *maps surjectively to an irreducible component of* Z_ϕ. *Moreover, the image in* $\mathcal{W}$ *of each irreducible component of* D_ϕ *is Zariski-dense in a component of* $\mathcal{W}$.

Proof. This is Proposition 6.4.2 of [8]. □

We now explain how the points of D_ϕ are in bijection with systems of eigenvalues of $\mathbf{T}$. For L a complete extension of K, say that a map $\lambda : \mathbf{T} \to L$ is an *L-valued system of eigenvalues* if there is an affinoid $X = \mathrm{Max}(R_X) \subseteq \mathcal{W}$, a point in $X(L)$ (giving a map $R_X \to L$) and $0 \neq m \in M_X \widehat{\otimes}_{R_X} L$ such that $tm = \lambda(t)m$ for all $t \in \mathbf{T}$. Say that an L-valued system of eigenvalues is *ϕ-finite* if $\lambda(\phi) \neq 0$.

Lemma 5.9. *There is a natural bijection between ϕ-finite systems of eigenvalues and L-points of D_ϕ.*

Proof. Because D_ϕ is separated, no pathologies occur when base extending to L and hence we may assume $L = K$. Recall that D_ϕ is covered by the $D(Y)$ for $Y \in \mathcal{C}$; choose $Y \in \mathcal{C}$ and let $X \subseteq \mathcal{W}$ be its image in $\mathcal{W}$. Choose a K-point P of X. This K-point corresponds to a map $R_X \to K$ and it suffices to construct a bijection between the K-points of $D(Y)$ lying above P and the ϕ-finite systems of eigenvalues coming from eigenvectors in $N \otimes_{R_X} K$, where $N \subseteq M_X$ is the subspace corresponding to Y. The result then follows from the following purely algebraic lemma. □

Lemma 5.10. *Let R be a commutative Noetherian ring and let N be a projective module of finite rank over R. Let T be a commutative subring of $\mathrm{End}_R(N)$. Let $\mathfrak{m}$ denote a maximal ideal of R, and let S denote the image of the natural map $T/\mathfrak{m}T \to \mathrm{End}_{R/\mathfrak{m}}(N/\mathfrak{m}N)$. Then the natural map $T/\mathfrak{m}T \to S$ induces a bijection between the prime ideals of $T/\mathfrak{m}T$ and the prime ideals of S.*

Proof. It suffices to show that the kernel of the map $T/\mathfrak{m}T \to S$ is nilpotent. After localising at $\mathfrak{m}$ we may assume that N is free; choose a basis for N. Let t be an element of T whose image in $\mathrm{End}_{R/\mathfrak{m}}(N/\mathfrak{m}N)$ is zero. Then all the matrix coefficients of t with respect to this basis are in $\mathfrak{m}$. Thinking of t as a matrix with coefficients in R, we see that t is a root of its characteristic polynomial, which is monic and all of whose coefficients other than its leading term are in $\mathfrak{m}R$. Hence t is nilpotent in $T/\mathfrak{m}T$ and we are home. □

Part II: The Coleman-Mazur eigencurve.

6 Overconvergent modular forms.

One can say much more about the eigenvariety D_ϕ in the specific case for which all this machinery was originally invented, namely the Coleman-Mazur eigencurve. Again we shall not give a complete treatment of this topic, but will refer to [9] for many of the basic definitions and results we need. The paper [9] gives constructions of two objects, called C and D, both in the case of level 1 and $p > 2$. The results in sections 2–3 of this paper are enough for us to be able to extend the construction of D to the case of an arbitrary level and an arbitrary prime p, and we shall give details of the construction here. Note that we do not need the results in sections 4–5 of this paper here, because the eigenvarieties constructed are over a 1-dimensional base, and the rigid analytic results that Coleman develops in section A5 of [10] are sufficient.

Fix a prime p, set $K = \mathbf{Q}_p$, and let $\mathcal{W}$ be weight space, that is the rigid space whose $\mathbf{C}_p$-points are naturally the continuous group homomorphisms $\mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ (see section 2 of [5] for more details on representability of such functors). Then $\mathcal{W}$ is the disjoint union of finitely many open discs, and there is a natural affinoid covering of $\mathcal{W}$ which on each component is a cover of the open disc by countably many closed discs. Coleman and Mazur restrict to the case $p > 2$, and for an affinoid Y in weight space define define M_Y to be the space of r-overconvergent modular forms of level 1, for some appropriate real number r. As Y gets bigger one has to consider forms which overconverge less and less; this is why we must include cases where (in the notation of section 5) the "links" α are not the identity. Finally the map ϕ is chosen to be the Hecke operator U_p, which is compact. See [9] for rigorous definitions of the above objects, and verification that they satisfy the necessary criteria for the machine to work. In [9] it is proved that (for $N = 1$ and $p > 2$) the resulting eigencurve D_ϕ is reduced, and flat over Z_ϕ (see [9], Proposition 7.4.5 and the remarks before Theorem 7.1.1 respectively). Our Lemma 5.9 is just the statement that points on the eigencurve are overconvergent systems of finite slope eigenvalues, and the existence of q-expansions assures us that, at least in the cuspidal case, points on the eigencurve correspond bijectively with normalised overconvergent eigenforms.

In fact much more is proved in [9], where two rigid spaces are constructed for each odd prime p: a curve C, constructed via deformation theory and the theory of pseudorepresentations, and a curve D constructed via glueing Hecke algebras as above. In Theorem 7.5.1 of [9] it is proven that D is isomorphic to the space C^{red}. Since the paper [9] appeared, various authors have assumed that the constructions in it would generalise to the cases $N > 1$ and $p = 2$. In fact, it seems to us that the following are the main reasons that $N = 1$ and $p > 2$ are assumed in [9]. Firstly, the theory of pseudorepresentations does not work quite so well in the case $p = 2$. Secondly, there are some issues to be resolved when writing down the local conditions at primes dividing N on the deformation theory side (we remark that Kisin tells us that both of these issues can be resolved without too much trouble). And thirdly one sometimes has to deal with eigenspaces for the action of $(\mathbf{Z}/4\mathbf{Z})^\times$ on a 2-adic Banach module when $p = 2$ on the Hecke algebra side, causing problems when looking for orthonormal bases. The first two issues will not concern us in this paper, as we do not talk about the construction of C, and the results in sections 2 and 3 of this paper are enough to deal with the third issue. In fact Chenevier has pointed out to us that one can also avoid the troubles caused by the third issue when constructing eigencurves for $p = 2$ by appealing to the corollary of lemma 1 in [15]. We do not construct a generalisation of C here, but we do show how to construct an eigencurve D for a general prime p and level N prime to p.

We first establish some generalities and notation. All our rigid spaces will be over $K = \mathbf{Q}_p$ in this section. Recall that $\mathcal{W}$ is the rigid space over $\mathbf{Q}_p$ representing maps $\mathbf{Z}_p^\times \to \mathbf{G}_m$. Define $q = p$ if $p > 2$, and $q = 4$ if $p = 2$. Define $D = (\mathbf{Z}/q\mathbf{Z})^\times$, regarded as a quotient of $\mathbf{Z}_p^\times$ in the natural way. Define $\widehat{D}$ to be the set of group homomorphisms $D \to \mathbf{C}_p^\times$. Set $\gamma = 1 + q \in \mathbf{Z}_p^\times$. The natural surjection $\mathbf{Z}_p^\times \to D$ has kernel $1 + q\mathbf{Z}_p$, which is topologically isomorphic to $\mathbf{Z}_p$, and is topologically generated by γ. The map $\mathbf{Z}_p^\times \to D$ induces an isomorphism between D and the roots of unity in $\mathbf{Z}_p^\times$, and hence the surjection splits and we have an isomorphism $\mathbf{Z}_p^\times \cong D \times \mathbf{Z}_p$; we shall thus identify $\mathbf{Z}_p^\times$ with $D \times \mathbf{Z}_p$. If $\chi \in \widehat{D}$ then the composite of χ with the natural projection $\mathbf{Z}_p^\times \to D$ is an element of $\mathcal{W}$, and one easily checks that distinct elements of $\widehat{D}$ are in different components of $\mathcal{W}$. Hence this construction establishes a bijection between the components of $\mathcal{W}$ and the group $\widehat{D}$. Let $\mathcal{W}_\chi$ denote the component of $\mathcal{W}$ corresponding to the character $\chi \in \widehat{D}$. Let $\mathbf{1}$ denote the trivial character of D (sending everything to 1) and let $\mathcal{B}$ denote the component $\mathcal{W}_{\mathbf{1}}$ of $\mathcal{W}$. Note that if $\chi \in \widehat{D}$ then multiplication by χ gives an isomorphism $\mathcal{W}_{\mathbf{1}} \to \mathcal{W}_\chi$.

For $n \geq 1$ let X_n denote the affinoid subdomain of $\mathcal{W}$ corresponding to group homomorphisms $\psi : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$ such that $|\psi(1+q)^{p^{n-1}} - 1| \leq |q|$. It is easily checked that for any $\chi \in \widehat{D}$, $X_n \cap \mathcal{W}_\chi$ is an affinoid disc, that $X_1 \subseteq X_2 \subseteq \ldots$, and that the X_i give an admissible cover of $\mathcal{W}$. The inclusion $X_i \subset \mathcal{W}$ induces a bijection of the connected components of X_i with the connected components of $\mathcal{W}$; if $\chi \in \widehat{D}$ then write $X_{i,\chi}$ for the closed disc $X_i \cap \mathcal{W}_\chi$. We remark here that if $k \in \mathbf{Z}$ and $\chi : (\mathbf{Z}/qp^{n-1}\mathbf{Z})^\times \to \mathbf{C}_p^\times$ then $\chi(1+q)^{p^{n-1}} = 1$ and hence the map $\psi \in \mathcal{W}$ defined by $\psi(x) = x^k\chi(x)$ is in $X_n(\mathbf{C}_p)$.

Define $R_i = \mathcal{O}(X_i)$. Then R_i is an affinoid for each i, and $R_i = \oplus_\chi R_{i,\chi}$, where χ runs through $\widehat{D}$ and $R_{i,\chi} = \mathcal{O}(X_{i,\chi})$.

In preparation for the application of our eigenvariety machine, we have to choose a family of radii of overconvergence.

Fortunately Coleman and Mazur have done enough for us here, even if $p = 2$. We give a brief description of the modular curves and affinoids that we shall use. Let $\mathbf{A}_f$ denote the finite adeles. For a compact open subgroup $\Gamma \subset \mathrm{GL}_2(\mathbf{A}_f)$ that contains the principal congruence subgroup Γ_N for some N prime to p, we define the compact modular curve $X(\Gamma)$ over $\mathbf{Q}_p$ in the usual way. Let us firstly assume that Γ is sufficiently small to ensure that the associated moduli problem on generalised elliptic curves has no non-trivial automorphisms (we will remove this assumption below). Now recall from section 3 of [4], for example, that for an elliptic curve E over a finite extension of $\mathbf{Q}_p$, there is a measure $v(E)$ of its supersingularity, and that $v(E) < p^{2-m}/(p+1)$ implies that E possesses a canonical subgroup of order p^m. So for $r \in \mathbf{Q}$ with $0 \leq r < p/(p+1)$ we define $X(\Gamma)_{\geq p^{-r}}$ to be the affinoid subdomain of the rigid space over $\mathbf{Q}_p$ associated to $X(\Gamma)$ whose non-cuspidal points parametrise elliptic curves E with a level Γ structure and such that $v(E) \leq r$. For example if $r = 0$ then $X(\Gamma)_{\geq p^{-r}}$ is the ordinary locus of $X(\Gamma)$.

For $m \geq 1$ there is a fine moduli space $X(\Gamma, \Gamma_1(p^m))$ (resp. $X(\Gamma, \Gamma_0(p^m))$) over $\mathbf{Q}_p$ whose non-cuspidal points parametrise elliptic curves equipped with a level Γ structure and a point (resp. cyclic subgroup) of order p^m over $\mathbf{Q}_p$-schemes. There are natural forgetful functors

$$X(\Gamma, \Gamma_1(p^m)) \to X(\Gamma, \Gamma_0(p^m)) \to X(\Gamma).$$

If $0 \leq r < p^{2-m}/(p+1)$ and E is an elliptic curve over a finite extension of $\mathbf{Q}_p$ with $v(E) \leq r$ then, as mentioned above, E has a canonical subgroup of order p^m. For r in this range we define $X(\Gamma, \Gamma_0(p^m))_{\geq p^{-r}}$ to be the components of the pre-image of $X(\Gamma)_{\geq p^{-r}}$ in $X(\Gamma, \Gamma_0(p^m))$ whose non-cuspidal points parametrise elliptic curves with the property that their given cyclic subgroup of order p^m equals their canonical subgroup, and we define the rigid space $X(\Gamma, \Gamma_1(p^m))_{\geq p^{-r}}$ to be the pre-image of $X(\Gamma, \Gamma_0(p^m))_{\geq p^{-r}}$ in $X(\Gamma, \Gamma_1(p^m))$.

All these spaces are affinoids; this follows from the fact that

$X(\Gamma)_{\geq p^{-r}}$ is an affinoid, being the complement of a non-zero finite number of open discs in a complete curve. There is a natural action of the finite group $(\mathbf{Z}/p^m\mathbf{Z})^\times$ on $X(\Gamma, \Gamma_1(p^m))$ and on $X(\Gamma, \Gamma_1(p^m))_{\geq p^{-r}}$ via the (weight 0) Diamond operators.

Finally if Γ is a compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$ containing Γ_N for some N prime to p, but which is not "sufficiently small", then choose some prime $l \nmid 2Np$ and define $\Gamma' := \Gamma \cap \Gamma_l$; then Γ' is a normal subgroup of Γ and Γ' is sufficiently small. Hence one may apply all the constructions above to Γ' and then define $X(\Gamma)_{\geq p^{-r}}$, $X(\Gamma, \Gamma_1(p^m))_{\geq p^{-r}}$ and so on by taking Γ/Γ'-invariants. The resulting objects are only coarse moduli spaces but this will not trouble us. A standard argument shows that this construction is independent of l. We define $X_1(p^m)_{\geq p^{-r}} := X(\mathrm{GL}_2(\widehat{\mathbf{Z}}), \Gamma_1(p^m))_{\geq p^{-r}}$ and $X_0(p^m)_{\geq p^{-r}} := X(\mathrm{GL}_2(\widehat{\mathbf{Z}}), \Gamma_0(p^m))_{\geq p^{-r}}$. Similarly if $N \geq 1$ is prime to p then we define $X_0(Np^m)_{\geq p^{-r}} := X(\Gamma_0(N), \Gamma_0(p^m))_{\geq p^{-r}}$, where $\Gamma_0(N)$ is as usual the matrices in $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ which are upper triangular mod N. Note that $X_1(q)_{\geq 1}$ is the curve that Coleman and Mazur refer to as $Z_1(q)$, and that the quotient of $X_1(p^m)_{\geq p^{-r}}$ by the action of $(\mathbf{Z}/p^m\mathbf{Z})^\times$ is $X_0(p^m)_{\geq p^{-r}}$. Note also that $X_0(p^m)_{\geq p^{-r}}$ is "independent of m", in the sense that the natural (forgetful) map $X_0(p^m)_{\geq p^{-r}} \to X_0(p)_{\geq p^{-r}}$ is an isomorphism (as both rigid spaces represent the same functor).

We now come to the definition of the radii of overconvergence r_i. Note that these numbers depend only on p and not on any level structure. We let $\mathbf{E}_p$ denote the function on $X_1(q)_{\geq 1} \times \mathcal{B}$ defined in Proposition 2.2.7 of [9] (briefly, $\mathbf{E}_p$ is the function which, when restricted to a classical even weight $k \geq 4$ in $\mathcal{B}$, corresponds to the function $E_k(q)/E_k(q^p)$, where $E_k(q)$ is the p-deprived ordinary old Eisenstein series of weight k and level p). In Proposition 2.2.7 of [9] it is proved that $\mathbf{E}_p$ is overconvergent over $\mathcal{B}$. The specialisations to a classical weight of $\mathbf{E}_p$ are fixed by the weight 0 Diamond operators, and hence $\mathbf{E}_p$ descends to an overconvergent function on $X_0(q)_{\geq 1} \times \mathcal{B} = X_0(p)_{\geq 1} \times \mathcal{B}$. Furthermore, the assertions about the q-expansion coefficients of $\mathbf{E}_p$

made in Proposition 2.2.7 of [9], and the q-expansion principle, are enough to ensure that $\mathbf{E}_p$ has no zeroes on $X_0(p)_{\geq 1} \times \mathcal{W}_{\mathbf{1}}$. Hence the inverse of $\mathbf{E}_p$ is a function on $X_0(p)_{\geq 1} \times \mathcal{B}$ and it is elementary to check that it is also overconvergent over $\mathcal{B}$. In particular, for any $i \geq 1$ there exists a rational $0 < r_i < 1/(p+1)$ such that the restrictions of both $\mathbf{E}_p$ and $\mathbf{E}_p^{-1}$ to $X_0(p)_{\geq 1} \times X_{i,\mathbf{1}}$ extend to functions on $X_0(p)_{\geq p^{-r_i}} \times X_{i,\mathbf{1}}$. We choose a sequence of rationals $r_1 \geq r_2 \geq r_3 \geq \ldots \geq 0$ such that each r_i has the aforementioned property. We may furthermore assume (for technical convenience) that $r_i < p^{2-i}/q(p+1)$ (although this may be implied by the other assumptions).

Remark. It is not important for us to establish concrete values for the r_i, so we do not. However for other applications where one wants to have explicit knowledge about how far one can extend overconvergent modular forms, it may in future be important to understand exactly how the r_i behave. For example, machine computations for $p \in \{3, 5, 7, 13\}$ and $i = 1$ show that it is *not* the case that r_1 can be taken to be an arbitrary rational number less than $1/(p+1)$, because there are classical Eisenstein eigenforms of level p which have zeroes in $\cup_{r<1/(p+1)} X_1(p)_{\geq p^{-r}}$. Can one take r_1 to be any positive rational less than $(p-1)/p(p+1)$? There are other interesting questions here, which we shall not attempt to answer here.[4]

We are now ready to begin our definitions of the Banach modules of overconvergent forms. These modules depend on an auxiliary level structure, which we now choose. Fix a compact open subgroup $\Gamma \subseteq \mathrm{GL}_2(\mathbf{A}_f)$ that contains the principal congruence subgroup Γ_N for some N prime to p. For each $n \geq 1$ we wish now to define a Banach module $M_n = M_{\Gamma,n}$ over $R_n := \mathcal{O}(X_n)$. Recall that $R_n = \oplus_{\chi \in \widehat{D}} R_{n,\chi}$. If $\chi \in \widehat{D}$ then we define the Banach module $B_{n,\chi}$ to be

$$B_{n,\chi} := R_{n,\chi} \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{O}(X(\Gamma, \Gamma_1(q))_{\geq p^{-r_n}})$$

[4]See however [7] for explicit calculations in the case $p = 2$, and note that these calculations may well generalise to odd primes p.

where r_n is the radius of convergence defined previously. Note that $B_{n,\chi}$ is naturally a potentially ONable Banach $R_{n,\chi}$-module, because the ring $\mathcal{O}(X(\Gamma,\Gamma_1(q)))$ can be viewed as a Banach space over a discretely-valued field and is hence potentially ONable by Proposition 1 of [15] and the remarks before it.

We give the module $B_{n,\chi}$ an $R_{n,\chi}$-linear action of the group $(\mathbf{Z}/q\mathbf{Z})^\times$ by letting it act trivially on $R_{n,\chi}$ and via the (weight 0) Diamond operators on $\mathcal{O}(X(\Gamma,\Gamma_1(q)))$. We define $M_{n,\chi}$ to be the direct summand of $B_{n,\chi}$ where $(\mathbf{Z}/q\mathbf{Z})^\times$ acts via χ. Note that by definition $M_{n,\chi}$ satisfies property (Pr) (one can check that it is even potentially ONable, but we shall not need this because of our extension of Coleman's theory). We remark in passing that if $\Gamma = \mathrm{GL}_2(\widehat{\mathbf{Z}})$ then $M_{n,\chi} = 0$ if $\chi(-1) \neq 1$. Finally we define $M_n = M_{\Gamma,n}$ to be the module over $R_n = \oplus_{\chi\in\widehat{D}} R_{n,\chi}$ whose $R_{n,\chi}$-part is $M_{n,\chi}$. We give M_n an R_n-linear action of $(\mathbf{Z}/q\mathbf{Z})^\times$ by letting it act via χ on $M_{n,\chi}$. We will shortly see that the fibre of M_n at a point $\kappa \in X_n$ can be naturally identified with the r_n-overconvergent forms of weight κ (although there are caveats regarding compatibility of this map with Diamond operators; see below).

The modules M_n we have described have the following functorial property: if Γ_1 and Γ_2 both satisfy the conditions imposed on Γ above (that is, they are of level prime to p), if $\gamma \in \mathrm{GL}_2(\mathbf{A}_f)$ has determinant which is a unit at p, and if $\gamma\Gamma_1\gamma^{-1} \subseteq \Gamma_2$, then there is a natural induced finite flat map $X(\Gamma_1) \to X(\Gamma_2)$ and the assumption on the determinant of γ means that if $0 \leq r \leq r_1$ then $X(\Gamma_1,\Gamma_1(q))_{\geq p^{-r}}$ is the pre-image of $X(\Gamma_2,\Gamma_1(q))_{\geq p^{-r}}$. Hence there is a natural inclusion $M_{\Gamma_2,n} \to M_{\Gamma_1,n}$. If furthermore $\gamma\Gamma_1\gamma^{-1} = \Gamma_2$ then this inclusion has an inverse and is hence an isomorphism. One can check using these ideas that if Γ_1 is a normal subgroup of Γ_2 then $M_{\Gamma_2,n}$ is the Γ_2/Γ_1-invariants of $M_{\Gamma_1,n}$. Note also that for $n \geq 1$ there is a natural map $M_n \to M_{n+1}\widehat{\otimes}_{R_{n+1}} R_n$, induced by restriction, which we shall later see is a link, in the sense of Part I.

We now make explicit the relation between these spaces and

Katz' spaces of overconvergent modular forms. Firstly we recall the definitions (in the form due to Coleman). If $\Gamma \subset \mathrm{GL}_2(\mathbf{A}_f)$ is a sufficiently small compact open subgroup then there is a sheaf commonly denoted ω on $X(\Gamma)$ which, on non-cuspidal points, is the pushforward of the differentials on the universal elliptic curve. A weight k modular form of level Γ, defined over $\mathbf{Q}_p$, is a global section of $\omega^{\otimes k}$. If L is a field extension of $\mathbf{Q}_p$ then a weight k modular form of level Γ defined over L is a global section of $\omega^{\otimes k}$ on the base change of $X(\Gamma)$ to L. If Γ contains Γ_N for some positive integer N prime to p and $0 \leq r < p/(p+1)$ then a p^{-r}-overconvergent modular form of level Γ and weight k defined over $\mathbf{Q}_p$ is a section of (the analytic sheaf associated to) $\omega^{\otimes k}$ on the rigid space $X(\Gamma)_{\geq p^{-r}}$, and similarly for L a complete extension of $\mathbf{Q}_p$. If Γ is not sufficiently small then one can still make sense of these definitions by replacing Γ by a sufficiently small normal subgroup Γ' and then taking Γ/Γ'-invariants, as Γ/Γ' will act on everything. Finally, if $0 \leq r < p^{2-n}/(p+1)$ then one can define p^{-r}-overconvergent modular forms of weight k and level $\Gamma \cap \Gamma_1(p^n)$ as sections of $\omega^{\otimes k}$ on $X(\Gamma, \Gamma_1(p^n))_{\geq p^{-r}}$ (again using the standard tricks if Γ is not sufficiently small). There is a natural action of $(\mathbf{Z}/p^n\mathbf{Z})^\times$ on these spaces via the (weight k) Diamond operators.

Fix $n \geq 1$ and let L be the field $\mathbf{Q}_p(\zeta_{p^{n-1}})$ generated by a primitive p^{n-1}th root of unity $\zeta_{p^{n-1}}$. Fix $k \in \mathbf{Z}$, and a character $\varepsilon : (\mathbf{Z}/qp^{n-1}\mathbf{Z})^\times \to L^\times$. Define $\psi : \mathbf{Z}_p^\times \to L^\times$ by $\psi(x) = x^k\varepsilon(x)$. Then ψ is an L-valued point of weight space and in fact $\psi \in X_n(L)$. Choose $\chi \in \widehat{D}$ such that $\psi \in \mathcal{W}_\chi$. Define $\kappa = \psi/\chi \in \mathcal{W}_{\mathbf{1}}$ and let $E_\kappa = 1 + \cdots$ be the associated Eisenstein series (see p39 of [9] and note that $\kappa \in \mathcal{W}_{\mathbf{1}} = \mathcal{B}$ so there are no problems with zeros of p-adic L-functions).

Proposition 6.1. *The q-expansion E_κ is the q-expansion of a p^{-r_n}-overconvergent weight k modular form of level $\Gamma_1(qp^{n-1})$ defined over L, which is non-vanishing on $X_1(qp^{n-1})_{\geq p^{-r_n}}$. Furthermore E_κ is in the ε/χ-eigenspace for the Diamond operators.*

Proof. The fact that the q-expansion E_κ is the q-expansion of a section of $\omega^{\otimes k}$ on $X_1(qp^{n-1})_{\geq p^{-r}}$ for some $r > 0$ is Corollary 2.2.6 of [9]. Now the fact that E_κ is an eigenvector for U_p, which increases overconvergence, implies that E_κ extends to a section of $\omega^{\otimes k}$ on of $X_1(qp^{n-1})_{\geq p^{-r}}$ for any rational r with $0 \leq r < p^{3-n}/q(p+1)$. For r in this range, there is a map $\mathrm{Frob} : X_1(qp^{n-1})_{\geq p^{-r/p}} \to X_1(qp^{n-1})_{\geq p^{-r}}$ which is finite and flat of degree p and which induces, by pullback, the map $F(q) \mapsto F(q^p)$ on modular forms (on non-cuspidal points the map sends (E, P) to $(E/C, \overline{Q})$ where C is the canonical subgroup of order p of E and $\overline{Q}$ is the image in E/C of any generator Q of the canonical subgroup of order qp^n of E such that $pQ = P$). By the q-expansion principle, E_κ has no zeroes on the ordinary locus $X_1(qp^{n-1})_{\geq 1}$. Let S denote the set of zeroes of E_κ on the non-ordinary locus of $X_1(qp^{n-1})_{\geq p^{-r_n}}$. It suffices to show that S is empty. We know that S is finite, because $X_1(qp^{n-1})_{\geq p^{-r_n}}$ is a connected affinoid curve and $E_\kappa \neq 0$ (as its q-expansion is non-zero). We also know that $E_\kappa(q)/E_\kappa(q^p)$ has no zeroes on $X_1(qp^{n-1})_{\geq p^{-r_n}}$, by definition of r_n, and that $r_n < p^{2-n}/q(p+1)$. Hence any zero of E_κ is also a zero of $E_\kappa(q^p) = \mathrm{Frob}^* E_\kappa$. But if S is non-empty then let P denote a point of S which is "nearest to the ordinary locus" (that is, such that $v'(P)$ is minimal, where v' is the composite of the natural projection $X_1(qp^{n-1}) \to X_0(p)$ and the function denoted v' in section 4 of [4]). Then P is also a zero of $E_\kappa(q^p)$ and hence if $P = \mathrm{Frob}(Q)$ for some point $Q \in X_1(qp^{n-1})_{\geq p^{-r_n/p}}$ then Q is also a zero of $E_\kappa(q)$, and furthermore Q is closer to the ordinary locus than P (in fact $v'(Q) = \frac{1}{p}v'(P) < v'(P)$ by Theorem 3.3(ii) of [4]), a contradiction.

The assertion about the Diamond operators is classical if $k \geq 2$ (see for example Proposition 7.1.1 of [13]). For general k it can be deduced as follows: by Theorem B4.1 of [10] applied with $i = 0$, the function on $X_1(q)_{\geq 1} \times \mathcal{B} \times \mathcal{B}$ denoted $E_\alpha(q)E_\beta(q)/E_{\alpha\beta}(q)$ in Theorem 2.2.2 of [9], when restricted to $X_1(q)_{\geq 1} \times \mathcal{B}^* \times \mathcal{B}^*$ (in the notation of the proof of Theorem 2.2.2 of [9]) is invariant under the natural action of $(\mathbf{Z}/q\mathbf{Z})^\times$ (acting trivially on $\mathcal{B}^\times$ and via Diamond operators on $X_1(q)$). But this action is continuous, so $(\mathbf{Z}/q\mathbf{Z})^\times$ acts trivially on $E_\alpha(q)E_\beta(q)/E_{\alpha\beta}(q)$. In particular, the function $E_\alpha(q)E_\beta(q)/E_{\alpha\beta}(q)$ descends to a function on $X_0(q)_{\geq 1}\times\mathcal{B}\times\mathcal{B} = X_0(qp^{n-1})_{\geq 1}\times\mathcal{B}\times\mathcal{B}$. The result for general k now follows from the result for $k \geq 2$. □

Corollary 6.2. *Let $k, \varepsilon, \psi, \chi, \kappa$ be as above, and let $\psi : R_n \to L$ also denote the homomorphism corresponding to the L-point of X_n induced by ψ. Then multiplication by E_κ induces an isomorphism between $M_{\Gamma,n} \otimes_{R_n} L$ (the tensor product formed via the homomorphism $\psi : R_n \to L$) and the space of p^{-r_n}-overconvergent modular forms of level Γ, weight k and character ε defined over L.*

Proof. Say $\psi \in \mathcal{W}_\chi$. Then unravelling the definitions gives that $M_{\Gamma,n} \otimes_{R_n} L = M_{n,\chi} \otimes_{R_{n,\chi}} L$ is equal to the χ-eigenspace of $\mathcal{O}(X(\Gamma, \Gamma_1(q))_{\geq p^{-r_n}}) \otimes_{\mathbf{Q}_p} L$, and hence the χ-eigenspace of $\mathcal{O}(X(\Gamma, \Gamma_1(qp^{n-1}))_{\geq p^{-r_n}}) \otimes_{\mathbf{Q}_p} L$, where χ here is regarded as a character of $(\mathbf{Z}/qp^{n-1}\mathbf{Z})^\times$ (note that the forgetful functor

$$X(\Gamma, \Gamma_1(qp^{n-1}))_{\geq p^{-r_n}} \to X(\Gamma, \Gamma_1(q))_{\geq p^{-r_n}}$$

is the map induced by quotienting $X(\Gamma, \Gamma_1(qp^{n-1}))_{\geq p^{-r_n}}$ out by the group $(1 + q\mathbf{Z}_p/1 + qp^{n-1}\mathbf{Z}_p)$). The result now follows from the fact that E_κ has weight k, character ε/χ and is non-vanishing on $X(\Gamma, \Gamma_1(qp^{n-1}))_{\geq p^{-r_n}}$. □

Motivated by this Corollary, we define the q-expansion of an element of M_n to be the following element of $R_n[[q]]$, as follows: it suffices to attach a q-expansion in $R_{n,\chi}[[q]]$ to an element of $M_{n,\chi}$, and hence it suffices to attach a q-expansion in $R_{n,\chi}[[q]]$ to an element of $B_{n,\chi}$. Now an element of $\mathcal{O}(X(\Gamma, \Gamma_1(q))_{\geq p^{-r_n}})$ has a q-expansion in $\mathbf{Q}_p[[q]]$ in the usual way, and hence an element of $B_{n,\chi}$ has a q-expansion in $R_{n,\chi}[[q]]$. This is not the q-expansion that we are interested in however—this q-expansion corresponds to a family of weight 0 overconvergent forms. we twist this q-expansion by multiplying it by $\mathbf{E}$, the q-expansion of the restricted Eisenstein family defined in Section 2.2 of [9]. Note that the restricted Eisenstein family is a family over $\mathcal{B}$ and so we have to explain how to regard it as a family over $X_{n,\chi}$; we do this by pulling back via the composite of the natural inclusion $X_{n,\chi} \to \mathcal{W}_\chi$ and the natural isomorphism $\mathcal{W}_\chi \to \mathcal{W}_{\mathbf{1}}$. The resulting power series is defined to be the q-expansion of m, and with this normalisation, the isomorphism of the previous corollary preserves q-expansions.

As we shall see in the next section, we will be defining Hecke operators on $M_{\Gamma,n}$ (at least for certain choices of Γ) so that they agree with the standard Hecke operators on overconvergent modular forms, via the isomorphism of the previous Corollary. We finish this section by remarking that the isomorphism of the previous corollary is however not compatible with Diamond operators in general, for two reasons—firstly, overconvergent forms of classical weight-character ψ naturally have an action of the group $(\mathbf{Z}/qp^{n-1}\mathbf{Z})^\times$, but the full space M_n only has an action of $(\mathbf{Z}/q\mathbf{Z})^\times$, and secondly even the actions of $(\mathbf{Z}/q\mathbf{Z})^\times$ do not in general coincide, as one is defined in weight 0 and the other in weight k so the two actions differ (at least for $p > 2$) by the kth power of the Teichmüller character.

7 Hecke operators and classical eigencurves

We now restrict to the case where Γ is either the congruence subgroup $\Gamma_0(N)$ (the subgroup of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ consisting of matrices which are upper triangular mod N) or $\Gamma_1(N)$ (the subgroup of $\Gamma_0(N)$ consisting of matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ mod N) of $\mathrm{GL}_2(\mathbf{A}_f)$, for some N prime to p. The associated moduli problems are then just the usual problems of representing cyclic subgroups or points of order N. We now define Hecke operators T_m for m prime to p, and a compact operator U_p, on the spaces

$M_{\Gamma,n}$ for $n \geq 1$. Almost all of the work has been done for us, in section B5 of [10] (where Hecke operators are defined on overconvergent forms over $\mathcal{B}^*$) and in section 3.4 of [9] (where this work is extended to $\mathcal{B}$). Note that the arguments in these references do not assume $p > 2$ or $N = 1$. We do not reproduce the arguments here, we just mention that the key point is that because the argument is *not* a geometric one, the construction of T_m is done at the level of q-expansions and the resulting definitions initially go from forms of level N to forms of level Nm. However one can prove that the resulting maps do in fact send forms of level N to forms of level N by noting that the result is true for forms of classical weight, where the Hecke operator can be defined via a correspondence, and deducing the general case by considering a trace map and noting that a family of forms that vanishes at infinitely many places must be zero.

The one lacuna in the arguments in the references above is that in both cases the operators are defined as endomorphisms of overconvergent forms, rather than r-overconvergent forms for some fixed r. What we need to do is to prove that the Hecke operators defined by Coleman and Mazur send r-overconvergent forms to r-overconvergent forms, for some appropriate choice of r. Let $\mathbf{E}$ denote the restricted Eisenstein family (that is, the usual family of Eisenstein series over $\mathcal{B}$), and for $\ell \neq p$ a prime number, let $\mathbf{E}_\ell(q)$ denote the ratio $\mathbf{E}(q)/\mathbf{E}(q^\ell)$ as in Proposition 2.2.7 of [9], thought of as an overconvergent function on $X_0(\ell p)_{\geq 1} \times \mathcal{B}$ (note that Coleman and Mazur only assert that this function lives on $X_1(q, \ell)_{\geq 1} \times \mathcal{B}$ but it is easily checked to be invariant under the Diamond operators at p).

Lemma 7.1. *The restriction of* $\mathbf{E}_\ell(q)$ *to* $X_0(\ell p)_{\geq 1} \times X_{n,\mathbf{1}}$ *extends to a non-vanishing function on* $X_0(\ell p)_{\geq p^{-rn}} \times X_{n,\mathbf{1}}$.

Proof. For simplicity in this proof, we say that a function on $X_0(p)_{\geq 1} \times X_{n,\mathbf{1}}$ is r-overconvergent if it extends to a function on $X_0(p)_{\geq p^{-r}} \times X_{n,\mathbf{1}}$, and similarly we say that such a function is r-overconvergent and non-vanishing if it extends to a non-vanishing function on $X_0(p)_{\geq p^{-r}} \times X_{n,\mathbf{1}}$.

We first prove that $\mathbf{E}_\ell(q)$ is r_n-overconvergent. Observe that Proposition 2.2.7 of [9] tells us that $\mathbf{E}_\ell(q)$ is r-overconvergent for some $r > 0$. Let us assume $r < r_n$ and explain how to analytically continue $\mathbf{E}_\ell(q)$ a little further. By definition of r_n, we know that $\mathbf{E}_p(q)$ is r_n-overconvergent and non-vanishing. Furthermore the non-trivial degeneracy map $X_0(\ell p) \to X_0(p)$ which on q-expansions sends $F(q)$ to $F(q^\ell)$ induces a morphism of rigid spaces $X_0(\ell p)_{\geq p^{-r_n}} \to X_0(p)_{\geq p^{-r_n}}$ and hence $\mathbf{E}_p(q^\ell)$ is also an r_n-overconvergent non-vanishing function. We deduce that the ratio $\mathbf{E}_p(q^\ell)/\mathbf{E}_p(q)$ is also an r_n-overconvergent non-vanishing function. But

$$\mathbf{E}_p(q^\ell)/\mathbf{E}_p(q) = \mathbf{E}_\ell(q^p)/\mathbf{E}_\ell(q)$$

and $\mathbf{E}_\ell(q)$ is r-overconvergent, and hence we see that $\mathbf{E}_\ell(q^p)$ is also r-overconvergent. Let $r' = \min\{pr, r_n\}$. We claim that $\mathbf{E}_\ell(q)$ is r'-overconvergent and this clearly is enough because repeated applications of this idea will analytically continue $\mathbf{E}_\ell(q)$ until it is r_n-overconvergent, which is what we want. But it is a standard fact that the U operator increases overconvergence by a factor of p, and hence if $\mathbf{E}_\ell(q^p)$ is r-overconvergent, then $\mathbf{E}_\ell(q) = U(\mathbf{E}_\ell(q^p))$ is r'-overconvergent.

Finally we show that $\mathbf{E}_\ell(q)$ is non-vanishing on $X_0(\ell p)_{\geq p^{-r_n}} \times X_{n,\mathbf{1}}$ and this follows from an argument similar to the non-vanishing statement proved in Proposition 6.1—if $\mathbf{E}_\ell(q)$ had a zero then choose a zero (x, κ) and then specialise to weight κ; we may assume that x is a zero closest to the ordinary locus in weight κ, and then $\mathbf{E}_\ell(q^p)$ has a zero closer to the ordinary locus in weight κ and hence $\mathbf{E}_\ell(q)/\mathbf{E}_\ell(q^p)$ would have a pole in weight κ, contradicting the fact that $\mathbf{E}_\ell(q)/\mathbf{E}_\ell(q^p) = \mathbf{E}_p(q)/\mathbf{E}_p(q^\ell)$ is r_n-overconvergent. □

We now have essentially everything we need to apply our eigenvariety machine. The preceding lemma and the arguments in section 3.4 of [9] can be used to define Hecke operators T_m

(m prime to p) and U_p on r_n-overconvergent forms over X_n. If X is any admissible affinoid open subdomain of $\mathcal{W}$ then there exists some $n \geq 1$ such that $X \subseteq X_n$; we choose the smallest such n and define the Banach module M_X to be the pullback to $\mathcal{O}(X)$ of the $\mathcal{O}(X_n)$-module M_n. We let $\mathbf{T}$ denote the abstract polynomial algebra over R generated by the Hecke operators T_m for m prime to p, the operator $\phi = U_p$, and the Diamond operators at N if $\Gamma = \Gamma_1(N)$. These Hecke operators are well-known to commute, as can be seen by checking on classical points. We need to check that the natural restriction maps α between spaces of overconvergent forms of different radii are all links, but this follows easily from the technique used in the standard proof that the characteristic power series of U_p on r-overconvergent forms is independent of the choices of $r > 0$. The key point is that the endomorphism U_p of r-overconvergent forms can be checked to factor as a continuous map from r-overconvergent forms to s-overconvergent forms, for some $s > r$, followed by the (compact) restriction map from s-overconvergent forms to r-overconvergent forms. In fact one can take s to be anything less than both pr and $p/(p+1)$. One deduces that for any $0 < r < r' < p/(p+1)$, the natural map from r'-overconvergent forms to r-overconvergent forms is a "link". Our conclusion is that the construction of the "D" eigencurve in [9] can be generalised to all p and $N \geq 1$ prime to p.

PART III: Eigenvarieties for Hilbert modular forms.

8 Thickenings of K-points and weight spaces

Let K be a non-archimedean local field (that is, a field either isomorphic to a finite extension of $\mathbf{Q}_p$ or to the field of fractions of $k[[T]]$ with k finite). Let $\mathcal{O}$ denote the integers of K, and let V denote the closed affinoid unit disc over K. As an example of a construction which will be used many times in the sequel, we firstly show that is not difficult to construct a sequence $U_1 \supset U_2 \supset \ldots$ of affinoid subdomains of V, defined over K, with the property that

$$\bigcap_{t\geq 1}\big(U_t(L)\big) = V(K) = \mathcal{O} \qquad (*)$$

for all complete extensions L of K. Note that $(*)$ implies that $U_t(K) = V(K)$ for all t (set $L = K$), but also that no non-empty K-affinoid subdomain of V can be contained in all of the U_t. The U_t should be thought of as a system of affinoid neighbourhoods of $V(K) = \mathcal{O}$ in V. The construction of the U_t is simple: Let $\pi \in K$ be a uniformiser and define $U_t = \bigcup_{\alpha\in X_t} B(\alpha, |\pi|^t)$, where X_t is a set of representatives in $\mathcal{O}$ for $\mathcal{O}/(\pi)^t$, and $B(\alpha, |\pi|^t)$ is the closed affinoid disc with centre α and radius $|\pi|^t$. Note that X_t is finite because K is a local field, and hence U_t is an affinoid subdomain of V: it is a finite union of affinoid subdiscs of V of radius $|\pi|^t$. It is easy to check moreover that the U_t satisfy $(*)$ (use the fact that $\mathcal{O}$ is compact to get the harder inclusion).

We in fact need a "twisted" n-dimensional version of this construction, which is more technical to state but which requires essentially no new ideas. Before we explain this generalisation, we make an observation about the possible radii of discs defined over non-archimedean fields. Let K be an arbitrary field complete with respect to a non-trivial non-archimedean norm. If L is a finite extension of K then there is a unique way to extend the norm on K to a norm on L, and hence there is a unique way to extend the norm on K to a norm on an algebraic closure $\overline{K}$ of K. Let $\left|\overline{K}^\times\right|$ denote the set $\left\{|x| : x \in \overline{K}^\times\right\}$. It is easily checked that this set is just $\{|y|^{1/d} : y \in K^\times, d \in \mathbf{Z}_{\geq 1}\}$. Note that for $r \in \left|\overline{K}^\times\right|$ the closed disc $B(0, r)$ with centre zero and radius r is a rigid space *defined over* K: if $r \leq 1$ then $r = |y|^{1/d}$ for some $y \in K^\times$ and $d \in \mathbf{Z}_{\geq 1}$, and one can construct $B(0, r)$ as the space associated to the affinoid algebra $K\langle T, S\rangle/(T^d - yS)$; the general case can be reduced to this by scaling. By using products of these discs, one sees that one can construct polydiscs with "fractional" radii over K.

Let M/M_0 be a fixed finite extension of non-archimedean local fields, and assume that the restriction of the norm on M is the norm on M_0. Let K denote any complete extension of M_0, again with the norm on K assumed to extend the norm on M_0. Assume moreover that K has the property that the image of any M_0-algebra homomorphism $M \to \overline{K}$ (an algebraic closure of K) lands in K. The U_t above will correspond to the case $M_0 = M = K$ of the construction below. Later on M_0 will be $\mathbf{Q}_p$ but we do not need to assume that we are in mixed characteristic yet.

Let I denote the set of M_0-algebra homomorphisms $M \to K$. We will use $|.|$ to denote the norms on both M and K, and there is of course no ambiguity here because any M_0-algebra map $i : M \to K$ will be norm-preserving. Let $\mathcal{O}$ now denote the integers of M (in particular $\mathcal{O}$ is no longer the integers of K), and let π be a uniformiser of M. Let V be the unit polydisc over K of dimension $|I|$, the number of elements of I, and think of the coordinates of V as being indexed by elements of I. Let $\mathcal{N}_K$ denote the set $\{|x| : x \in \overline{K}^\times, |x| \leq 1\}$ and let $\mathcal{N}_K^\times$ denote $\mathcal{N}_K \backslash \{1\} = \{|x| : x \in \overline{K}^\times, |x| < 1\}$.

If $\alpha = (\alpha_1, \alpha_2, \ldots,) \in K^I$ and $r \in \mathcal{N}_K$ then we define $B(\alpha, r)$ to be the K-polydisc whose L-points, for L any complete extension of K, are

$$B(\alpha, r)(L) = \{(x_1, x_2, \ldots) \in L^I : |x_i - \alpha_i| \leq r \text{ for all } i\}.$$

For example we have $V = B(0, 1)$. Note that $B(\alpha, r)$ is defined over K by the comments above.

There is a natural map $M \to K^I$ which on the ith component sends m to $i(m)$, and we frequently write m_i for $i(m)$. Furthermore, we implicitly identify $m \in M$ with its natural image $(m_i)_{i \in I}$ in K^I. In particular, if $\alpha \in \mathcal{O}$ then α can be thought of as a K-point of V.

Definition. *If* $r \in \mathcal{N}_K$ *then define* $\mathbf{B}_r := \bigcup_{\alpha \in \mathcal{O}} B(\alpha, r)$*, and if furthermore* $r \in \mathcal{N}_K^\times$ *then define* $\mathbf{B}_r^\times := \bigcup_{\alpha \in \mathcal{O}^\times} B(\alpha, r)$*.*

We see that $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ are finite unions of polydiscs, because $B(\alpha, r) = B(\beta, r)$ if $|\alpha - \beta| \leq r$, and $\mathcal{O}$ is compact. In particular $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ are K-affinoid subdomains of V. The space $\mathbf{B}_r$ should be thought of as a thickening of $\mathcal{O}$ in V; similarly $\mathbf{B}_r^\times$ should be thought of as a thickening of $\mathcal{O}^\times$. One can check that for all complete extensions L of K, we have

$$\bigcap_{r \in \mathcal{N}_K} \mathbf{B}_r(L) = \mathcal{O}$$

and

$$\bigcap_{r \in \mathcal{N}_K^\times} \mathbf{B}_r^\times(L) = \mathcal{O}^\times.$$

The proof follows without too much difficulty from the fact that $\mathcal{O}$ and $\mathcal{O}^\times$ are compact subsets of the metric space K^I. One can also check that for any $r \in \mathcal{N}_K$, the space $\mathbf{B}_r$ is an affinoid subgroup of $(\mathbf{A}^1)^I$, the product of I copies of the additive group, and that if $r \in \mathcal{N}_K^\times$ then $\mathbf{B}_r^\times$ is an affinoid subgroup of $\mathbf{G}_m^I$, the product of I copies of the multiplicative group. An example of the idea continually used in the argument is that if $(y_i) \in \mathbf{B}_r^\times(L)$ for L some complete extension of K, then (y_i) is close to some element α of $\mathcal{O}^\times$, and hence (y_i^{-1}) is close to α^{-1}. See the lemma below for other examples of this type of argument.

We record some elementary properties of $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ that we shall use later. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\mathrm{M}_2(\mathcal{O})$ with $|c| < 1$, $|d| = 1$ and $\det(\gamma) \neq 0$. Define $m \in \mathcal{N}_K$ by $|\det(\gamma)| = m$. Choose $r \in \mathcal{N}_K$, and $t \in \mathbf{Z}_{>0}$ such that $|c| \leq |\pi^t|$.

Lemma 8.1. *(a) There is a map of rigid spaces* $\mathbf{B}_r \to \mathbf{B}_{r|\pi^t|}^\times$ *which on points sends* (z_i) *to* $(c_i z_i + d_i)$ *(where here as usual* c_i *denotes the image of* $c \in M$ *in* K *via the map* i *and so on).*

(b) There is a map of rigid spaces $\mathbf{B}_r \to \mathbf{B}_{rm}$ *which on points sends* (z_i) *to* $\left(\frac{a_i z_i + b_i}{c_i z_i + d_i}\right)$*.*

Proof. (a) Clearly there is a map $V \to V$ sending (z_i) to $(c_i z_i + d_i)$; we must check that the image of $\mathbf{B}_r$ is contained within $\mathbf{B}^{\times}_{r|\pi^t|}$. Because all the rigid spaces in question are finite unions of affinoid polydiscs, it suffices to check this on L-points, for L any complete extension of K. So let (z_i) be an L-point of $\mathbf{B}_r$. Then there exists $\alpha \in \mathcal{O}$ such that $|z_i - \alpha_i| \leq r \leq 1$ for all $i \in I$. In particular $|z_i| \leq 1$ so $|c_i z_i + d_i| = 1$, and so $c_i z_i + d_i$ is invertible. Note now that $\beta := c\alpha + d \in \mathcal{O}^{\times}$ and for $i \in I$ we have $|(c_i z_i + d_i) - \beta_i| = |c_i(z_i - \alpha_i)| \leq |c| r \leq |\pi^t| r$, which is what we wanted.

(b) A similar argument works, the key point being that if $(z_i) \in \mathbf{B}_r(L)$ and $\alpha \in \mathcal{O}$ is chosen such that $|z_i - \alpha_i| \leq r$ for all i, then defining $\beta = \frac{a\alpha+b}{c\alpha+d} \in \mathcal{O}$, we see that $\left|\frac{a_i z_i + b_i}{c_i z_i + d_i} - \beta_i\right| = |\det(\gamma)_i (z_i - \alpha_i)| \leq |\det(\gamma)| r = mr$ (as $|c_i z_i + d_i| = 1 = |c\alpha + d|$ as in (a)) and this is enough. □

Now let M_0, M, K be as before, and let $p > 0$ denote the residue characteristic of K. Let Γ be a profinite abelian group containing an open subgroup topologically isomorphic to $\mathbf{Z}_p^d$ for some d. If U is a rigid space over K, let $\mathcal{O}(U)$ denote the ring of rigid functions on U. Note that $\mathcal{O}$ is still the integer ring of M but this should not cause confusion. We say that a group homomorphism $\Gamma \to \mathcal{O}(U)$ (resp. $\Gamma \to \mathcal{O}(U)^{\times}$) is *continuous* if, for all affinoid subdomains X of U, the induced map $\Gamma \to \mathcal{O}(X)$ (resp. $\Gamma \to \mathcal{O}(X)^{\times}$) is continuous. We recall some results on representability of certain group functors. By a K-group we mean a group object in the category of rigid spaces over K.

Lemma 8.2. *(a) The functor from K-rigid spaces to groups, sending a space X to the group $\mathcal{O}(X)$ under addition (resp. the group $\mathcal{O}(X)^{\times}$ under multiplication) is represented by the K-group $\mathbf{A}^1$ (resp. $\mathbf{G}_m$), the analytification of the affine line (resp. the affine line with zero removed).*

(b) If Γ is as above, then there is a separated K-group $\mathcal{X}_\Gamma$ representing the functor which sends a K-rigid space U to the

group of continuous group homomorphisms $\Gamma \to \mathcal{O}(U)^\times$*. Moreover* $\mathcal{X}_\Gamma$ *is isomorphic to the product of an open unit polydisc and a finite rigid space over* K*.*

(c) If L *is a complete extension of* K *then the base change of* $\mathcal{X}_\Gamma$ *to* L *represents the functor on* L*-rigid spaces sending* U *to the group of continuous group homomorphisms* $\Gamma \to \mathcal{O}(U)^\times$*.*

Proof. (a) If X is a rigid space and $f \in \mathcal{O}(X)$ then for all affinoid subdomains $U \subseteq X$, f induces a unique map $U \to \mathbf{A}^1$ because there exists $0 \neq \lambda \in K$ such that λf is power-bounded on U, and then there is a unique map $K\langle T\rangle \to \mathcal{O}(U)$ sending T to λf by Proposition 1.4.3/1 of [1]; everything is compatible and glues, and the result for $\mathbf{A}^1$ follows easily. Moreover, if f is in $\mathcal{O}(X)^\times$ then for every affinoid subdomain U of X, it is possible to find an affinoid annulus in $\mathbf{G}_m$ containing $f(U)$, as both $f(U)$ and $(1/f)(U)$ lie in an affinoid subdisc of $\mathbf{A}^1$. Hence $f \in \mathcal{O}(X)^\times$ gives a map $X \to \mathbf{G}_m$, and the result in the multiplicative case now follows without too much difficulty.

(b) The existence of $\mathcal{X}_\Gamma$ is Lemma 2(i) of [5]. We recall the idea of the proof: the structure theorem for topologically finitely-generated profinite abelian groups shows that Γ is topologically isomorphic to a product of groups which are either finite and cyclic, or copies of $\mathbf{Z}_p$. Now by functorial properties of products in the rigid category, it suffices to show representability in the cases $\Gamma = \mathbf{Z}_p$ and Γ a finite cyclic group. The case $\Gamma = \mathbf{Z}_p$ is treated in Lemma 1 of [5] and the remarks after it, which show that the functor is represented by the open unit disc centre 1, and the case of Γ cyclic of order n is represented by the analytification of μ_n over K. That χ_Γ has the stated structure is now clear.

(c) By functoriality it is enough to verify these base change properties in the two cases $\Gamma = \mathbf{Z}_p$ and Γ finite cyclic; but in both of these cases the result is clear. □

We now assume that $M_0 = \mathbf{Q}_p$, and hence that M is a finite extension of $\mathbf{Q}_p$. We assume (merely for notational ease) that the norms on M_0, M and K are all normalised such that

$|p| = p^{-1}$. We remind the reader that if t is an element of an affinoid K-algebra and $|t| < 1$ then the power series for $\log(1+t)$ converges, and if $|t| < p^{-1/(p-1)}$ then the power series for $\exp(t)$ converges; furthermore log and exp give isomorphisms of rigid spaces from the open disc with centre 1 and radius $p^{-1/(p-1)}$ to the open disc with centre 0 and radius $p^{-1/(p-1)}$. We would like to use logs to analyse $\mathbf{B}_r^\times$ and hence are particularly interested in the spaces $\mathbf{B}_r^\times$ for $r < p^{-1/(p-1)}$; we call such r "sufficiently small". For these r, we see that the component of $\mathbf{B}_r^\times$ containing 1 is isomorphic, via the logarithm on each coordinate, to the component of $\mathbf{B}_r$ containing 0.

Recall that $\mathcal{O}$ is the integers of M, and hence $\Gamma = \mathcal{O}^\times$ satisfies the conditions just before Lemma 8.2. If $n \in \mathbf{Z}^I$ then there is a group homomorphism $\mathcal{O}^\times \to K^\times = \mathbf{G}_m(K)$, which sends α to $\prod_i \alpha_i^{n_i}$. It is easily checked (via exp and log) that if r is sufficiently small then this map is the K-points of a map of K-rigid spaces $\mathbf{B}_r^\times \to \mathbf{G}_m$. For more arithmetically complicated continuous maps from $\mathcal{O}^\times$ to invertible functions on affinoids, we might have to make r smaller still before such an analytic extension exists, but the proposition below shows that we can always do this. An important special case of this proposition is the case of an arbitrary continuous homomorphism $\mathcal{O}^\times \to K^\times$, but the proof is essentially no more difficult if $K^\times$ is replaced by the invertible functions on an arbitrary affinoid, so we work in this generality.

Proposition 8.3. *If X is a K-affinoid space and $n : \mathcal{O}^\times \to \mathcal{O}(X)^\times$ is a continuous group homomorphism, then there is at least one $r \in \mathcal{N}_K^\times$ and, for any such r, a unique map of K-rigid spaces $\beta_r : \mathbf{B}_r^\times \times X \to \mathbf{G}_m$, such that for all $\alpha \in \mathcal{O}^\times$, $n(\alpha)$ is the element of $\mathcal{O}(X)^\times$ corresponding (via Lemma 8.2(a)) to the map $X \to \mathbf{G}_m$ obtained by evaluating β_r at the K-valued point α of $\mathbf{B}_r^\times$.*

Definition. *We call β_r a* thickening *of n.*

Proof of Proposition. It suffices to prove that there exists at least one sufficiently small r such that if B is the component of $\mathbf{B}_r^\times$ containing the K-point 1, then there is a unique $\beta : B \times X \to \mathbf{G}_m$ with a property analogous to that of β_r above for all $\alpha \in \Delta := \{\alpha \in \mathcal{O}^\times : |\alpha - 1| \leq r\}$ (or even in some subgroup of finite index). Via the logarithm map one sees that Δ is isomorphic to $\mathcal{O}$. It hence suffices to prove that for any continuous group homomorphism $\chi : \mathcal{O} \to \mathcal{O}(X)^\times$ there exists $N \in \mathbf{Z}_{\geq 0}$ such that the induced homomorphism $p^N\mathcal{O} \to \mathcal{O}(X)^\times$ is induced by a unique map of K-rigid spaces $B(0, p^{-N}) \times X \to \mathbf{G}_m$. Here $B(0, p^{-N})$ denotes the polydisc in V with radius p^{-N}, and $p^N\mathcal{O}$ is embedded as a subset of the K-points of $B(0, p^{-N})$ in the usual way. Now observe that if $d = [M : \mathbf{Q}_p]$ then as a topological group, $\mathcal{O}$ is isomorphic to $\mathbf{Z}_p^d$. Hence, if one fixes a K-Banach algebra norm on $\mathcal{O}(X)$ and a $\mathbf{Z}_p$-basis $e_1, e_2, \ldots, e_d$ of $\mathcal{O}$, one sees using Lemma 1 of [5] that there exists a positive integer N such that $|\chi(p^N e_j) - 1| < p^{-1/(p-1)}$ for all j. Observe now that $\log(\chi)$ is a continuous group homomorphism $p^N\mathcal{O} \to \mathcal{O}(X)$, and $\mathcal{O}(X)$ is a K-vector space; furthermore, the image of $p^N\mathcal{O}$ will land in a finite-dimensional K-subspace of $\mathcal{O}(X)$. It is a standard fact (linear independence of distinct field embeddings) that the continuous group homomorphisms $\mathcal{O} \to K$ form a finite-dimensional K-vector space with basis the set I, now regarded as the ring homomorphisms $\mathcal{O} \to K$, and hence there exists $f_1, f_2, \ldots, f_d \in \mathcal{O}(X)^\times$ such that for all $\alpha \in p^N\mathcal{O}$ we have $\chi(\alpha) = \exp\left(\sum_i \alpha_i f_i\right)$, where α_i denotes $i(\alpha) \in K$. By increasing N if necessary, we may assume that $\left|p^N f_i\right| < p^{-1/(p-1)}$ for all i, and we claim that this N will work. To construct $\beta : B(0, p^{-N}) \times X \to \mathbf{G}_m$ it suffices, by Lemma 8.2(a), to construct a unit in the affinoid $\mathcal{O}(X)\langle T_1, T_2, \ldots, T_d\rangle = \mathcal{O}\left(B(0, p^{-N}) \times X\right)$ which specialises to $\chi(\alpha)$ via the map sending T_i to α_i/p^N, for all $\alpha \in p^N\mathcal{O}$. The unit $\exp\left(\sum_i p^N T_i f_i\right)$ is easily seen to do the trick.

For uniqueness it suffices (again via exp and log) to prove that a map of rigid spaces $f : \mathbf{B}_1 \to \mathbf{A}^1$ which sends every element of $\mathcal{O}$ to 0 must be identically 0, that is, that $\mathcal{O}$ is Zariski-dense in $\mathbf{B}_1$. It suffices to show that f vanishes on a small polydisc centre 0, and one can check this on points. Again choose a $\mathbf{Z}_p$-basis $(e_1, e_2, \ldots, e_d)$ of $\mathcal{O}$ as a $\mathbf{Z}_p$-module. It suffices to prove that for all complete extensions L of K, f is zero on all L-points of $\mathbf{B}_1$ of the form $z_1e_1 + z_2e_2 + \ldots + z_de_d$ with $z_i \in \mathcal{O}_L$, as this contains all the L-points of a small polydisc in $\mathbf{B}_1$ by a determinant calculation. Note that if all the z_β are in $\mathbf{Z}_p$ then certainly $f(z_1e_1 + z_2e_2 + \ldots) = 0$. Now fix $z_\beta \in \mathbf{Z}_p$ for $\beta \geq 2$ and consider the function on the affinoid unit disc over L sending z_1 to $f(z_1e_1 + z_2e_2 + \ldots)$. This is a function on a closed 1-ball that vanishes at infinitely many points, and hence it is identically zero. Now fix $z_1 \in \mathcal{O}_L$ and $z_\beta \in \mathbf{Z}_p$ for $\beta \geq 3$, and let z_2 vary, and so on, to deduce that f is identically 0. □

For applications, we want to consider products of the $\mathbf{B}_r$ and $\mathbf{B}_r^\times$ constructed above. Let F denote a number field, with integers $\mathcal{O}_F$, and let p be a prime. Let $\mathcal{O}_p$ denote $\mathcal{O}_F \otimes \mathbf{Z}_p$, the product of the integer rings in the completions of F at all the primes above p, and let K_0 be the closure in $\overline{\mathbf{Q}}_p$ of the compositum of the images of all the field homomorphisms $F \to \overline{\mathbf{Q}}_p$. Then K_0 is a finite Galois extension of $\mathbf{Q}_p$. Then K_0 contains the image of any field homomorphism $F_v \to \overline{\mathbf{Q}}_p$, where v is any place of F above p, so we are in a position to apply the previous constructions with $M_0 = \mathbf{Q}_p$, $M = F_v$, and K any complete extension of K_0.

Let J denote the set of places of F above p, and let I denote the set of field homomorphisms $F \to K$. Note that each $i \in I$ extends naturally to a map $i : F_p \to K$ where $F_p = F \otimes \mathbf{Q}_p = \oplus_{j \in J} F_j$. For $j \in J$, let I_j denote the subset of I consisting of $i : F_p \to K$ which factor through the completion $F \to F_j$. Then I is the disjoint union of the I_j. For $r \in (\mathcal{N}_K)^J$ and $j \in J$ write r_j for the component of r at j. Let $\mathbf{B}_r$ (resp. $\mathbf{B}_r^\times$ if $r_j < 1$ for all j) denote the rigid space over K which is the product over $j \in J$ of the rigid spaces $\mathbf{B}_{r_j}$ (resp. $\mathbf{B}_{r_j}^\times$) defined above. Then $\mathbf{B}_r$ (resp. $\mathbf{B}_r^\times$) is a thickening of $\mathcal{O}_p$ (resp. $\mathcal{O}_p^\times$) in the unit g-polydisc over K, where now $g = [F : \mathbf{Q}]$. Indeed, it is easily checked that for all complete extensions L of K we have

$$\mathbf{B}_r(L) = \{z \in L^I \ : \text{there is } \alpha \in \mathcal{O}_p \text{ with } |z_i - \alpha_i| \leq r_i\}$$

and, when $r_i < 1$ for all i,

$$\mathbf{B}_r^\times(L) = \{z \in L^I \ : \text{there is } \alpha \in \mathcal{O}_p^\times \text{ with } |z_i - \alpha_i| \leq r_i\}$$

just as before, where, for $\alpha \in \mathcal{O}_p$, α_i denotes $i(\alpha) \in K$.

Now assume that F is totally real. Let G denote a subgroup of $\mathcal{O}_F^\times$ of finite index, and let Γ_G be the quotient of $\mathcal{O}_p^\times \times \mathcal{O}_p^\times$ by the closure of the image of G via the map $\gamma \mapsto (\gamma, \gamma^2)$. Then Γ_G is topologically isomorphic to $\left(\mathcal{O}_p^\times/\overline{G}\right) \times \mathcal{O}_p^\times$, so its dimension is related to the defect of Leopoldt's conjecture for the pair (F, p) (in particular, the dimension is at least $g+1$ and conjecturally equal to $g+1$). Let $\mathcal{X}_{\Gamma_G}$ be the rigid space associated to Γ_G in Lemma 8.2(b), and let $\mathcal{W}$ to be the direct limit $\lim \mathcal{X}_{\Gamma_G}$ as G varies over the set of subgroups of finite index of $\mathcal{O}_F^\times$, partially ordered by inclusion. The fact that $\mathcal{W}$ exists is an easy consequence of Lemma 2(iii) of [5], which shows that the transition morphisms are closed and open immersions: if $G_1 \subseteq G_2 \subseteq \mathcal{O}_F^\times$ are subgroups of $\mathcal{O}_F^\times$ of finite index and $\Gamma_i = \Gamma_{G_i}$, then there is a surjection $\Gamma_1 \to \Gamma_2$ with finite kernel, and the corresponding map $\mathcal{X}_{\Gamma_2} \to \mathcal{X}_{\Gamma_1}$ is a closed immersion which geometrically identifies $\mathcal{X}_{\Gamma_2}$ with a union of components of $\mathcal{X}_{\Gamma_1}$. In particular we see that the $\mathcal{X}_{\Gamma_G}$, as G varies through the subgroups of $\mathcal{O}_F^\times$ of finite index, form an admissible cover of $\mathcal{W}$. A K-point of $\mathcal{W}$ corresponds to a continuous group homomorphism $\mathcal{O}_p^\times \times \mathcal{O}_p^\times \to K^\times$ whose kernel contains a subgroup of $\mathcal{O}_F^\times$ of finite index (we always regard $\mathcal{O}_F^\times$ as being embedded in $\mathcal{O}_p^\times \times \mathcal{O}_p^\times$ via the map $\gamma \mapsto (\gamma, \gamma^2)$). More generally, we define a *weight* to be a continuous group homomorphism $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$, for X any affinoid, such that the kernel of κ contains a subgroup of $\mathcal{O}_F^\times$ of finite index.

If U is an affinoid K-space and $U \to \mathcal{W}$ is a map of rigid spaces, then (because the $\mathcal{X}_{\Gamma_G}$ cover $\mathcal{W}$ admissibly) there is a subgroup G of finite index of $\mathcal{O}_F^\times$ such that the image of U is contained within $\mathcal{X}_{\Gamma_G}$. In particular, by the universal property of $\mathcal{X}_{\Gamma_G}$ there is an induced continuous group homomorphism $\Gamma_G \to \mathcal{O}(U)^\times$, which induces a continuous group homomorphism $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(U)^\times$. By composing this map with the map $\mathcal{O}_p^\times \to \mathcal{O}_p^\times \times \mathcal{O}_p^\times$ sending γ to $(\gamma, 1)$, we get a continuous group homomorphism $n : \mathcal{O}_p^\times \to \mathcal{O}(U)^\times$, which can be written as a product over $j \in J$ of continuous group homomorphisms $n_j : \mathcal{O}_{F_j}^\times \to \mathcal{O}(U)^\times$. Hence by Proposition 8.3 there exists $r \in \left(\mathcal{N}_K^\times\right)^J$ and a map $\mathbf{B}_r^\times \times U \to \mathbf{G}_m$ giving rise to n. We call such a map a *thickening* of n. Because we have only set up our Fredholm theory on Banach modules, we will have to somehow single out one such thickening, which we do (rather arbitrarily) in the definition below. First we single out a discrete subset $\mathcal{N}_d^\times$ of $\left(\mathcal{N}_K^\times\right)^J$ as follows: let π_j denote a uniformiser of F_j and define $\mathcal{N}_d^\times \subset \left(\mathcal{N}_K^\times\right)^J$ to be the product over $j \in J$ of the sets $\{|\pi_j^t| : t \in \mathbf{Z}_{>0}\}$. We equip $\mathcal{N}_d^\times$ with the obvious partial ordering.

Definition. *Let X be an affinoid and let $\kappa = (n, v) : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ be a weight. We define $r(\kappa)$ to be the largest element of $\mathcal{N}_d^\times$ such that the construction above works. Explicitly, we choose $r(\kappa)_j = |\pi_j^{t_j}|$ with $t_j \in \mathbf{Z}_{>0}$, and the t_j are chosen as small as possible such that the maps $n_j : \mathcal{O}_{F_j}^\times \to \mathcal{O}(X)^\times$ are induced by maps $\mathbf{B}_{r(\kappa)_j}^\times \times X \to \mathbf{G}_m$ and hence the map $n : \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ is induced by a map $\mathbf{B}_{r(\kappa)}^\times \times X \to \mathbf{G}_m$.*

This construction applies in particular when X is an affinoid subdomain of $\mathcal{W}$ (the inclusion $X \to \mathcal{W}$ induces a map κ as above). Note however that as the image of X in $\mathcal{W}$ gets larger, the $r(\kappa)_j$ will get smaller—there is in general no universal r and map $\mathbf{B}_r^\times \times \mathcal{W} \to \mathbf{G}_m$. Note also that the construction applies if X is a point, and in this case κ corresponds to a point of $\mathcal{W}$.

9 Classical automorphic forms.

Our exposition of the classical theory follows [12] for the most part. We recall the notation of the latter part of the previous section, and add a little more. Recall that F is a totally real field of degree g over $\mathbf{Q}$, and $\mathcal{O}_F$ is the integers of F. We fix an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_p$; then we can think of I as the set of all infinite places of F, or as all the field embeddings $F \to \overline{\mathbf{Q}}_p$. Note that any such map i extends to a map $i : F_p := F \otimes \mathbf{Q}_p \to \overline{\mathbf{Q}}_p$. Recall that K is a complete extension of K_0, the compositum of the images $i(F)$ of F as i runs through I, and we may also think of I as the set of all field homomorphisms $F \to K$. We let J denote the set of primes of $\mathcal{O}_F$ dividing p. If $j \in J$ then let F_j denote the completion of F at j and let $\mathcal{O}_j$ denote the integers in F_j. We set $\mathcal{O}_p := \mathcal{O}_F \otimes \mathbf{Z}_p$; then $F_p = \oplus_{j \in J} F_j$ and $\mathcal{O}_p = \oplus_{j \in J} \mathcal{O}_j$. Choose once and for all uniformisers π_j of the local fields F_j for all j, and let $\pi \in F_p$ denote the element whose jth component is π_j. We will also use π to denote the ideal of $\mathcal{O}_F$ which is the product of the prime ideals above p. Note that some constructions (for example the Hecke operators U_{π_j} defined later) will depend to a certain extent on this choice, but others (for example the eigenvarieties we construct) will not.

Any $i \in I$ gives a map $F_p \to K$ and this map factors through the projection $F_p \to F_j$ for some $j := j(i) \in J$; hence we get a natural surjection $I \to J$. If S is any set then this surjection induces a natural injection $S^J \to S^I$, where as usual S^I denotes the set of maps $I \to S$. We continue to use the following very useful notation: if $(a_j) \in S^J$ and $i \in I$ then by a_i we mean a_j for $j = j(i)$.

Now let D be a quaternion algebra over F ramified at all infinite places. Let us assume that D is split at all places above p.[5] Let $\mathcal{O}_D$ denote a fixed maximal order of D, and fix an iso-

[5] One can almost certainly develop some of the theory as long as at least one place above p is split, although one might have to fix the weights at the ramified places.

morphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} = \mathrm{M}_2(\mathcal{O}_{F_v})$ for all finite places v of F where D splits (here F_v is the completion of F at v and $\mathcal{O}_{F_v}$ is the integers in this completion). In particular we fix an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_p = \mathrm{M}_2(\mathcal{O}_p)$, and this induces an isomorphism $D_p := D \otimes_F F_p = \mathrm{M}_2(F_p)$.

We recall the classical definitions of automorphic forms for D. If $n \in \mathbf{Z}^I_{\geq 0}$ then we define L_n to be the K-vector space with basis the monomials $\prod_{i\in I} Z_i^{m_i}$, where $m \in \mathbf{Z}^I_{\geq 0}$, $0 \leq m_i \leq n_i$, and where the Z_i are independent indeterminates. If $t \in \mathbf{Z}^J_{\geq 1}$ then define $\mathbf{M}_t$ to be the elements (γ_j) of $\mathrm{M}_2(\mathcal{O}_p) = \prod_{j\in J} \mathrm{M}_2(\mathcal{O}_j)$ with the property that if $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ then $\det(\gamma_j) \neq 0$, $\pi_j^{t_j}$ divides c_j, and π_j does not divide d_j. Then $\mathbf{M}_t$ is a monoid under multiplication. By $\mathbf{M}_1$ we mean the monoid $\mathbf{M}_t$ for $t = (1, 1, \ldots, 1)$. If $v \in \mathbf{Z}^I$ and $n \in \mathbf{Z}^I_{\geq 0}$ then define the right $\mathbf{M}_1$-module $L_{n,v}$ to be the K-vector space L_n equipped with the action of $\mathbf{M}_1$ defined by letting $(\gamma_j) = \left(\left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)\right)_{j\in J}$ send $\prod_i Z_i^{m_i}$ to $\prod_i (c_i Z_i + d_i)^{n_i} (a_i d_i - b_i c_i)^{v_i} \left(\frac{a_i Z_i + b_i}{c_i Z_i + d_i}\right)^{m_i}$ and extending K-linearly (note that here we are using the notation a_i for the image of $a_{j(i)}$ in K via the map i, as explained above). Note that in fact the same definition gives an action of $\mathrm{GL}_2(F_p)$ on $L_{n,v}$, but we never use this action.

The natural maps $\mathcal{O}_F \to \mathcal{O}_p \to \mathrm{M}_2(\mathcal{O}_p)$ (via the diagonal embedding) induce an embedding from $\mathcal{O}_F^\times$ into $\mathbf{M}_1$. An easy check shows that the totally positive units in $\mathcal{O}_F^\times$ act trivially on $L_{n,v}$ if $n + 2v \in \mathbf{Z}$.

Define $\mathbf{A}_{F,f}$ to be the finite adeles of F and $D_f := D \otimes_F \mathbf{A}_{F,f}$. If $x \in D_f$ then let $x_p \in D_p = \mathrm{M}_2(F_p)$ denote the projection onto the factor of D_f at p. If $t \in \mathbf{Z}^J_{\geq 1}$ then we say that a compact open subgroup $U \subset D_f^\times$ has *wild level* $\geq \pi^t$ if the projection $U \to D_p^\times$ is contained within $\mathbf{M}_t$. If $t = (1, 1, \ldots, 1)$ then we drop it from the notation and talk about compact open subgroups of wild level $\geq \pi$.

Say $D_f^\times = \coprod_{\lambda=1}^{\mu} D^\times \tau_\lambda U$. Then the groups $\Gamma_\lambda := \tau_\lambda^{-1} D^\times \tau_\lambda \cap U$

are finitely-generated and moreover $\tau_\lambda\Gamma_\lambda\tau_\lambda^{-1} \subset D^\times$ is commensurable with $\mathcal{O}_D^\times$ and hence with $\mathcal{O}_F^\times$. Hence Γ_λ is also commensurable with $\mathcal{O}_F^\times$. If $\mathfrak{n}$ is an ideal of $\mathcal{O}_F$ which is coprime to $\mathrm{disc}(D)$ then we define $U_0(\mathfrak{n})$ (resp. $U_1(\mathfrak{n})$) in the usual way as being matrices in $(\mathcal{O}_D \otimes \widehat{\mathbf{Z}})^\times$ which are congruent to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$) mod $\mathfrak{n}$. Note that for many such choices of U we see that the Γ_λ are all contained within $\mathcal{O}_F^\times$ (see, for example, Lemma 7.1 of [12] and the observation that, in Hida's notation, the groups $\overline{\Gamma}^i(U)$ are finite because D is totally definite). However we do not need to assume this because of our generalisation of Coleman's Fredholm theory.

Say $t \in \mathbf{Z}_{\geq 1}^J$, U is a compact open of wild level $\geq \pi^t$, and A is any right $\mathbf{M}_t$-module, with action written $(a, m) \mapsto a.m$. If $f : D_f^\times \to A$ and $u \in U$ then define $f|u : D_f^\times \to A$ by $(f|u)(g) := f(gu^{-1}).u_p$. Now set

$$\mathcal{L}(U, A) := \left\{ f : D^\times \backslash D_f^\times \to A \,:\, f|u = f \text{ for all } u \in U \right\}.$$

Note that $f \in \mathcal{L}(U, A)$ is determined by $f(\tau_\lambda)$ for $1 \leq \lambda \leq \mu$, and one checks easily that the map $f \mapsto (f(\tau_\lambda))_{1 \leq \lambda \leq \mu}$ induces an isomorphism

$$\mathcal{L}(U, A) \to \bigoplus_{\lambda=1}^{\mu} A^{\Gamma_\lambda}.$$

In particular, the functor $\mathcal{L}(U, -)$ is left exact. We remark that in the circumstances that will interest us later on, A will be an ONable Banach module over an affinoid in characteristic zero, the Γ_λ will all act via finite groups, and the invariants will hence be a Banach module with property (Pr). Indeed, this phenomenon was the main reason for extending Coleman's theory from ONable modules to modules with property (Pr).

If $\eta \in D_f^\times$ and $\eta_p \in \mathbf{M}_t$ then one can define an endomorphism $[U\eta U]$ of $\mathcal{L}(U, A)$ as follows: decompose $U\eta U = \coprod_i Ux_i$ (a finite union) and define

$$f|[U\eta U] := \sum_i f|x_i.$$

This operator is called the Hecke operator associated to η.

Now let $n \in \mathbf{Z}^I_{\geq 0}$ and $v \in \mathbf{Z}^I$ be such that $n + 2v \in \mathbf{Z}$. Set $k = n + 2$ and $w = v + n + 1$. Then $k - 2w \in \mathbf{Z}$ and $k \geq 2$ (that is, $k_i \geq 2$ for all i), and conversely given k and w with these properties one can of course recover n and v. We finish by recalling the definition of classical automorphic forms for D in this context. Let $U \subset D_f^\times$ be a compact open subgroup of wild level $\geq \pi$.

Definition. *The space of classical automorphic forms $S^D_{k,w}(U)$ of weight (k, w) and level U for D is the space $\mathcal{L}(U, L_{n,v})$.*

This space is a finite-dimensional K-vector space. It is not, in the strict sense, a classical space of forms, because we have twisted the weight action from infinity to p. On the other hand if one chooses a field homomorphism $K \to \mathbf{C}$ then $S^D_{k,w}(U) \otimes_K \mathbf{C}$ is isomorphic to a classical space of Hilbert modular forms, as described in, for example, section 2 of [12]. We remark also that because the full group $\mathrm{GL}_2(F_p)$ acts naturally on $L_{n,v}$, our assumption that U has wild level $\geq \pi$ is unnecessary at this point. However, the forms that we shall p-adically interpolate will always have wild level $\geq \pi$, because of the standard phenomenon that to p-adically analytically interpolate forms on GL_2 one has to drop an Euler factor.

10 Overconvergent automorphic forms.

Let X be an affinoid over K, and let $\kappa = (n, v) : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ be a weight. In this section we will define $\mathcal{O}(X)$-modules of r-overconvergent automorphic forms of weight κ. In this generality, κ really is a family of weights; one important case to keep in mind is when X is a point, so n and v are continuous group homomorphisms $\mathcal{O}_p^\times \to K^\times$, the resulting spaces will then be Banach spaces over K and will be automorphic forms of a fixed weight. One important special case of this latter situation is when n and v are of the form $\alpha \mapsto \prod_i \alpha_i^{m_i}$ where the m_i are integers; the resulting spaces of automorphic forms will have a "classical weight" and there will be a natural finite-dimensional subspace corresponding to a space of classical automorphic forms as defined in the previous section.

The group homomorphism $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ induces a map $f : X \to \mathcal{W}$ by Lemma 8.2(b) and (c). We extend the map $v : \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ to a group homomorphism $v : F_p^\times \to \mathcal{O}(X)^\times$ by defining $v(\pi_j) = 1$ for all $j \in J$. Note that this extension depends on our choice of π_j (it is analogous to Hida's choices of $\{x^v\}$ in [12]) but subsequent definitions will not depend seriously on this choice (in particular the eigenvariety we construct will not depend on this choice). Note also that the supremum seminorm of every element in the image of n or v is 1.

Now for $r \in (\mathcal{N}_K)^J$ define $\mathcal{A}_{\kappa,r}$ to be the K-Banach algebra $\mathcal{O}(\mathbf{B}_r \times X)$. Note that $\mathcal{A}_{\kappa,r}$ does not yet depend on κ but we will define a monoid action below which does. Let us assume for simplicity that X is reduced (this is not really necessary, but will be true in practice and also gives us a canonical choice of norm on $\mathcal{O}(X)$, namely the supremum norm). Endow $\mathcal{A}_{\kappa,r}$ with the supremum norm. As usual write $\kappa = (n, v)$ and $n = \prod_{j \in J} n_j$ with $n_j : \mathcal{O}_{F_j}^\times \to \mathcal{O}(X)^\times$.

Definition. *We say that* $t = (t_j)_{j \in J} \in \mathbf{Z}_{>0}^J$ *is* good *for the pair* (κ, r) *if for each* $j \in J$ *there is thickening of* n_j *to a map* $\mathbf{B}^\times_{r_j |\pi_j^{t_j}|} \times X \to \mathbf{G}_m$.

Equivalently, $t \in \mathbf{Z}_{>0}^J$ is good if $r|\pi^t| <= r(\kappa)$ in $(\mathcal{N}_K)^J$ with the obvious partial order. Given any (κ, r) as above, there will exist good $t \in \mathbf{Z}_{>0}^J$ by Proposition 8.3 (indeed, there will exist a unique minimal good t). The point of the definition is that if t is good for (κ, r) then we can define a right action of $\mathbf{M}_t$ on $\mathcal{A}_{\kappa,r}$ (we denote the action by a dot) by letting $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}_t$ act as follows: if $h \in \mathcal{A}_{\kappa,r}$ and $(z, x) \in \mathbf{B}_r(L) \times X(L)$ for L any complete extension of K then

$$(h.\gamma)(z,x) := \\ n(cz+d,x)\left(v\left(\det(\gamma)\right)(x)\right)h\left((az+b)/(cz+d),x\right).$$

This is really a definition "on points" but it is easily checked that $h.\gamma \in \mathcal{A}_{\kappa,r}$, using Lemma 8.1, and that the definition does give an action. It is elementary to check that for fixed $\gamma \in \mathbf{M}_t$, the map $\mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,r}$ defined by $h \mapsto h.\gamma$ is a continuous $\mathcal{O}(X)$-module homomorphism (but it is not in general a ring homomorphism if κ is non-trivial). The fact that n and v take values in elements of $\mathcal{O}(X)^\times$ with supremum norm 1 easily implies that $\gamma : \mathcal{A}_{\kappa,r} \to \mathcal{A}_{\kappa,r}$ is norm-decreasing. One also checks using Lemma 8.1(b) that if $|\det(\gamma_j)| = m_j$ then γ induces a continuous norm-decreasing $\mathcal{O}(X)$-module homomorphism $\mathcal{A}_{\kappa,rm}$ to $\mathcal{A}_{\kappa,r}$.

We now have enough to define our Banach modules of overconvergent modular forms. This definition is ultimately inspired by [16], a preprint which sadly may well never see the light of day but which contained the crucial idea of beefing up a polynomial ring to a restricted power series ring in order to move from the classical to the overconvergent setting.

Definition. *Let X be a reduced affinoid over K and let $X \to \mathcal{W}$ be a morphism of rigid spaces, inducing $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$. If $r \in (\mathcal{N}_K)^J$, if t is good for (κ, r), and if U is a compact open subgroup of $D_f^\times$ of wild level $\geq \pi^t$, then define the space of r-overconvergent automorphic forms of weight κ and level U to be the $\mathcal{O}(X)$-module*

$$\mathbf{S}_\kappa^D(U;r) := \mathcal{L}(U, \mathcal{A}_{\kappa,r}).$$

We remark that, just as in the case of "classical" overconvergent modular forms, the hypotheses of the definition imply that if κ is a weight near the boundary of weight space (that is, such that $r(\kappa)$ is small), then $r|\pi^t|$ must be small and hence for each j either r_j is small or there must be some large power of π_j in the level.

If $f \in \mathbf{S}_\kappa^D(U;r)$ then f is determined by $f(\tau_\lambda)$ for $\lambda = 1,\ldots,\mu$. Moreover, if $u \in U$ then so is u^{-1}, and hence both u_p and u_p^{-1} are in $\mathbf{M}_t$. In particular, both u_p and its inverse are norm-decreasing, and hence u_p is norm-preserving. We deduce that for $d \in D^\times$, $\tau \in D_f^\times$ and $u \in U$ we have $|f(d\tau u)| = |f(\tau).u_p| = |f(\tau)|$ and hence $|f(g)| \leq \max_\lambda f(\tau_\lambda)$ for all $g \in D_f^\times$. In particular we can define a norm on $\mathbf{S}_\kappa^D(U;r)$ by $|f| = \max_{g\in D_f^\times}|f(g)|$, and the isomorphism

$$\mathbf{S}_\kappa^D(U;r) \to \bigoplus_{\lambda=1}^{\mu}(\mathcal{A}_{\kappa,r})^{\Gamma_\lambda}$$

defined by $f \mapsto (f(t_\lambda))_\lambda$ is norm-preserving. Next observe that the group Γ_λ contains, with finite index, a subgroup of $\mathcal{O}_F^\times$ of finite index, and hence Γ_λ acts on $\mathcal{A}_{\kappa,r}$ via a finite quotient. Hence $\mathbf{S}_\kappa^D(U;r)$ is a direct summand of an ONable Banach $\mathcal{O}(X)$-module and our Fredholm theory applies.

11 Classical forms are overconvergent.

Fix $n \in \mathbf{Z}_{\geq 0}^I$ and $v \in \mathbf{Z}^I$ such that $n + 2v \in \mathbf{Z}$. Set $k = n+2$ and $w = v+n+1$ as usual. Define $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to K^\times$ by $\kappa(\alpha,\beta) = \prod_i \alpha_i^{n_i}\beta_i^{v_i}$. Note that κ is trivial on the totally positive units in $\mathcal{O}_F^\times$ (embedded via $\gamma \mapsto (\gamma,\gamma^2)$ as usual) and hence κ is a K-point of $\mathcal{W}$. With notation as above, we are taking $L = K$ and X a point. The map $n : \mathcal{O}_p^\times \to K^\times$ defined by $\alpha \mapsto \prod_i \alpha_i^{n_i}$ extends to a map of rigid spaces $\mathbf{B}_r^\times \to \mathbf{G}_m$ for any $r \in \left(\mathcal{N}_K^\times\right)^J$, so $r(\kappa)_j = |\pi_j|$ for all $j \in J$. If $r \in (\mathcal{N}_K)^J$ then there is a natural injection $L_{n,v} \to \mathcal{A}_{\kappa,r} = \mathcal{O}(\mathbf{B}_r)$ induced from the natural inclusion $\mathbf{B}_r \subset (\mathbf{A}^1)^I$ and one checks easily that this is an $\mathbf{M}_1$-equivariant inclusion. If $U \subset D_f^\times$ is a compact open subgroup of level $\geq \pi$ then we get an inclusion

$$S_{k,w}^D(U) = \mathcal{L}(U, L_{n,v}) \subseteq \mathcal{L}(U,\mathcal{A}_{\kappa,r}) = \mathbf{S}_\kappa^D(U;r)$$

between the finite-dimensional space of classical forms and the typically infinite-dimensional space of overconvergent ones.

This relationship between classical and overconvergent modular forms is however not quite the one that we want in general. When we construct our eigenvarieties in this setting, we will want the level structure at p to be $U_0(\pi)$, and hence we need to explain how to interpret finite slope classical forms of conductor π^2 and above, or forms with non-trivial character at p, as forms of level π and some appropriate weight. Briefly, the trick is that we firstly load the character at p into the weight of the overconvergent form, thus reducing us to level $U_0(\pi^n)$, and then decrease r and decrease the level to $U_0(\pi)$. Note that a variant of this trick is used to construct the classical eigencurve—although there the level structure is reduced only to $\Gamma_1(p)$ (or $\Gamma_1(4)$ if $p = 2$) because the p-adic zeta function may have zeroes on points of weight space which are not contained in the identity component.

Let us explain these steps in more detail. Let U_0 be a compact open subgroup of $D_f^\times$ of the form $U' \times \mathrm{GL}_2(\mathcal{O}_p)$, choose $t \in \mathbf{Z}_{\geq 1}^J$, and let U_1 denote the group $U_0 \cap U_1(\pi^t)$. Then U_1 is a normal subgroup of $U_0 \cap U_0(\pi^t)$; let Δ denote the quotient group. The map $\mathcal{O}_p^\times \to \mathrm{GL}_2(\mathcal{O}_p) \subset U_0$ sending d to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$ identifies Δ with the quotient $(\mathcal{O}_p/\pi^t)^\times$. If L is a complete extension of K, if $n \in \mathbf{Z}_{\geq 0}^I$ and $v \in \mathbf{Z}^I$ are chosen such that $n+2v \in \mathbf{Z}$, and if $k = n+2$ and $w = v + n + 1$ as usual, then for $f \in S_{k,w}^D(U_1) = \mathcal{L}(U_1, L_{n,v})$ and $u \in U_0 \cap U_0(\pi^t)$ we have $f|u^{-1} \in S_{k,w}^D(uU_1u^{-1}) = S_{k,w}^D(U_1)$ and hence there is a left action of $U_0 \cap U_0(\pi^t)$ on the finite-dimensional L-vector space $S_{k,w}^D(U_1)$ defined by letting u act by $f \mapsto f|u^{-1}$. This action is easily seen to factor through Δ, and is just the Diamond operators at primes above p in this setting. If L contains enough roots of unity then $S_{k,w}^D(U_1)$ is a direct sum of eigenspaces for this action. Choose a character $\varepsilon : \Delta \to L^\times$ and let ε also denote the induced character of $\mathcal{O}_p^\times$. Now define $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to L^\times$ by $\kappa(\alpha, \beta) = \varepsilon(\alpha) \prod_i \alpha_i^{n_i} \beta_i^{v_i}$. The fact that $n + 2v \in \mathbf{Z}$ means that κ vanishes on a subgroup of $\mathcal{O}_F^\times$ of finite index, and hence κ is a weight. One checks that $|\pi^t| \leq r(\kappa)$ and hence that if $r \in (\mathcal{N}_K)^J$ then t is good for (r, κ), so the spaces $\mathcal{L}(U_0, \mathcal{A}_{\kappa,r})$ and $\mathcal{L}(U_1, \mathcal{A}_{\kappa,r})$ are well-defined. Moreover, the natural map $L_{n,v} \to \mathcal{A}_{\kappa,r}$ is equivariant for the action of the submonoid of $\mathbf{M}_t$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\pi_j^{t_j}|(d_j - 1)$. Hence if $\mathcal{L}(U_1, L_{n,v})(\varepsilon)$ denotes the ε-eigenspace of $\mathcal{L}(U_1, L_{n,v})$ under the action of Δ, then we get an induced map $\mathcal{L}(U_1, L_{n,v})(\varepsilon) \to \mathcal{L}(U_1, \mathcal{A}_{\kappa,r})$ and unravelling the definitions one checks easily that the image of $\mathcal{L}(U_1, L_{n,v})(\varepsilon)$ is in fact contained in $\mathcal{L}(U_0, \mathcal{A}_{\kappa,r})$ (the point being that the map $L_{n,v} \to \mathcal{A}_{\kappa,r}$ is not $U_0(\pi^t)$-equivariant, and the two actions differ by ε). This construction embeds classical forms with non-trivial character at primes above p into overconvergent forms with $U_0(\pi^t)$ level structure at p, and should be thought of as the replacement in this setting of the construction of moving from a classical form of level p^n and character ε to an overconvergent function of level p^n and trivial character, by dividing by an appropriate Eisenstein series with character ϵ. Note the phenomenon, also present in the classical case, that forms in distinct eigenspaces of $S_{k,w}^D(U_1)$ for the Diamond operators above p actually become overconvergent eigenforms of distinct weights in this setting.

We now explain the relationship between forms of level $U_0(\pi)$ and forms of level $U_0(\pi^r)$ for any $r \geq 1$. Let $X \to \mathcal{W}$ be a map from a reduced affinoid to weight space, and let $\kappa = (n, v) : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)$ be the induced weight. Let U be a compact open subgroup of $D_f^\times$ of the form $U' \times \mathrm{GL}_2(\mathcal{O}_p)$. Say $r \in (\mathcal{N}_K)^J$, $s \in \mathbf{Z}_{\geq 0}^J$ and $t \in \mathbf{Z}_{\geq 1}^J$ are chosen such that there is a thickening of n to $\mathbf{B}_{r|\pi^{s+t}|}^\times \times X \to \mathbf{G}_m$. Then we have defined spaces of $r|\pi^s|$-overconvergent weight κ automorphic forms of level $U \cap U_0(\pi^t)$ and also r-overconvergent weight κ automorphic forms of level $U \cap U_0(\pi^{t+s})$. We now show that these spaces are canonically isomorphic.

Proposition 11.1. *There is a canonical isomorphism*

$$\mathcal{L}(U \cap U_0(\pi^t), \mathcal{A}_{\kappa,r|\pi^s|}) \cong \mathcal{L}(U \cap U_0(\pi^{t+s}), \mathcal{A}_{\kappa,r}).$$

Remark. We will see later that this isomorphism preserves the action of various Hecke operators when $U = U_1(\mathfrak{n})$ or $U_0(\mathfrak{n})$.

Proof. The proof is an analogue of [5], Lemma 4, part 4, in this setting. We explain the construction of maps in both directions; it is then easy to check that these maps are well-defined and inverse to one another. As usual let π^s denote the element of $\mathcal{O}_p$ whose component at $j \in J$ is $\pi_j^{s_j}$. Then $\left(\begin{smallmatrix} \pi^s & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an element of $\mathrm{GL}_2(\mathcal{O}_p)$ and hence we can think of it as an element of $D_f^\times$.

If $f \in \mathcal{L}(U \cap U_0(\pi^t), \mathcal{A}_{\kappa,r|\pi^s|})$ then define $h : D_f^\times \to \mathcal{A}_{\kappa,r}$ by $h(g) = f\left(g\left(\begin{smallmatrix} \pi^{-s} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right).\left(\begin{smallmatrix} \pi^s & 0 \\ 0 & 1 \end{smallmatrix}\right)$; note that if $\phi \in \mathcal{O}(\mathbf{B}_{r|\pi^s|} \times X)$ then $\phi.\left(\begin{smallmatrix} \pi^s & 0 \\ 0 & 1 \end{smallmatrix}\right)$ can be thought of as an element of $\mathcal{O}(\mathbf{B}_r \times X)$ as if $z \in \mathbf{B}_r(L)$ then $\pi^s z \in \mathbf{B}_{r|\pi^s|}(L)$. One checks that $h \in \mathcal{L}(U \cap U_0(\pi^{t+s}), \mathcal{A}_{\kappa,r})$.

Slightly harder work is the map the other way. First note that $\mathbf{B}_{r|\pi^s|}$ is the disjoint union of $\pi^s\mathbf{B}_r + \alpha$ as $\alpha \in \mathcal{O}_p$ runs through a set of coset representatives S for $\mathcal{O}_p/\pi^s$; hence $\mathcal{A}_{\kappa,r} = \bigoplus_{\alpha \in S} \mathcal{O}((\pi^s\mathbf{B}_r + \alpha) \times X)$. Now if $h \in \mathcal{L}(U \cap U_0(\pi^{t+s}), \mathcal{A}_{\kappa,r})$ then define $f : D_f^\times \to \mathcal{A}_{\kappa,r|\pi^s|}$ as follows: for $g \in D_f^\times$ we define $f(g)$ on $(\pi^s\mathbf{B}_r + \alpha) \times X$ by $f(g)(\pi^s z + \alpha, x) = h\left(g\left(\begin{smallmatrix} \pi^s & \alpha \\ 0 & 1 \end{smallmatrix}\right)\right)(z, x)$. One checks easily that this is well-defined (that is, independent of choice of coset representatives S). A little trickier is that $f \in \mathcal{L}(U \cap U_0(\pi^t), \mathcal{A}_{\kappa,r|\pi^s|})$, the hard part being to check that $f|u = f$ for $u \in U \cap U_0(\pi^t)$. We give a sketch of the idea, which is just algebra. Firstly one checks easily that $f|u = f$ if $u = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)$ with $\gamma \in \mathcal{O}_p$. Now say $u \in U \cap U_0(\pi^t)$, and choose $\alpha \in \mathcal{O}_p$. We must check that $(f(g))(\pi^s z + \alpha, x) = ((f|u)(g))(\pi^s z + \alpha, x)$ for all $x \in X$ and $z \in \mathbf{B}_r$ (again we present the argument on points but of course this suffices). The trick is knowing how to unravel the right hand side. Because $u' := \left(\begin{smallmatrix} 1 & -\alpha \\ 0 & 1 \end{smallmatrix}\right) u \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \in U \cap U_0(\pi^t)$, there exists $\beta \in \mathcal{O}_p$ such that $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) u' = v$ with $v_p = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $\pi^s | b$. In particular $\left(\begin{smallmatrix} \pi^{-s} & 0 \\ 0 & 1 \end{smallmatrix}\right) v \left(\begin{smallmatrix} \pi^s & 0 \\ 0 & 1 \end{smallmatrix}\right) = v' \in U \cap U_0(\pi^{t+s})$. Hence

$$\begin{aligned} ((f|u)(g))(\pi^s z + \alpha, x) &= \left(f|v\left(\begin{smallmatrix} 1 & -\alpha \\ 0 & 1 \end{smallmatrix}\right)\right)(g)(\pi^s z + \alpha, x) \\ &= (f|v)\left(g\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\right)(\pi^s z, x) \end{aligned}$$

and by expanding out the definition of $f|v$ and then using the definition of f in terms of h, one checks readily that this equals $(h|v')\left(g\left(\begin{smallmatrix}\pi^s & \alpha\\ 0 & 1\end{smallmatrix}\right)\right)(z,x)$. We are now home, as $h|v' = h$.

Finally one checks easily that the above associations $f \mapsto h$ and $h \mapsto f$ are inverse to one another. $\square$

No doubt one can now mimic the constructions of section 7 of [5] to deduce the existence of various canonical maps between spaces of overconvergent forms, and relate the kernels of these maps to spaces of classical forms; these maps, analogous to Coleman's θ^{k-1} operator, will not be considered here for reasons of space, as they are not necessary for the construction of eigenvarieties. The reader interested in these things might like, as an exercise, to verify that overconvergent forms of small slope are classical in this setting, following section 7 of [5].

12 Hecke operators.

Let X be a reduced affinoid over K and let $\kappa = (n, v) : X \to \mathcal{W}$ be a morphism of rigid spaces. If $r \in (\mathcal{N}_K)^J$, if $\rho = |\pi^t| \in \mathcal{N}_d^\times$ is such that n has a thickening to $\mathbf{B}_{r\rho}$, and if U is a compact open subgroup of $D_f^\times$ of wild level $\geq \pi^t$, then we have defined the r-overconvergent automorphic forms of weight κ and level U. If v is a finite place of F where D splits then we define $\eta_v \in D_f^\times$ to be the element which is the identity at all places away from v, and the matrix $\left(\begin{smallmatrix}\pi_v & 0\\ 0 & 1\end{smallmatrix}\right)$ at v, where $\pi_v \in F_v$ is a uniformiser. If v is prime to p then will not matter which uniformiser we choose, but if $v|p$ then for simplicity we use the uniformiser which we have already chosen earlier (this is really just for notational convenience though—a different choice would only change the operators we define by units). Let us assume in this section that U is a compact open of the form $U_0(\mathfrak{n}) \cap U_1(\mathfrak{r})$ for some integral ideals $\mathfrak{n}$ and $\mathfrak{r}$ of $\mathcal{O}_F$, both prime to $\mathrm{disc}(D)$, with $\pi|\mathfrak{n}$ and π coprime to $\mathfrak{r}$. In this case, the resulting Hecke operator $T_v = [U\eta_v U]$, acting on $\mathcal{L}(U, \mathcal{A}_{\kappa,r})$, is easily checked to be independent of the choice of π_v, as long as v is prime to p. If furthermore v is prime to $\mathfrak{nr}$ then we may regard π_v as an element of the centre of $D_f^\times$ and we define S_v to be the resulting Hecke operator $[U\pi_v U]$.

A standard argument shows that the endomorphisms T_v and S_v all commute with one another. Furthermore, we have

Lemma 12.1. *The isomorphism of Proposition 11.1 is Hecke equivariant.*

Proof. For the Hecke operators away from p this is essentially immediate. At primes above p things are slightly more delicate, because for $t_j \geq 1$ the natural left coset decomposition of the double coset $U_0(\pi_j^{t_j})\left(\begin{smallmatrix}\pi_j & 0\\ 0 & 1\end{smallmatrix}\right)U_0(\pi_j^{t_j})$ is $\coprod_{\alpha\in\mathcal{O}_j/\pi_j} U_0(\pi_j^{t_j})\left(\begin{smallmatrix}\pi_j & 0\\ \alpha\pi_j^{t_j} & 1\end{smallmatrix}\right)$ which depends on t_j. However, one checks easily that if (in the notation of Proposition 11.1) $f \in \mathcal{L}(U \cap U_0(\pi^t), \mathcal{A}_{r|\pi^s|})$ and h is the element of $\mathcal{L}(U \cap U_0(\pi^{t+s}), \mathcal{A}_r)$ associated to f in the proof, then $T_v h$ is indeed associated to $T_v f$, for all $v|p$, the calculation boiling down to the fact that

$$\begin{pmatrix}\pi_j & 0\\ \alpha\pi_j^{t_j} & 1\end{pmatrix}\begin{pmatrix}\pi^s & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\pi^s & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\pi_j & 0\\ \alpha\pi_j^{t_j+s_j} & 1\end{pmatrix}.$$

□

If p factors in F as $\prod_j \mathfrak{p}_j^{e_j}$ then let U_j denote the Hecke operator $T_{\mathfrak{p}_j}$, let U_π denote $\prod_{j\in J} U_j$, and let η_j denote the matrix $\eta_{\mathfrak{p}_j}$.

Lemma 12.2. *The map $U_\pi : \mathbf{S}_\kappa^D(U;r) \to \mathbf{S}_\kappa^D(U;r)$ is the composite of the natural inclusion $\mathbf{S}_\kappa^D(U;r) \to \mathbf{S}_\kappa^D(U;r|\pi|)$ and a continuous norm-decreasing map $\mathbf{S}_\kappa^D(U;r|\pi|) \to \mathbf{S}_\kappa^D(U;r)$. The inclusion $\mathbf{S}_\kappa^D(U;r) \to \mathbf{S}_\kappa^D(U;r|\pi|)$ is norm-decreasing and compact, and hence U_π, considered as an endomorphism of $\mathbf{S}_\kappa^D(U;r)$, is also norm-decreasing and compact.*

Proof. One checks easily that U_π is the Hecke operator $[U\eta U]$ associated to the matrix $\eta := \prod_j \eta_j$. If one decomposes $U\eta U$ into a finite disjoint union $\coprod_\delta Ux_\delta$ of cosets, then $\det((x_\delta)_p)/\det(\eta_p)$ is a unit at all places of F above p, and hence by Lemma 8.1(b) the endomorphism of $\mathcal{A}_{\kappa,r}$ induced by $(x_\delta)_p$ can be factored as the inclusion $\mathcal{A}_{\kappa,r} \subset \mathcal{A}_{\kappa,r|\pi|}$ followed by a norm-decreasing map $\mathcal{A}_{\kappa,r|\pi|} \to \mathcal{A}_{\kappa,r}$. The inclusion $\mathcal{A}_{\kappa,r} \subset \mathcal{A}_{\kappa,r|\pi|}$ is induced by the inner inclusion of affinoids $\mathbf{B}_{r|\pi|} \to \mathbf{B}_r$ and is hence compact and norm-decreasing; the result now follows easily. □

13 The characteristic power series of U_π.

We now have enough data to define the ingredients for our eigenvariety machine in this case. Let $\mathfrak{n}$ be an integral ideal of $\mathcal{O}_F$ prime to p and to $\mathrm{disc}(D)$ (this latter hypothesis is not really necessary, but we enforce it for simplicity's sake), and set $U_0 = U_0(\mathfrak{n})$ or $U_1(\mathfrak{n})$. Define $U = U_0 \cap U_0(\pi)$; then U has wild level $\geq \pi$. If $X \subset \mathcal{W}$ is an affinoid subdomain, then set $R_X = \mathcal{O}(X)$, let $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(X)^\times$ denote the corresponding weights, and let $r = r(\kappa)$. Let $\mathbf{T}$ be the set of Hecke operators T_v (for v running through all the finite places of F where D splits) and S_v (for v running through all the finite places of F prime to $\mathfrak{n}p$ where D splits) defined above, and let ϕ denote the operator U_π. Define $M_X = \mathbf{S}_\kappa^D(U;r) = \mathcal{L}(U, \mathcal{A}_{\kappa,r})$. If $Y \subseteq X$ is an affinoid subdomain and κ' is the weight corresponding to Y then $r(\kappa) \leq r(\kappa')$ and hence there is an inclusion $\mathbf{B}_{r(\kappa)} \subseteq \mathbf{B}_{r(\kappa')}$. There is a canonical isomorphism $M_X \widehat{\otimes}_{R_X} R_Y = \mathcal{L}(U, \mathcal{A}_{\kappa',r(\kappa)})$, and the inclusion $\mathbf{B}_{r(\kappa)} \to \mathbf{B}_{r(\kappa')}$ induces an injection $\mathcal{A}_{\kappa',r(\kappa')} \to \mathcal{A}_{\kappa',r(\kappa)}$ and hence an injection $\alpha : M_Y \to M_X \widehat{\otimes}_{R_X} R_Y$. It is easy to check that this injection commutes with the action of all the Hecke operators T_v and S_v. We now check that α is a link; the argument is a slight variant on the usual one because we have allowed non-parallel radii of convergence in our definitions and hence have to make essential use of r-overconvergent forms with $r \notin \mathcal{N}_d^\times$.

Lemma 13.1. *If $U = U_0 \cap U_0(\pi)$ as above, if $Y \subseteq \mathcal{W}$ is a reduced affinoid with corresponding weight $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to \mathcal{O}(Y)^\times$, and* *if* $r, r' \in (\mathcal{N}_K)^J$ *with* $r, r' \leq r(\kappa)$ *and* $r_j \leq r'_j$ *for all* j*, then the natural map* $\mathbf{S}^D_\kappa(U; r') \to \mathbf{S}^D_\kappa(U; r)$ *is a link.*

Proof. It suffices to prove that α is a primitive link when $r'_j|\pi_j| < r_j \leq r'_j$ for all j. But this is not too hard: let c be the compositum of the (compact) restriction map $\mathbf{S}^D_\kappa(U; r) \to \mathbf{S}^D_\kappa(U; r'|\pi|)$ and the continuous norm-decreasing map $\beta : \mathbf{S}^D_\kappa(U; r'|\pi|) \to \mathbf{S}^D_\kappa(U; r')$ in the statement of Lemma 12.2; then it is not hard to check that αc and $c\alpha$ are both U_π as endomorphisms of their respective spaces. □

We may now apply our eigenvariety machine, and deduce the existence of an eigenvariety parametrising systems of Hecke eigenvalues on overconvergent automorphic forms, and in particular p-adically interpolating classical automorphic forms for D. The eigenvariety itself is a rigid space, the geometry of which we know very little about—indeed if we do not know Leopoldt's conjecture then we do not even know its dimension.

If one were to check (and it is no doubt not difficult, following the ideas of section 7 of [5]) that overconvergent forms of small slope were classical, then the existence of the eigenvariety implies results of Gouvêa-Mazur type for classical Hilbert modular forms over F, if $[F : \mathbf{Q}]$ is even (although there are probably more elementary ways of attacking analogues of the Gouvêa-Mazur conjectures in this setting—see for example the recent thesis of Aftab Pande).

References

[1] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.

[2] N. Bourbaki. *Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie*

générale. Chapitres III et IV. Actual. Sci. Ind., no. 916. Hermann & Cie., Paris, 1942.

[3] Nicolas Bourbaki. *Espaces vectoriels topologiques. Chapitres 1 à 5.* Masson, Paris, new edition, 1981. Éléments de mathématique. [Elements of mathematics].

[4] Kevin Buzzard. Analytic continuation of overconvergent eigenforms. *J. Amer. Math. Soc.*, 16(1):29–55 (electronic), 2003.

[5] Kevin Buzzard. On p-adic families of automorphic forms. *Progress in Mathematics*, 224:23–44, 2004.

[6] Kevin Buzzard and Frank Calegari. The 2-adic eigencurve is proper. Accepted for publication in Documenta Math.

[7] Kevin Buzzard and L. J. P. Kilford. The 2-adic eigencurve at the boundary of weight space. *Compos. Math.*, 141(3):605–619, 2005.

[8] Gaëtan Chenevier. Familles p-adiques de formes automorphes pour $\mathrm{GL}(n)$. *Journal für die reine und angewandte Mathematik*, 570:143–217, 2004.

[9] R. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[10] Robert F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

[11] Brian Conrad. Modular curves and rigid-analytic spaces. *Pure Appl. Math. Q.*, 2(1):29–110, 2006.

[12] Haruzo Hida. On p-adic Hecke algebras for GL_2 over totally real fields. *Ann. of Math. (2)*, 128(2):295–384, 1988.

[13] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[14] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[15] Jean-Pierre Serre. Endomorphismes complètement continus des espaces de Banach p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (12):69–85, 1962.

[16] Glenn Stevens. Overconvergent modular symbols. Preprint.

[17] Atsushi Yamagami. On p-adic families of Hilbert cusp forms of finite slope. Preprint.