What does a general unitary group look like?

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February 7, 2012

Abstract

Just some random calculations on the general unitary group.

Last modified 08/01/2010.

1 Definition.

Let $E/F$ be a (separable) quadratic extension of fields of characteristic not equal to 2, and we’ll use a bar to denote conjugation. To define a unitary group we need a non-degenerate Hermitian sesquilinear form; this will be given by an $n$ by $n$ matrix $J \in \text{GL}_n(E)$ with $J = J^t$. The $R$-points of the general unitary group $\text{GU}(J)$ are then

$$\text{GU}(J)(R) = \{ g \in \text{GL}_n(E \otimes_F R) : gJg^t = \lambda J \}.$$  

Here $\lambda \in E \otimes_F R$ if you like, but taking conjugate-transpose of everything we see instantly that $\overline{\lambda} = \lambda$ and hence we may as well assume $\lambda \in R$.

If $R$ is in fact an $E$-algebra then $E \otimes_F R = R \oplus R$, conjugation becomes “switch the factors”, and

$$\text{GU}(J)(R) = \{ (g, h) \in \text{GL}_n(R \oplus R) : (g, h)(J, J^t)(h^t, g^t) = \lambda(J, J^t) \}$$

and the resulting two equations in $\text{GL}_n(R)$ are equivalent (one is the transpose of the other), and equivalent to $h = \lambda J^tg^{-1}J^{-1}$. Hence $\text{GU}(J)(R) = \text{GL}_n(R) \times R^\times$, the isomorphism sending $(g, h)$ to $(g, \lambda)$ and the other way sending $(g, \lambda)$ to $(g, \lambda J^tg^{-1}J^{-1})$.

The based root data of this group can be computed over $E$. Here the group becomes $\text{GL}_n \times \text{GL}_1$ and so we have the usual story: we can use the upper triangular matrices in $\text{GL}_n$ and so on, and get the usual $X^* = X^*(T) \oplus \mathbb{Z}$ and so on ($T$ the torus in $\text{GL}_n$).

Now we need to compute $\mu$, the action of the Galois group on this gadget. Well, let’s use $\text{GL}_n \times \text{GL}_1$ coordinates. Imagine we start with an element $(b, \lambda)$ of the standard Borel. Hitting this with Galois gives the element..., let’s move to the other coordinates. This sends $(b, \lambda)$ to $(b, \lambda J^tb^{-1}J^{-1})$. We now have to work out what Galois does to this. This is a bit subtle. Galois acts non-linearly and is different to the “switching” we saw earlier: the “switching”, which we used a bar for, was all coming from the Galois action on the $E$ in $E \otimes_F R$. We are now considering $R = \overline{E}$ and Galois is acting on the right. Let’s let $c$ denote this. Before we had $e \otimes \overline{r} = \overline{r} \otimes r$. If $R = E$ then we have $E \otimes R = E \oplus E$ via the map $e \otimes r \mapsto (er, er)$. Via this identification we see that $e \otimes \overline{r}$ becomes $(e\overline{r}, \overline{er})$ which is “switch and hit with Galois”, so the Galois action on $\text{GU}(J)(R)$ when $R = E$ sends (using our $\text{GL}_n \times \text{GL}_1$ coordinates) $(g, \lambda)$ to $(\lambda J^{-t}g^{-1}J^{-1}, \mu)$ and we can work out $\mu$ by translating back into unitary group coordinates, where we see $\lambda = \mu$. So the Galois action on $\text{GL}_n(E) \times \text{GL}_1(E)$, regarded as the $E$-points of a variety over $F$, sends $(g, \lambda)$ to $(\lambda J^{-t}g^{-1}J^{-1}, \lambda)$, and lo and behold the fixed points are precisely the $(g, \lambda)$ with $gJg^t = \lambda J$, which is just what we expected now we’ve warmed up.

Right, so how do we compute $\mu_G$? It suffices to compute $\mu_G(c)$. We use the $\text{GL}_n \times \text{GL}_1$ model. Here’s how it goes. We start with $(b, \lambda)$ in the Borel. We hit with $c$ and get $(\lambda J^{-t}b^{-1}J^{-1}, \lambda)$. We conjugate until we’re back in $B$, so we may as well go to $(\lambda J^{-t}b^{-1}J^{-1}, \lambda)$, with $\Phi$ the antidiagonal matrix with alternating $+1$s and $-1$s up the antidiagonal. So now we can see how Galois is acting on $\text{GL}_n \times \text{GL}_1$: it’s induced by the map sending $(b, \lambda)$ to $(\lambda J^{-t}b^{-1}J^{-1}, \lambda)$.
2 The case \( n = 2 \).

In this case the character sending \((\mathrm{diag}(\mu, \nu), \lambda)\) to \(\mu^a \nu^b \lambda^c\) gets sent, by Galois, to the character sending it to \((\lambda \nu^{-1})^a (\lambda \mu^{-1})^b \lambda^c\), which is \(\mu^{-b} \nu^{-a} \lambda^{a+b+c}\). So Galois is represented (on \(X^*\)) by the matrix
\[
e := \begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}.
\]

Note that the root, corresponding to \(a = 1, b = -1, c = 0\), remains fixed.

It’s easy to check that Galois is represented on the dual based root datum by the transpose of this matrix, namely
\[
e^t := \begin{pmatrix}
0 & -1 & 1 \\
-1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Now I claim that these based root data together with their Galois actions are not isomorphic. Here is a very short proof. Any isomorphism of based root data had better send the positive root to the positive root! The positive root in both cases is \((1, -1, 0)\) and note that in both cases this is fixed by Galois. Because the trace of \(e\) is 1 and \(e^2 = 1\), both \(e\) and \(e^t\) have a 1-dimensional eigenspace for the eigenvalue \(-1\). For \(e\) this eigenspace is spanned by \((1, 1, -1)\) and for \(e^t\) it’s spanned by \((1, 1, 0)\). Any isomorphism had better hence send one of these vectors to plus or minus the other. But modulo 2 one of these vectors is congruent to the roots, and the other one isn’t, and the isomorphism sends a root to a root, so it can’t exist.

Conclusion: the based root datum for \(GU(2)\), with its Galois action, is not self-dual.

3 \(L\)-groups.

Back to the general case now. If we think of an element of the torus \(T\) (over the algebraic closure as \((\mu_1, \mu_2, \ldots, \mu_n)\) then the action of \(c\) on the torus of the general unitary group sends (with our \(GL_n \times GL_1\) coordinates as usual) \(((\mu_1, \ldots, \mu_n), \lambda)\) to \(\lambda (\mu_1^{-1}, \ldots, \mu_n^{-1}), \lambda)\). So the matrix representing \(c\) on \(X^*\) is just the obvious generalisation of the \(c\) above (an anti-diagonal of \(-1s\), on top of a row of \(1s\)), and so \(c\) on the dual based root datum is an anti-diagonal of \(-1s\) next to a column of \(1s\). We need to translate this into an action of \(c\) on \(GL_n \times GL_1\), and one checks that it’s this: \(c(g, \lambda) = (\Phi g^{-t} \Phi^{-1}, \det(g) \lambda)\).

I want to assert that for \(n = 2\) this group is not isomorphic to the CHT group, namely \(GL_2 \times GL_1\) with \(c(g, \lambda) = (\lambda g^{-t}, \lambda)\).

So let’s say there’s an isomorphism taking one \(c\) into the other. Note that it’s easy to understand all maps \(GL_2 \times GL_1 \rightarrow GL_2 \times GL_1\) because each such is the sum of a 2-dimensional and a 1-dimensional representation of \(GL_2 \times GL_1\), and we can list these using standard facts about representation theory of reductive groups. Note also that inverse-transpose doesn’t come into it, because \(g \in GL_2\) is conjugate to a twist of its inverse-transpose.

Next we have to understand all the possibilities for complex conjugation on the \(L\)-group. Let’s list the elements of order 2 in the non-identity component of the \(L\)-group. They’re of the form \((Y, \mu)\) with \((Y, \mu)(\Phi Y^{-t} \Phi^{-1}, \det(Y) \mu) = 1\), so we have \(Y \Phi Y^{-t} = \Phi\) and \(\mu^2 \det(Y) = 1\). Well, such things are going to exist in general. Let’s fix one, and let’s let \(C\) denote the corresponding automorphism of \(GL_n \times GL_1\). Explicitly, we have
\[
C(g, \lambda) = (Y, \mu)(c(gY, \lambda)) = (Y, \mu)(\Phi g^{-t} Y^{-t} \Phi^{-1}, \det(g) \det(Y) \lambda \mu)
= (Y \Phi g^{-t} Y^{-t} \Phi^{-1}, \det(g) \lambda) = (Y \Phi g^{-t} \Phi^{-1} Y^{-1}, \det(g) \lambda)
\]

Hmm, so it’s just come out as conjugation anyway... actually, that was always going to happen wasn’t it.

Let’s say the isomorphism from the CHT group to the \(L\)-group sends \((g, \lambda)\) to
\[
i(g, \lambda) := (X g X^{-1} \det(g)^a \lambda^b, \det(g)^c \lambda^d)
\]

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for some integers $a, b, c, d$. Applying the isomorphism $i$ and then the $L$-group's $c$, starting with $(g, \lambda)$, we get $C(XgX^{-1} \det(g)^a \lambda^b, \det(g)^c \lambda^d)$, which is

$$(Y \Phi X^{-t} g^{-t} X^t \det(g)^{-a} \lambda^{-b} \Phi^{-1} Y^{-1}, \det(g)^{1+2a+c} \lambda^{2b+d}).$$

Applying on the other hand $c$ and then the isomorphism gives us the isomorphism applied to $(\lambda g^{-t}, \lambda)$, which is

$$(X \lambda g^{-t} X^{-1} \lambda^{2a+b} \det(g)^{-a} \lambda^{2c+d} \det(g)^{-c}).$$

Now these two last displayed things can’t be equal, because look at the $\det(g)$ factor in the $GL_1$ component.

Conclusion: the CHT group really is not the $L$-group of the general unitary group when $n = 2$. 