What is in the book "Automorphic forms on GL(2)" by Jacquet and Langlands?

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This note, sadly, is unfinished, which is a shame because I did actually make it through to the last-but-one section, in some sense, and the note looked like it was shaping up well. Last modified 25/06/2010.

Introduction.

The book "Automorphic forms on GL(2) by Jacquet and Langlands weighs in at over 550 pages. What is in it? Here's a blow by blow account.

1 Section 1.

Let F be a local field (that is, either a finite extension of $\mathbf{Q}_p$, or of $\mathbf{R}$, or of $\mathbf{F}_q((t))$). Let K be either $F \oplus F$, or a separable quadratic extension of F, or $M_2(F)$, or the non-split quaternion algebra over F. The first goal in §1 is, following Weil, to construct, in a "natural" way, a complex representation of $\mathrm{SL}_2(F)$ associated to K and then, in the cases where K isn't $M_2(F)$, to furthermore associate a representation of $\mathrm{GL}_2(F)$ to any irreducible finite-dimensional representation of $K^\times$. If F is non-archimedean then this representation will be smooth and admissible (but not always irreducible, although we will have such a good understanding of it that we'll be able to decompose it).

The first step is to let $\mathcal{S}(K)$ denote the Schwarz-Bruhat functions on K, which is, I think, just the locally-constant $\mathbf{C}$-valued functions on K with compact support if F is non-arch, and is, I think, the C^∞ $\mathbf{C}$-valued functions all of whose derivatives are rapidly decreasing if F is arch. The crucial observation is that a trick of Weil enables J-L to write down a natural action of $\mathrm{SL}_2(F)$ on this space; the action depends on the choice of a non-trivial additive character $\psi_F : F \to S^1$ (equivalently, of a choice of isomorphism of F with its Pontrjagin dual) but apart from that is pretty natural; the trick is that there is an explicit presentation of $\mathrm{SL}_2(F)$ so one can write down a representation by saying how the upper triangular matrices act and how $w = (0, 1; -1, 0)$ acts (the latter by Fourier transform). The representation is extendible to a unitary representation of $\mathrm{SL}_2(F)$ on $L^2(K)$.

This representation is much too big though, and depends only on K rather than any choice of a representation of K. So next let's assume K is either a separable quadratic extension of F or a non-split quaternion algebra over F and let's fix Ω, a finite-dimensional complex representation of $K^\times$. If K' denotes the norm one elements of $K^\times$ then one looks at the subspace of $\mathcal{S}(K) \otimes \Omega$ where "K' acts via Ω" (see p10 for a more precise definition) and one checks that this space is $\mathrm{SL}_2(F)$-stable, giving us a a representation $\mathcal{S}(K, \Omega)$ of $\mathrm{SL}_2(F)$, and this extends naturally (via inserting a central character, basically) to a representation of $G^+ := \mathrm{GL}_2^+(F)$, the (index 1 or 2) subgroup of $\mathrm{GL}_2(F)$ consisting of matrices whose determinant is in $N(K^\times)$ with N the norm. This representation is called r_Ω. We'll see later that (at least in many cases) it's smooth, admissible, and only has finitely many J-H constituents.

Next (bottom of p12) they write down some "models" of r_Ω. At the minute it's a representation of G^+ (an index one or two subgroup of $\mathrm{GL}_2(F)$) on a space of Ω-valued functions on K. But we want to consider "Whittaker and Kirillov (see next section) models" of r_Ω (they aren't exactly Whittaker and Kirillov models in every case, but they're very close). So for every element Φ in $\mathcal{S}(K,\Omega)$ they associate a function $W_\Phi : G^+ \to \Omega$ simply by $W_\Phi(g) = (g.\Phi)(1)$, and a function $\phi_\Phi : N(K^\times) \to \Omega$ by $\phi_\Phi(a) = W_\Phi(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right))$. The associations $\Phi \mapsto W_\Phi$ and $\Phi \mapsto \phi_\Phi$ are injective. The W map realises r_Ω as a space of functions W_Φ on G^+ satisfying $W_\Phi(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)g) = \psi_F(x)W_\phi(g)$, which is very "Whittaker model"ish (see next section). And the ϕ map realises r_Ω as a representation of G^+ on a certain space of functions on $N(K^\times)$, each element ϕ of which satisfies $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right).\phi(b) = \phi(ba)$ and $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right).phi(b) = \psi_F(xb)\phi(b)$, which is very "Kirillov model"ish. Because the action of the "Borel" $B_+ \subset G^+$ is easy to write down on this space (the action of the unipotent depends on ψ_F but big deal, everything does), to understand r_Ω we just need to see what w does (and, I guess, work out precisely which functions the representation is defined on). J-L introduce the Mellin Transform $\hat{\phi}$ of $\phi := \phi_\Phi$, this being the integral of $\phi.\mu$ over $N(K^\times)$, where μ is a quasi-character of $F^\times$. This is a function of s (the usual trick, a la Tate) which converges for $\Re(s)$ sufficiently large. Surprise surprise, it has analytic continuation to the space of all quasi-characters, and the Mellin transform of $\phi_{w.\Phi}$ at μ is a simple analytic function of s times the Mellin transform of ϕ_Φ at $|.|/\mu$ (that is, "$1-s$"). They only do this last bit for K a separable quadratic extension of F, but say they'll do it later for the quaternion algebra case. [1] 1

They then extend the ideas to $K = F \oplus F$. In this case the maps $\Phi \mapsto W_\Phi$ and $\Phi \mapsto \phi_\Phi$ are no longer injective, so the upshot is that they have maps from r_Ω to various "model" spaces and I'm not sure whether the canonical thing is r_Ω or the image. Anyway, the model spaces satisfy the usual Whittaker and Kirillov properties, the Mellin transforms are only meromorphic, but the same story basically holds.

2 Section 2.

This rather long section develops the theory of smooth admissible irreducible representations of $\mathrm{GL}_2(F)$, for F non-arch local. After some basic preliminaries the heart of the section is the theory of the Kirillov and Whittaker models. The authors then draw some important consequences for the theory (in particular they prove statements about representations that don't mention the models, by using the models).

The first part of the section is easy enough. They fix a Haar measure on $\mathrm{GL}_2(F)$ which gives $\mathrm{GL}_2(\mathcal{O}_F)$ ($\mathcal{O}_F$ the integers of F) measure 1. They define the Hecke algebra $\mathcal{H}_F$ to be the locally constant functions on $\mathrm{GL}_2(F)$ with compact support, give it an algebra structure via convolution, associate an idempotent $\xi \in \mathcal{H}_F$ to each irreducible (finite-dimensional) smooth representation of $\mathrm{GL}_2(\mathcal{O}_F)$ in the obvious way (the idempotent will be supported on $\mathrm{GL}_2(\mathcal{O}_F)$) and then prove an equivalence between the theories of smooth admissible representations of $\mathrm{GL}_2(F)$ and smooth admissible representations of $\mathcal{H}_F$, and a Schur's Lemma.

They then get onto the meat of the section—the Kirillov and Whittaker model stuff. It's just sort-of "blundered through" without any overview of where they're going though, without some clear statement of what they're going to prove, and the main results are scattered about the section. Let me attempt to summarise the main things they prove. Note that the proofs are elementary but sometimes quite long.

Firstly they note that any smooth admissible irreducible representation π of $\mathrm{GL}_2(F)$ is either infinite-dimensional or one-dimensional. They also prove the following useful technical tool: if π is infinite-dimensional then there is no $0 \neq v \in \pi$ with $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)v = v$ for all $x \in F$.

They now fix $\psi_F : F \to S^1$ a non-trivial additive character and define a natural representation of $B_F := \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$ on the space of (all) functions $F^\times \to \mathbf{C}$ by, if $\gamma = \left(\begin{smallmatrix} a & x \\ 0 & 1 \end{smallmatrix}\right)$, setting $(\gamma\phi)(t) = \psi_F(xt)\phi(at)$. They prove that if π (now always assumed smooth irreducible and admissible) is infinite-dimensional, then the set π' of $v \in \pi$ with the property that for some n the integral

$$\int_{(\varpi)^{-n}} \psi_F(-x)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)v\,\mathrm{d}x = 0$$

[1] When if ever do they do this?

(here ϖ is a uniformiser of course, and the brackets denote "fractional ideal") form a codimension 1 space (2.12(iii)), and, identifying the quotient π/π' with $\mathbf{C}$ via a map A, the map sending $v \in \pi$ to the function $\phi_v : F^\times \to \mathbf{C}$ defined by $\phi_v(a) = A(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)v)$ gives an *injection* of π into the functions $F^\times \to \mathbf{C}$ (2.8(ii)) which commutes with the action of B_F. They show that the image contains $\mathcal{S}(F^\times, \mathbf{C})$, the locally constant functions with compact support (2.9(i)), and that $\mathcal{S}(F^\times, \mathbf{C})$ has finite codimension in the space (2.16.1). They associate a formal power series in $\mathbf{C}((t))$ to every element of π, which I hope is just a technical tool so I won't explain this bit (p44) (I'm hoping it's just used to prove that π/π' is 1-dimensional and to prove 2.13 and 2.19; we'll get to these later but their statements don't mention the power series) and then prove that for any infinite-dimensional π there is in fact a *unique* space V of functions $F^\times \to \mathbf{C}$ and a unique action of G on V extending the given action of B_F, such that V and π are isomorphic.

Definition. The V just mentioned above is the *Kirillov model* of π.

Note that we have quite an explicit construction of it, assuming that π' really does have codimension 1.

Now imagine V is the Kirillov model of π. For $\phi \in V$ define $W_\phi : \mathrm{GL}_2(F) \to \mathbf{C}$ by $W_\phi(g) = (g.\phi)(1)$. This is an injection (2.14(i)) from V to a certain space of functions from $\mathrm{GL}_2(F)$ to $\mathbf{C}$. If we give the image the induced action of $\mathrm{GL}_2(F)$ then we see that $(\gamma.W_\phi)(g)$ is, by definition, $W_{\gamma.zphi}(g) = (g\gamma.\phi)(1) = W_\phi(g\gamma)$, so the sensible thing to do is to let $\mathrm{GL}_2(F)$ act on all functions $\mathrm{GL}_2(F) \to \mathbf{C}$ by right translation, and then this injection $\phi \mapsto W_\phi$ is $\mathrm{GL}_2(F)$-equivariant.

Definition. The space of functions above (the image of π) is the *Whittaker model* of π.

The theorem is that the Whittaker model of π is the unique space of functions $W : \mathrm{GL}_2(F) \to \mathbf{C}$ with $\mathrm{GL}_2(F)$ acting via right translation, and such that $W\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)g = \psi_F(x)W(g)$ (2.14(ii)). Using Whittaker models the authors prove in 2.16 and 2.17 that the functions in the Kirillov model are $\mathcal{S}(F^\times, \mathbf{C})$ (rather than these functions plus a non-zero finite-dimensional extra bit) iff π isn't a submodule of a representation induced from the Borel. They call such representations *absolutely cuspidal*. The proof is quite an interesting read (not least because some pages of it are only about 1/10th the size of the print of the other pages): the point is that an induced representation is a space of functions $\mathrm{GL}_2(F) \to \mathbf{C}$ but this time satisfying $f(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)g) = f(g)$, which is not "Whittaker"ish at all. Lemma 2.15.2 shows (if you use a magnifying glass) that any linear map from a Whittaker model to a space of functions with this property had, when composed with the map $f \mapsto f(1)$, better kill $\mathcal{S}(F\times, \mathbf{C})$. In particular this shows that Whittaker models are rather different to "principal series models".

One of the major achievements of the chapter, I think (in the sense that as far as I know this was original) is the definition of a local L-factor $L(s, \pi)$ associated to π (Theorem 2.18). The idea is pretty simple: given W in (the Whittaker model of) π and $g \in \mathrm{GL}_2(F)$, a certain integral of a function involving g and W and $|a|^s$ gives a function $\Psi(g, s, W)$ which converges for $\Re(s)$ sufficiently large. It turns out that there's a unique function $L(s, \pi) = 1/P(q^{-s})$ (P a polynomial with constant term 1, q the size of the residue field) with $\Phi(g, s, W) := \Psi(g, s, W)/L(s, \pi)$ a holomorphic function of s for all g, W and such that there's at least one W in the model such that $\Phi(1, s, W) = x^s$ for some positive real x (2.18(iii)). Along the way they prove (2.18(i)) that the contragredient of π is isomorphic to its twist by the inverse of its central character (I think this is specific to GL_2 and the proof uses the Whittaker model). They check that $L(s, \pi)$ does not depend on the choice of ψ_F.

Using the Whittaker model of both π and its contragredient (which are easily related by the comments just above) they get functions $\Phi(g, s, W)$ and $\tilde{\Phi}(g, s, W)$ which, recall, are holomorphic for all s (note that I use the same notation W for Whittaker functions in both spaces: this is via an implicit identification of the two spaces which I won't bore you with—see p77). It turns out, presumably analogous to what Tate did (I should look) that there's a unique function $\epsilon(s, \pi, \psi_F)$ (see 2.18(iv)) such that

$$\tilde{\Phi}(wg, 1 - s, W) = \epsilon(s, \pi, \psi_F)\Phi(g, s, W)$$

for all g, W, where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Note that this ϵ factor does depends on ψ_F but only in a rather trivial way (p77).

The upshot so far is that for π infinite-dimensional (smooth admiss irred) and ψ_F a non-trivial additive character, there are canonical functions $L(s,\pi)$ (the reciprocal of a polynomial in q^{-s}) and $\epsilon(s,\pi,\psi_F)$ (I don't know much about this one). Hmm, I should say that they only prove this for absolutely cuspidal representations here (where on the way they check that the L-function is identically 1); they prove the result for principal series reprsentations in the next chapter.

Now using the funny power series attached to elements of π which I didn't talk about at all, and the fact that if π and π' have the same central character and the same bunch of power series attached to them then they're isomorphic (2.15), and the fact that these power series for π can be read off from the L-function and ϵ factor in some way (p79), we deduce the following result from JL 2.18:

Theorem. (Cor 2.19) If π and π' are smooth irred admiss infinite-dimensional with the same central character, then π and π' are isomorphic iff for all $\chi : F^\times \to \mathbf{C}^\times$ we have

$$\frac{L(1-s,\chi^{-1}\otimes\tilde{\pi})\epsilon(s,\chi\otimes\pi,\psi_F)}{L(s,\chi\otimes\pi)} = \frac{L(1-s,\chi^{-1}\otimes\tilde{\pi}')\epsilon(s,\chi\otimes\pi',\psi_F)}{L(s,\chi\otimes\pi')}.$$

They only complete the proof of 2.18 for absolutely cuspidal representations; the principal series proofs are in the next chapter. Note that for absolutely cuspidal representations $L(s,\pi)=1$ and the theorem above implies that two abs cuspidal reps π_1 and π_2 with the same central character are isomorphic iff $\epsilon(s,\chi\otimes\pi_1,\psi_F)=\epsilon(s,\chi\otimes\pi_2,\psi_F)$ for all χ. This sort of observation will be useful to us in section 4 when JL will do "explicit local Jacquet-Langlands".

They finish by proving the following cool fact: an absolutely cuspidal representation of $\mathrm{GL}_2(F)$ with unitary central character is unitary. They do it by using the Kirillov model and simply writing down a Hermitian form.

3 Section 3: the principal series for non-arch fields.

After all that work, section 3 is much easier. Let $B(\mu_1,\mu_2)$ denote the usual principal series (using normalised induction), so it's functions f on $\mathrm{GL}_2(F)$ with $f(\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)g)=\mu_1(a)\mu_2(d)|a/d|^{1/2}f(g)$ (we'll call such f functions in "the induction model"). They write down a simple bilinear map $B(\mu_1,\mu_2)\times B(\mu_1^{-1},\mu_2^{-1})\to\mathbf{C}$ thus:

$$\langle\phi_1,\phi_2\rangle=\int_{\mathrm{GL}_2(\mathcal{O}_F)}\phi_1(k)\phi_2(k)\mathrm{d}k,$$

and show it's non-degenerate, thus proving that the admissible dual of the first factor is the second.

Next, and much more interestingly, they go on to show the relationship between the representation $r_{(\mu_1,\mu_2)}$ of chapter 1 and the induction space. It's not obvious! The point, as I've mentioned, is that the W model transforms via ψ_F under left multiplication by $\left(\begin{smallmatrix} 1 & x \\ 0 & 1\end{smallmatrix}\right)$, whereas the induction model consists of functions which are invariant under this.

An element of $r_{(\mu_1,\mu_2)}$ is a function Φ on $F\oplus F$, and the trick is that instead of going from Φ to W_Φ via

$$W_\Phi(g)=\int_{t\in F^\times}\mu_1(t)\mu_2^{-1}(t)(g\Phi)(t,t^{-1})\mathrm{d}t$$

they pass to

$$f_\Phi(g)=\int_{t\in F^\times}\mu_1(t)\mu_2(t)^{-1}|t|(g\Phi)(0,t)\mathrm{d}^\times t$$

and show that this is in the induction model. They define a partial Fourier transform

$$\tilde{\Phi}(a,b)=\int_F\Phi(a,y)\psi_F(by)\mathrm{d}y$$

and check that there's a well-defined map $W_\Phi\mapsto f_{\tilde{\Phi}}$ (Proposition 3.2). In particular we now have two completely different models for the principal series, a Whittaker-like one and an induction one. Now using both of these models, they prove the usual facts: $B(\mu_1,\mu_2)$ is either irreducible or has length 2 with one 1-d JH factor (which happens iff the ratio of the characters is $|.|^{\pm 1}$), and $B(\mu_1, \mu_2)$ is isomorphic to $B(\mu_2, \mu_1)$ when one (equivalently, both) is irreducible (by checking that the Whittaker models are the same spaces of functions). They introduce notation $B_f(\mu_1, \mu_2)$ for the 1-dimensional piece, when the representation is reducible, and $B_s(\mu_1, \mu_2)$ for the Steinberg piece, and show that $B_s(\mu_1, \mu_2)$ is isomorphic to $B_s(\mu_2, \mu_1)$, and compute $B_f(\mu_1, \mu_2)$.

Recall that J-L did some L-function and ϵ-factor stuff in the previous chapter, but the proofs were only complete for absolutely cuspidal representations. They finish the proof here for subquotients of principal series representations, and they even do much better: if $\mu_1/\mu_2 \neq |.|^{\pm 1}$ and $\pi = B(\mu_1, \mu_2)$ then they compute $L(\pi, s) = L(\mu_1, s)L(\mu_2, s)$ and similarly for the epsilon factor. This (Proposition 3.5) completes the proof of 2.18 for principal series. The proof is an explicit calculation using the Kirillov model, surprise surprise. They finally introduce notation $\pi(\mu_1, \mu_2)$ (the full principal series if irreducible, the 1-dimensional factor if not) and $\sigma(\mu_1, \mu_2)$ (the special rep when the PS is reducible), and prove 2.18 for $\sigma(\mu_1, \mu_2)$, again on the way explicitly computing the L and ϵ factors (see Proposition 3.6; the answer depends on whether the mu_i are ramified or not). They also define L and ϵ factors for the full (reducible) principal series representation (in exactly the same way as for the irreducible ones—just take the product of the factors of the two quasi-characters), but perhaps this is just some technical thing that we don't care about? Not sure. We finally finish the proof of 2.18.

They next prove something which will presumably be useful later on: Proposition 3.8. It says firstly that for π an irreducible representation there's an integer m such that if the "order" of χ (this is as far as I can see the conductor!) is more than m then $L(s, \chi \otimes \pi) = 1$ (the proof of this is easy: for principal series and special we have explicit formulae, and for absolutely cuspidal the L-function is always 1 anyway). It says secondly that for π_1 and π_2 with the same central character, there's some integer m such that for all χ with conductor bigger than m we have $\epsilon(s, \chi \otimes \pi_1, \psi_F) = \epsilon(s, \chi \otimes \pi_2, \psi_F)$ and thirdly that if the central character of π is $\mu_1\mu_2$ then for all χ of sufficiently large conductor (this depends on μ_1 and μ_2) we have $\epsilon(s, \chi \otimes \pi, \psi_F) = \epsilon(s, \chi\mu_1, \psi_F)\epsilon(s, \chi\mu_2, \psi_F)$. To prove these last two parts we crucially use these funny power series associated to π, which seem to hold so much information.

They end by checking that an irreducible π has a $\mathrm{GL}_2(\mathcal{O}_F)$-fixed vector iff it's $\pi(\mu_1, \mu_2)$ with μ_i unramified. This is clear for principal series and special reps, and they use the Kirillov model to check that it can't happen for an absolutely cuspidal representation. Finally they prove some funny results which presumably have global applications (3.10 and 3.11) but which I don't understand the significance of.

[**proofread up to here.**]

3.1 Examples of L and ϵ factors.

I always feel that I'm not on top of these things. Actually, L-factors I am on top of. For GL_1 we have $L(\chi, s) = 1$ if χ isn't unramified, and it's *something like* $(1 - q^{-s-t})^{-1}$ if $\chi(x) = |x|^t$ (I'm too lazy to check the signs but this is close enough and passes basic sanity tests). Epsilon factors though—these are interesting even in the GL_1 case. According to my Tate's Thesis notes, if χ on $\mathbf{Q}_p^\times$ has conductor p^n with $n \geq 1$ then the epsilon factor is of the form $A.B^s$ with A a non-zero constant (a power of p times a Gauss sum) and $B = p^n$. The epsilon factor is 1 if χ is unramified though (I suspect that in the general grossencharacter case one might have to be careful here because it might depend on the conductor of the additive character we have chosen).

For GL_2 the story goes like this. For principal series, the L and ϵ factors are just products of the characters involved in the induction. In particular it now seems reasonable that a principal series π is determined by its L and ϵ factors; the twists with non-trivial local L-function are telling you what the characters are.

For Steinberg representations, the L-function is just the L-function of one of the characters used to build the Steinberg (so it's trivial if the characters are ramified, and a 1-dimensional Euler factor if they're unramfied) and the ϵ factors are just the product of the ϵ factors of both characters if they're ramified, and something like $-p^{s+t-1/2}$ if the characters are unramified and of the form $|x|^{t+1/2}$ and $|x|^{t-1/2}$.

Finally, in the abs cuspidal case, the L-function is always 1 and the ϵ factor is a rich piece of information. Really the only thing I know about it is that of ω is the central character of absolutely cuspidal π then $\epsilon(\pi \otimes \chi) = \epsilon(\chi\omega)$ for all sufficiently highly ramified χ. Oh—I should say that that they're always of the form $A.B^s$ in general!

4 Section 4: Examples of absolutely cuspdidal representations.

In section 1 we gave constructions of representations r_Ω of $\mathrm{GL}_2(F)$, where Ω was a representation of another group—namely $K^\times$, for $K = F \oplus F$, or a separable quadratic extension of F, or the non-split quaternion algebra over F. We then just left these gadgets aside, and now I see why: in this section we prove things about these gadgets but as absolutely essential tools we use L and ϵ factors, and the Kirillov models of these representations (which we can get easily from the explicit form of the representations). First they do the quaternion algebra case. In this case Ω is an irreducible representation of $K^\times$ so it's finite-dimensional, and r_Ω is built as a representation of $\mathrm{GL}_2(F)$ on a space $\mathcal{S}(K, \Omega)$ of Ω-valued functions on K. If Ω is one-dimensional then $\Omega = \mathbf{C}$ WLOG, and a Kirillov-like model for this space is easily realised, from which they check quickly that r_Ω is Steinberg. Call this representation $\pi(\Omega)$. If however Ω has dimension greater than one, then any line L in Ω gives a subspace $\mathcal{S}(K, L)$ of r_Ω and it's this space which is irreducible. Call it $\pi(\Omega)$. We basically have the Kirillov model of $\pi(\Omega)$ and can see that it's absolutely cuspidal by inspection (the space coincides with $\mathcal{S}(F^\times)$. L and ϵ factors are easily defined, by integrating certain functions in our space, and are named $L(s, \Omega)$ and $\epsilon(s, \Omega, \psi)$, but these are quickly checked to coincide with $L(s, \pi(\Omega))$ and $\epsilon(s, \pi(\Omega), \psi)$. One actually uses these definitions to check that $\pi(\tilde{\Omega})$ is isomorphic to $\widetilde{\pi(\Omega)}$.

Then they do the same thing for K a separable quadratic extension of F. If ω is a quasi-character of $K^\times$ then r_ω is a representation of $\{g \in \mathrm{GL}_2(F) : \det(g) \in N(K^\times)\}$ (index two in $\mathrm{GL}_2(F)$) and we induce up to $\mathrm{GL}_2(F)$ to get $\pi(\omega)$. The theorem is that $\pi(\omega)$ is irreducible and admissible, that if ω factors through the norm then $\pi(\omega)$ is principal series, and if it doesn't then it's absolutely cuspidal. The tool is the Kirillov model again. The L and ϵ factors are computed, and in the absolutely cuspdial case the L-factor is checked to agree with the L-factor of ω, and the ϵ factors are also closely related, although there are subtleties: one depends on ψ_F and the other on ψ_K (but one gets ψ_K from ψ_F and the trace map) but there is also another fudge factor $\lambda(K/F, \psi_F)$ (the constant γ defined on p4) which presumably is not always one otherwise they would say so!

5 Section 5: Representations of $\mathrm{GL}_2(\mathbf{R})$.

The first thing they introduce are some Hecke algebras. The first, $\mathcal{H}_1$, is the C^∞ compactly-supported functions on $\mathrm{GL}_2(\mathbf{R})$ which are furthermore $O_2(\mathbf{R})$-finite on both sides. That to me looks like some sort of analogue of the unramified Hecke algebra. Once you pick a Haar measure then you can use convolution to give you a ring structure. But there's also a second Hecke algebra $\mathcal{H}_2$ consisting of the functions on $O_2(\mathbf{R})$ which are finite sums of matrix elements of irreducible representations of $O_2(\mathbf{R})$. Any such function gives rise to a measure on $O_2(\mathbf{R})$ (normalise Haar measure so that the measure of the group is 1) and hence a measure on $\mathrm{GL}_2(\mathbf{R})$. Under convolution these measures form an algebra $\mathcal{H}_2$ (I'm not sure I fully understand how to convolute measures. . .). The actual Hecke algebra $\mathcal{H}_\mathbf{R}$ is the sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. I wonder whether it's the same as what Flath does in Corvallis at the end of section 3: he wants to define $\mathcal{H}$ to be the left and right K-finite distributions on G with support in K and then shows that an admissible $\mathcal{H}$-module is precisely an admissible $(\mathfrak{g}, K)$-module. J-L define an admissible $\mathcal{H}_R$-module, define the admissible dual of an admissible $\mathcal{H}_R$-module,