Overconvergent Siegel Modular Symbols

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Chapter 1

Introduction

In [8], Chenevier defines overconvergent p-adic automorphic forms on any twisted form of $\mathrm{GL}_n/\mathbb{Q}$ compact at infinity cohomologically by embedding classically constructed irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ in certain infinite dimensional p-adic Banach spaces. He also defines and proves results about Hecke operators on these spaces of forms, including an analogue of Coleman's 'forms of small slope are classical' result, and constructs an 'eigenvariety' of finite slope eigenforms. Chenevier's work is a higher dimensional analogue of the study of p-adic overconvergent modular forms for subgroups of SL_2 developed by Serre, Katz, Dwork, Hida, Gôuvea-Mazur and Coleman. Also central in the process of developing the theory in higher dimensions is work of Ash-Stevens and Emerton.

In what follows we adapt Chenevier's ideas to the case of Siegel modular forms, modular forms for subgroups of the symplectic group Sp_{2n}, defining a cohomological model for p-adic overconvergent Siegel modular forms. Further we define a Hecke operator U_p and prove a 'forms of small slope are classical' result. We also define explicitly maps analogous to Coleman's θ^{k+1}-maps of [10] between these spaces of cohomological p-adic overconvergent Siegel forms in the case $n = 2$.

First we recall some definitions and motivation from the theory of Siegel modular forms.

Let K be a field and consider the group $\mathrm{GSp}_{2n}(K)$ of $2n$ by $2n$ matrices defined by

$$\mathrm{GSp}_{2n}(K) := \left\{ M \in M_{2n}(K) \left| c(M)J = M^T JM, c(M) \in K^* \right. \right\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, which forces $\det M \neq 0$ for $M \in \mathrm{GSp}_{2n}(K)$.

Notice also that $\mathrm{GSp}_{2n}(K)$ is closed under transpose as $c(M)J = M^T JM$ thus $(c(M))^{-1}J^{-1} = M^{-1}J^{-1}(M^T)^{-1}$ and as $J^{-1} = -J$ we have $MJM^T = c(M)J$, thereby giving an equivalent condition for the definition of $\mathrm{GSp}_{2n}(K)$. The following characterisation is also equivalent: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(K)$, $A, B, C, D \in M_n(K)$, if and only if AB^T and CD^T are symmetric and $AD^T - BC^T = c(M)I$. Let $\mathrm{Sp}_{2n}(K)$ be the subgroup of $\mathrm{GSp}_{2n}(K)$ defined by $c(M) = 1$.

Define Siegel upper half space by

$$Z_n = \{Z \in M_n(\mathbb{C}) | Z^T = Z, \mathrm{Im}(Z) > 0\}$$

where here > 0 denotes positive definite.
If we define

$$\mathrm{GSp}^+_{2n}(R) = \{M \in \mathrm{GSp}_{2n}(R) | c(M) > 0\}$$

for R a subring of $\mathbb{R}$ then $\mathrm{GSp}^+_{2n}(\mathbb{R})$ acts on Z_n by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} z = (Az + B)(Cz + D)^{-1}, \qquad z \in Z_n.$$

If W_t denotes the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight t and we denote by ρ_t the representation $\rho_t : \mathrm{GL}_n \times \mathbb{C}^* \to GL(W_t)$ where $\mathbb{C}^*$ acts via $\lambda \to \lambda^{\frac{1}{2}n(n+1) - \sum t_i}$ and $\Gamma \subset \mathrm{GSp}^+_{2n}(\mathbb{Q})$ is a discrete congruence subgroup then we denote by $S_t(\Gamma)$ the space of holomorphic functions $f : Z \to W_t$ such that

- $f|\gamma = f, \qquad \forall \gamma \in \Gamma$
- $\lim_{\lambda \to +\infty} (f|\gamma) \begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix} = 0, \qquad \forall \gamma \in \mathrm{GSp}^+_{2n}(\mathbb{Q}), \qquad z \in Z_{n-1}$

where for $\gamma \in \mathrm{GSp}^+_{2n}(\mathbb{R})$ we define

$$(f|\gamma)(z) = [\rho(Cz + D, c(\gamma))]^{-1} f(\gamma z), \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Call $f \in S_t(\Gamma)$ a vector valued *Siegel cusp form for* Γ *of weight* t. It is well known that the space $S_t(\Gamma)$ is a finite dimensional $\mathbb{C}$-vector space.

We now look at the relation between Siegel cusp forms and group cohomology. This generalizes the maps of Eichler-Shimura in the classical case. With t as above, if $\Gamma \subset \mathrm{GSp}_{2n}^{+}(\mathbb{Z})$ is a discrete congruence subgroup then there is a Hecke equivariant natural map

$$S_t(\Gamma) \hookrightarrow H^{\frac{1}{2}n(n+1)}(\Gamma, V_{t-(n+1)t_0})$$

where $t_0 = [1, \ldots, 1]$ and V_λ is the irreducible $\mathrm{Sp}_{2n}(\mathbb{C})$ module of highest weight λ.

If Γ is torsion free then this is a special case of [20] §2.3 and [11] Theorem 10. If Γ has torsion then by standard arguments $\exists N \in \mathbb{N}$ such that

$$\Gamma_N = \{g \in \mathrm{Sp}_{2n}(\mathbb{Z}) | g \equiv I_{2n} \bmod N\} \subset \Gamma$$

with Γ_N torsion free and of course $|\Gamma/\Gamma_N| < \infty$. Then we have a Hecke equivariant natural map

$$S_t(\Gamma_N) \hookrightarrow H^{\frac{1}{2}n(n+1)}(\Gamma_N, V_{t-(n+1)t_0})$$

where this map respects the action of Γ/Γ_N and thus

$$\begin{aligned} S_t(\Gamma) = S_t(\Gamma_N)^{\Gamma/\Gamma_N} \hookrightarrow H^{\frac{1}{2}n(n+1)}(\Gamma_N, V_{t-(n+1)t_0})^{\Gamma/\Gamma_N} \\ = H^{\frac{1}{2}n(n+1)}(\Gamma, V_{t-(n+1)t_0}) \end{aligned}$$

where the last equality follows from the Inflation-Restriction sequence.

The philosophy guiding the content herein is to embed the finite dimensional irreducible algebraic representation of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ of highest weight t into a suitably defined infinite dimensional p-adic Banach space to which Hecke operators extend. Overconvergent Siegel modular forms will then be modelled by the appropriate group cohomology group of this infinite dimensional space. A 'small slopes are classical' result will then allow us to recover classical forms by way of their slopes with respect to the Hecke operator at p.

Chapter 2

Certain Representations of GSp_{2n}

Throughout this section we follow [8].

Consider variables X_{ij}, $1 \leq i, j \leq 2n$ and consider the matrix X with $(X)_{ij} = X_{ij}$. Let $R_G = K[X_{ij}, D, D^{-1}]/I$ where I is the ideal of $K[X_{ij}, D, D^{-1}]$ generated by the relations among the X_{ij} and D defined by $DJ = X^T JX$ and $D^n = \det(X)$.

For example, if $n = 2$ then I is generated by $\gamma_0 = X_{11}X_{23} - X_{13}X_{21} + X_{12}X_{24} - X_{14}X_{22}$, $\gamma_1 = X_{31}X_{43} - X_{41}X_{33} + X_{32}X_{44} - X_{34}X_{42}$, $\gamma_2 = X_{11}X_{33} - X_{13}X_{31} + X_{12}X_{34} - X_{14}X_{32} - D$, $\gamma_3 = X_{11}X_{43} - X_{13}X_{41} + X_{12}X_{44} - X_{14}X_{42}$, $\gamma_4 = X_{21}X_{33} - X_{23}X_{31} + X_{22}X_{34} - X_{24}X_{32}$, and $\gamma_5 = X_{21}X_{43} - X_{23}X_{41} + X_{22}X_{44} - X_{24}X_{42} - D$.

$\mathrm{GSp}_{2n}(K)$ induces two actions on $K[X_{ij}, D, D^{-1}]$, one which we will call the Left action and one which we will call the Right action (though they are both left actions). These are induced by the following changes of variables:

$$g_l X = (g^{-1}X)_{ij} \qquad g_r X = (Xg)_{ij}, \qquad g \in \mathrm{GSp}_{2n}(K).$$

Since these actions preserve I, they descend to actions on R_G.

We now restrict our attention to the $K = \mathbb{Q}_p$, for some prime p. Thus GSp_{2n} and Sp_{2n} will denote $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ and $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ respectively unless otherwise noted.As described in [15] Sp_{2n} has a Borel subgroup H realised as the semidirect product $H = UT$ where

$$T = \left\{ \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{pmatrix} : \Gamma \in GL_n(\mathbb{Q}_p) \text{ is diagonal} \right\},$$

and where U is again a semidirect product, $U = VL$, where

$$V = \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} : B \in M_n(\mathbb{Q}_p), B = B^t \right\},$$

$$L = \left\{ \begin{pmatrix} X & 0 \\ 0 & X^{-t} \end{pmatrix} : X \in GL_n(\mathbb{Q}_p), X \text{ lower triangular and unipotent} \right\}.$$

Thus the unipotents of H are U which one checks easily are the matrices

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is lower triangular, D is upper triangular, A and D are unipotent, $AD^T = 1$ and AB^T is symmetric.

Let $\overline{U}$ denote the tranpose of U. Then similarly $\overline{U}$ is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ B & D \end{pmatrix}$$

where A is upper triangular, D is lower triangular, A and D are unipotent, $AD^T = 1$ and BD^T is symmetric.

If we define $R_G^{\overline{U}}$ to be the elements of R_G fixed by the Left action of $\overline{U}$ then we will study the Right action of GSp_{2n} on $R_G^{\overline{U}}$.

For $1 \le m \le n$ put increasing m-tuples of $\{1, 2, \ldots, 2n\}$ in lexographical order and define $J = J(m) = \binom{2n}{m}$. Then define

$$Y_{mj} = \begin{vmatrix} X_{n+1-m,j_1} & X_{n+1-m,j_2} & \ldots & X_{n+1-m,j_m} \\ X_{n+2-m,j_1} & X_{n+2-m,j_2} & \ldots & X_{n+2-m,j_m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,j_1} & X_{n,j_2} & \ldots & X_{n,j_m} \end{vmatrix}$$

for $j = (j_1, \ldots, j_m)$, $1 \le j_1 < j_2 < \ldots < j_m \le 2n$.

Lemma 2.0.1. *For each $m \in \mathbb{N}$, $1 \le m \le n$ and for any m-tuple $j = (j_1, \ldots, j_m)$, $1 \le j_1 < j_2 < \ldots < j_m \le 2n$, Y_{mj} is in $R_G^{\overline{U}}$.*

Proof: A simple calculation confirms that for $g \in \mathrm{GSp}_{2n}$, $g^{-1} = (g^{-1}_{ij})$, we have

$$g_l \begin{vmatrix} X_{i_1,j_1} & \dots & X_{i_1,j_m} \\ X_{i_2,j_1} & \dots & X_{i_2,j_m} \\ \vdots & \ddots & \vdots \\ X_{i_m,j_1} & \dots & X_{i_m,j_m} \end{vmatrix} = \sum_{k=(k_1,\dots,k_m)} \begin{vmatrix} g^{-1}_{i_1,k_1} & \dots & g^{-1}_{i_1,k_m} \\ g^{-1}_{i_2,k_1} & \dots & g^{-1}_{i_2,k_m} \\ \vdots & \ddots & \vdots \\ g^{-1}_{i_m,k_1} & \dots & g^{-1}_{i_m,k_m} \end{vmatrix} \begin{vmatrix} X_{k_1,j_1} & \dots & X_{k_1,j_m} \\ X_{k_2,j_1} & \dots & X_{k_2,j_m} \\ \vdots & \ddots & \vdots \\ X_{k_m,j_1} & \dots & X_{k_m,j_m} \end{vmatrix}.$$

for all m-tuples $i = (i_1, \dots, i_m)$, $j = (j_1, \dots, j_m)$, $1 \leq i_1 < \dots < i_m \leq 2n$, $1 \leq j_1 < \dots < j_m \leq 2n$.

Thus for $1 \leq m \leq n$, choosing $i = (n+1-m, n+2-m, \dots, n)$, we see that for $g \in \overline{U}$ the only nonvanishing

$$\begin{vmatrix} g^{-1}_{n+1-m,k_1} & \dots & g^{-1}_{n+1-m,k_m} \\ g^{-1}_{n+2-m,k_1} & \dots & g^{-1}_{n+2-m,k_m} \\ \vdots & \ddots & \vdots \\ g^{-1}_{n,k_1} & \dots & g^{-1}_{n,k_m} \end{vmatrix}$$

is for $k = (n+1-m, n+2-m, \dots, n)$ where it equals 1 thus confirming that Y_{mj} is in $R_G^{\overline{U}}$ for any m-tuple $j = (j_1, \dots, j_m)$, $1 \leq j_1 < \dots < j_m \leq 2n$. $\square$

Observe also that $\mathbb{Q}_p[Y_{ij}]$, the $\mathbb{Q}_p$-subalgebra of R_G generated by the Y_{ij}'s is invariant under the Right action of GSp_{2n}. This follows as $\sum_j \mathbb{Q}_p Y_{ij}$ is easily seen to be preserved by the Right action of GSp_{2n}.

2.1 Weights

The group GSp_{2n} has torus

$$T_G = \{\mathrm{diag}(d_1, \dots, d_n, cd_1^{-1}, cd_n^{-1}), d_i \neq 0, c \neq 0\}$$

and since T_G normalizes $\overline{U}$, then the Left action of T_G preserves $R_G^{\overline{U}}$ and thus induces a Left action of T_G on $R_G^{\overline{U}}$. We also restrict to obtain the Right action of T_G on $R_G^{\overline{U}}$.

Let $t = [a_1, \dots, a_n, z] \in \mathbb{Z}^{n+1}$ denote the character $T_G \to \mathbb{Q}_p^*$ defined by

$$t(\mathrm{diag}(d_1, \dots, d_n, cd_1^{-1}, \dots, cd_n^{-1})) = \prod_i d_i^{a_i} \cdot c^z$$

and denote $t_i = [0, \ldots, 1, \ldots, 0]$ with a 1 in the i-th position for $1 \leq i \leq n$, let μ denote $[0, \ldots, 0, 1]$ with a 1 in the $n+1$-st position and $t_i = \mu - t_{i-n}$ for $n+1 \leq i \leq 2n$.

If $d \in T$ and $1 \leq n \leq n$ we have

$$d_l Y_{mj} = \prod_{i=1}^{m} t_{n+1-i}(d)^{-1} Y_{mj}$$

and

$$d_r Y_{mj} = \prod_{i=1}^{m} t_{j_i}(d) Y_{mj}.$$

We say $f \in R_G$ is of Left (resp. Right) weight t if T_G acts on the Left (resp. Right) by t^{-1} (resp. t). If we denote the vector space of all elements of Left weight t by $(R_G)_t$ the action of T is diagonalisable and we get a decomposition

$$R_G = \bigoplus_{t \in \mathbb{Z}^{n+1}} (R_G)_t.$$

We now define $R = R_G/(D-1) \cong \mathbb{Q}_p[X_{ij}]/I$, where I is the ideal of relations generated by $J = X^T J X$. Then R does not have a Left or Right action of GSp_{2n} but does inherit the restricted Left and Right actions of Sp_{2n}. These actions are of course the natural actions induced by the changes of variables

$$g_l X = (g^{-1}X)_{ij} \qquad g_r X = (Xg)_{ij}, \qquad g \in \mathrm{Sp}_{2n}(K).$$

However, we note the following lemma:

Lemma 2.1.1. *The subalgebras $\mathbb{Q}_p[Y_{ij}] \subset R$ and $\mathbb{Q}_p[Y_{ij}] \subset R_G$ are isomorphic as $\mathbb{Q}_p$-algebras via the natural reduction map. Thus $\mathbb{Q}_p[Y_{ij}] \subset R$ retains the Right GSp_{2n}-module structure of $\mathbb{Q}_p[Y_{ij}] \subset R_G$.*

Proof: We must show that there are no new relations between the Y_{ij} introduced by reducing from R_G to R. This is equivalent to showing that the ideal of relations between the Y_{ij} and D in R_G is generated by relations between the Y_{ij} alone.

Given a polynomial relation $P(Y_{ij}, D) = 0$ in R_G we can write $P = P_{t_1} + \ldots + P_{t_k}$, with $P_{t_i} \in (R_G)_{t_i}$ for distinct t_i and with the P_{t_i} polynomials in the Y_{ij} and D. Then, since R_G is a direct sum of the $(R_G)_t$ we know that each $P_{t_i} = 0$ in R_G. Furthermore, since the Y_{ij} are of Left weights of the form $[a_1, \ldots, a_n, 0]$ and D is of weight $[0, \ldots, 0, 1]$ we know that P_{t_i} times a suitable power of D is a relation amongst the Y_{ij} alone.

Applying this reduction to a finite set of generators for the ideal of relations between the Y_{ij} and D shows that the ideal of relations between the Y_{ij} and D in R_G is generated by relations between the Y_{ij} alone. $\square$

Now for $t = [a_1, \ldots, a_n] \in \mathbb{Z}^n$ define R_t to be the image of $\oplus(R_G)_{t'}$ in R, where $t' \in \mathbb{Z}^{n+1}$ ranges over $\{[a_1, \ldots, a_n, z], z \in \mathbb{Z}\}$. Then R_t is of course the space of elements on which the torus

$$T = \{\mathrm{diag}(d_1, \ldots, d_n, d_1^{-1}, \ldots, d_n^{-1}), d_i \neq 0\}$$

of Sp_{2n} acts on the Left by t^{-1} where $t \in \mathbb{Z}^n$ denotes the character $T \to \mathbb{Q}_p^*$ defined by restricting $[a_1, \ldots, a_n, 0]$ to T. Again, by abuse of notation let t_i henceforth denote $[0, \ldots, 1, \ldots, 0] \in \mathbb{Z}^n$ with a 1 in the i-th position, for $1 \leq i \leq n$ and $t_i = -t_{i-n}$ for $n+1 \leq i \leq 2n$.

We define $R^{\overline{U}}$ to be the elements of R fixed by the Left action of $\overline{U}$ and $R_t^{\overline{U}}$ the elements in R_t fixed by $\overline{U}$. Note that the Y_{ij} are again in $R^{\overline{U}}$. We get a direct sum decomposition

$$R = \bigoplus_{t \in \mathbb{Z}^n} R_t$$

and we see below that we have a sub-decomposition

$$R^{\overline{U}} = \bigoplus_{t \in \mathbb{Z}^n} R_t^{\overline{U}}.$$

We note that the Y_{ij} are each of Left weight t for some t satisfying $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ with $a_i \in \mathbb{Z}$. We will call such weights *positive increasing* and denote this condition by $t \geq 0$.

For ease of notation let $F := R^{\overline{U}}$. We are now ready to describe F completely.

2.2 Invariants

Proposition 2.2.1. *[13] We have $F = \bigoplus_{t\geq 0} R_t^{\overline{U}}$. If $t = [a_1, \ldots, a_n] \geq 0$ then $R_t^{\overline{U}}$ is the irreducible algebraic representation of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ of highest weight t. $R_t^{\overline{U}}$ is generated over $\mathbb{Q}_p$ by all monomials in the variables Y_{ij} of Left weight t and has highest weight vector $\prod_{i=1}^n Y_{i,\jmath_i}^{(a_i - a_{i-1} - \ldots - a_1)}$, where $\jmath_i$ denotes the i-tuple $(n-i+1, n-i+2, \ldots, n)$.*

Proof: We refer to classical results in [13] §12.1.4.

With notation as in [13] we establish that the dominant weights $P_{++}(G)$ are indeed the weights $t \geq 0$ with our notation as above. One confirms that the Lie algebra of U is generated by

$$a_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, 1 \leq j < i \leq n,$$

$$b_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}, 1 \leq i < j \leq n,$$

and

$$c_i = \begin{pmatrix} 0 & E_{ii} \\ 0 & 0 \end{pmatrix}, 1 \leq i \leq n.$$

We see a_{ij} is in root space $t_i t_j^{-1}$, $i > j$, b_{ij} is in root space $t_i t_j$ and c_i is in root space t_i^2. Thus these are the positive roots corresponding the U.

One checks that $\alpha_1 := t_1^2, \alpha_2 := t_1^{-1} t_2, \ldots, \alpha_n := t_{n-1}^{-1} t_n$ are simple roots and we get, with notation as in [13],

$$H_1 = \mathrm{diag}(1, 0, \ldots, -1, 0, \ldots, 0)$$

with -1 in the $n+1$-st position, and

$$H_i = \mathrm{diag}(0, \ldots, -1, 1, 0, \ldots, 1, -1, 0, \ldots, 0), i > 1$$

with -1 in the $i-1$-st and $n+i$-th positions and 1 in the i-th and $n+i-1$-st positions. For $\mu = \sum k_i t_i$, $< \mu, H_i > \geq 0$, $\forall i$ is equivalent to $0 \leq k_1 \leq k_2 \leq \ldots \leq k_n$ as desired.

Theorems 12.1.9 and 12.1.10 in [13] §12.1.4 establish that F is the direct sum of all irreducible algebraic representations of Sp_{2n}, each with multiplicity one. Furthermore, these results confirm that the space $R_t^{\overline{U}}$, $t \geq 0$, is the irreducible algebraic representation of highest weight t. Now we must only confirm that the monomials of Left weight t in the variables Y_{ij} generate $R_t^{\overline{U}}$. Since for each i, $\sum_j \mathbb{Q}_p Y_{ij}$ is preserved by the (Right) action of Sp_{2n}, it follows easily that the subspace generated by monomials in the Y_{ij}'s of Left weight t is stable under the (Right) action of Sp_{2n}. Then by irreducibility of $R_t^{\overline{U}}$, the monomials in the Y_{ij}'s of Left weight t generate $R_t^{\overline{U}}$. It is easily noted that $\prod_{i=1}^{n} Y_{i,j_i}^{(a_i - a_{i-1} - \ldots - a_1)}$ is of highest (Right) weight t. $\square$

For ease of notation, let us reparameterize the weights in the following fashion: Let $\delta_1 = [0, \ldots, 0, 1]$, $\delta_2 = [0, \ldots, 0, 1, 1]$ and similarly through to $\delta_n = [1, \ldots, 1, 1]$. Then let $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ denote $\sum_{i=1}^{n} b_i \delta_i = [b_n, b_n + b_{n-1}, \ldots, b_n + b_{n-1} + \ldots + b_1]$.Thus Y_{ij} is of Left weight δ_i for $1 \leq i \leq n$.

Note that as $F = \mathbb{Q}_p[Y_{ij}]$ we have seen above that it retains a Right action by GSp_{2n}. Since the Left action of $\overline{U}$ on F is by definition trivial, by *the action of* Sp_{2n} *(resp.* GSp_{2n}*) on* F we will henceforth mean the Right action of Sp_{2n} (resp. GSp_{2n}) on F and will denote $g_r Y_{ij}$ simply by gY_{ij}.

Let $V := \mathbb{Q}_p^{2n}$ be the standard representation of $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ with its canonical basis and give $\bigwedge^i(V)$ its canonical basis Z_{ij}, $1 \leq j \leq J(i)$ ordered lexographically. If we define $B := \mathrm{Sym}\left(\bigoplus_{i=1}^{n} \bigwedge^i(V)\right)$ we have the following:

Proposition 2.2.2. *We have a map of* $\mathbb{Q}_p$ *algebras* $\phi : B \to F$ *induced by* $Z_{ij} \mapsto Y_{ij}$ *which respects the action of* GSp_{2n}. *The map* ϕ *is surjective.*

Proof: A simple calculation checks that $\phi(gZ_{ij}) = g\phi(Zij) = gY_{ij}$ for $g \in \mathrm{GSp}_{2n}$ and this suffices to check that ϕ respects the action of GSp_{2n}. The map ϕ is surjective since its image contains $\mathbb{Q}_p[Y_{ij}] = F$ with equality by the previous proposition. $\square$

2.3 An Example

In the case $n = 2$ the following calculations use Magma to compute all the relations between the Y_{ij}. Furthermore, we use the Proposition 2.2.1 and the Weyl character formula to calculate the Hilbert Polynomial of $\mathbb{Q}_p[Y_{ij}]$.

```
P<x11,x12,x13,x14,x21,x22,x23,x24,x31,x32,x33,x34,x41,x42,x43,x44,
D,E,y11,y12,y13,y14,y21,y22,y23,y24,y25,y26> := PolynomialRing(Rationals(),28);

I:=ideal<P|x11*x23-x13*x21+x12*x24-x22*x14,x31*x43-x41*x33+x32*x44-x42*x34,
x11*x33-x13*x31+x12*x34-x14*x32-D,x11*x43-x13*x41+x12*x44-x14*x42,
x21*x33-x23*x31+x22*x34-x24*x32,x21*x43+x22*x44-x23*x41-x24*x42-D,y11-x21,
y12-x22,y13-x23,y14-x24, y21 - x11*x22+x12*x21, y22-(x11*x23-x21*x13),
y23 -(x11*x24-x21*x14), y24-(x12*x23-x13*x22),y25-(x12*x24-x14*x22),
y26-(x13*x24-x14*x23), E*D-1,
x11*x22*x33*x44 - x11*x22*x34*x43 - x11*x23*x32*x44
+x11*x23*x34*x42 + x11*x24*x32*x43 - x11*x24*x33*x42
-x12*x21*x33*x44 + x12*x21*x34*x43 + x12*x23*x31*x44
-x12*x23*x34*x41 - x12*x24*x31*x43 + x12*x24*x33*x41
+x13*x21*x32*x44 - x13*x21*x34*x42 - x13*x22*x31*x44
+x13*x22*x34*x41 +x13*x24*x31*x42 - x13*x24*x32*x41
-x14*x21*x32*x43 + x14*x21*x33*x42 +x14*x22*x31*x43
-x14*x22*x33*x41 - x14*x23*x31*x42 + x14*x23*x32*x41 - D*D>;

J:= EliminationIdeal(I,18);

J;
```

This code returns that the ideal of relations between the Y_{ij} is generated by the following relations: $Y_{22}+Y_{25}$,$Y_{11}Y_{24}-Y_{12}Y_{22}+Y_{13}Y_{21}$, $-Y_{11}Y_{22}-Y_{12}Y_{23}+Y_{14}Y_{21}$, $Y_{11}Y_{26}-Y_{13}Y_{23}+Y_{14}Y_{22}$, $Y_{12}Y_{26}+Y_{13}Y_{22}+Y_{14}Y_{24}$, $Y_{21}Y_{26}+Y_{23}Y_{24}+Y_{22}^2$.

Note: Although we performed these calculations over the rationals the same relations generate the ideal of relations between the Y_{ij} over $\mathbb{Q}_p$ for the following reason:
The following holds for $n \in \mathbb{N}$ and not just $n = 2$.
Let I be the ideal in $\mathbb{Z}[X_{ij}, D, D^{-1}]_{1\le i,j\le 2n}$ defined by $DJ = X^T JX$ and

$D^n = det(X)$, $X = (X_{ij})$, J as above. Let the Y_{ij}'s be defined as above and J be the ideal $I \cap \mathbb{Z}[Y_{ij}]$ in $\mathbb{Z}[Y_{ij}]$. Then as $\mathbb{Z}[Y_{ij}]$ is Noetherian we have $J = (f_1, \dots, f_M)$ some $M \in \mathbb{N}$.
Then as $\mathbb{Q}_p$ is a flat $\mathbb{Z}$-module tensoring the exact sequence

$$0 \to J \to \mathbb{Z}[Y_{ij}] \to \mathbb{Z}[X_{ij}, D, D^{-1}]/I \to 0$$

with $\mathbb{Q}_p$ gives

$$0 \to J \otimes \mathbb{Q}_p \to \mathbb{Q}_p[Y_{ij}] \to \mathbb{Q}_p[X_{ij}, D, D^{-1}]/(I \otimes \mathbb{Q}_p) \to 0.$$

The ideal of relations between the Y_{ij} is generated by relations in $\mathbb{Z}[Y_{ij}]$.

The previous calculation shows that the kernel of the map ϕ for $n = 2$ is the ideal of B generated by $Z_{22} + Z_{25}$, $Z_{11}Z_{24} - Z_{12}Z_{22} + Z_{13}Z_{21}$, $-Z_{11}Z_{22} - Z_{12}Z_{23} + Z_{14}Z_{21}$, $Z_{11}Z_{26} - Z_{13}Z_{23} + Z_{14}Z_{22}$, $Z_{12}Z_{26} + Z_{13}Z_{22} + Z_{14}Z_{24}$, $Z_{21}Z_{26} + Z_{23}Z_{24} + Z_{22}^2$.

Let h denote the Hilbert polynomial $\mathbb{Q}_p[Y_{ij}]$. Proposition 2.2.1 tells us that F_t, $t = (a, b)$, is generated by monomials of degree a in the Y_{1j}'s and b in the Y_{2j}'s. Thus $h(d)$, the number of linearly independent monomials of degree d in the variables Y_{ij}, is equal to $\dim(\oplus_{a+b=d}F_{(a,b)})$.

Using the Weyl character formula we compute that

$$\dim F_{(a,b)} = \frac{(b+1)(a+1)(a+2b+3)(a+b+2)}{6}.$$

So we calculate

$$\begin{aligned} h(d) &= \dim\left(\bigoplus_{a+b=d} F_{(a,b)}\right) \\ &= \sum_{a=0}^{d} \frac{(d-a+1)(a+1)(a+2(d-a)+3)(a+(d-a)+2)}{6} \\ &= \frac{d^5}{24} + \frac{5d^4}{12} + \frac{13d^3}{8} + \frac{37d^2}{12} + \frac{17d}{6} + 1. \end{aligned}$$

This agrees with the following Magma calculation:

```
Q<y11,y12,y13,y14,y21,y22,y23,y24,y26> := PolynomialRing(Rationals(),9);


K:= ideal<Q|y11*y24 - y12*y22 + y13*y21,
```

```
-y11*y22-y12*y23+y14*y21,y11*y26-y13*y23+y14*y22,

y12*y26+y13*y22+y14*y24,y21*y26+y23*y24+y22*y22>;


HilbertPolynomial(K);
```

which returns:

$$h(d) = \frac{d^5}{24} + \frac{5d^4}{12} + \frac{13d^3}{8} + \frac{37d^2}{12} + \frac{17d}{6} + 1.$$

2.4 Integrality

Now define B^0 and F^0 to be the $\mathbb{Z}_p$-subalgebras of B and F generated by the Z_{ij} and Y_{ij} respectively. We have a direct sum decomposition $B = \bigoplus_{t \in \mathbb{N}^n} B_t$ by defining B_t to be the $\mathbb{Q}_p$-vector space generated by monomials of total degree a_i in the variables Z_{ij}, for each $1 \leq i \leq n$, for $t = (a_1, \ldots, a_n)$. Clearly our decomposition of B into weight spaces restricts to a decomposition $B^0 = \bigoplus_{t \in \mathbb{N}^n} B_t^0$ where B_t^0 is the $\mathbb{Z}_p$-submodule of B_t with coefficients in $\mathbb{Z}_p$.

Both B^0 and F^0 are stable under the action of $M_{2n}(\mathbb{Z}_p) \cap \mathrm{GSp}_{2n}(\mathbb{Q}_p)$. Let us denote by Δ the monoid (with respect to matrix multiplication) of matrices $M \in \mathrm{GSp}_{2n}(\mathbb{Q}_p) \cap M_{2n}(\mathbb{Z}_p)$ such that for each $1 \leq i \leq n$,

$$\begin{vmatrix} M_{n-i+1,n-i+1} & \ldots & M_{n-i+1,n} \\ \vdots & \ddots & \vdots \\ M_{n,n-i+1} & \ldots & M_{nn} \end{vmatrix}$$

has p-adic norm $\geq$ the norm of any other

$$\begin{vmatrix} M_{j_1,k_1} & \ldots & M_{j_1,k_i} \\ \vdots & \ddots & \vdots \\ M_{j_i,k_1} & \ldots & M_{j_i,k_i} \end{vmatrix},$$

for any $1 \leq j_1 < \ldots < j_i \leq 2n$ and $1 \leq k_1 < \ldots < k_i \leq 2n$, and strictly greater if $(k_1, \ldots, k_i) = (n-i+1, \ldots, n)$.
Straightforward calculation confirms that this is a monoid.

Define the congruence subgroup of Sp_{2n}

$$\Gamma_0(p) = \{\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z})| \quad \gamma \equiv \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \bmod p,$$
$$A \text{ lower triangular, } D \text{ upper triangular}\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_{2n}) \in \mathbb{Z}_{\geq 0}^{2n}$, we call α *admissible* if $\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_1 \leq \alpha_{n+1} \leq \ldots \leq \alpha_{2n}$ and $\alpha_1 + \alpha_{n+1} = \alpha_2 + \alpha_{n+2} = \ldots = \alpha_n + \alpha_{2n}$. For α admissible we let u^α denote $\mathrm{diag}(p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_{2n}})$. All α are henceforth assumed admissible.

Fix $\alpha_0 = (n-1, n-2, \ldots, 0, n, n+1, \ldots, 2n)$. Also let $\beta_i = \sum_{j=1}^{i} \alpha_{n+1-j}$, for $1 \leq i \leq n$.

Lemma 2.4.1. *For all* $\gamma \in \Delta$,
(i) Fix $1 \leq i \leq n$, $\gamma Z_{ij_i} \in p^m(\mathbb{Z}_p^* Z_{ij_i} + p\sum_{j \neq j_i} \mathbb{Z}_p Z_{ij})$, *with* $m \geq 0$ *where* j_i *is as in Proposition 2.2.1 and* $|\gamma Z_{ij}| \leq p^{-m}$ *with* $\gamma Z_{ij} \in \sum_{1 \leq k \leq J(i)} \mathbb{Z}_p Z_{ik}$, $\forall j = 1, \ldots, J(i)$.
(ii) If $1 \leq i \leq n$, *and* $j = (j_1, \ldots, j_i)$ *then* $u^\alpha Z_{ij} = p^{\sum_k \alpha_{j_k}} Z_{ij}$. *In particular* $u^\alpha Z_{ij} \in p^{\beta_i} \mathbb{Z}_p Z_{ij}$, $u^{\alpha_0} Z_{ij} \in p^{\frac{i(i-1)}{2}} \mathbb{Z}_p Z_{ij}$.

Proof: These are all immediate. If $\gamma = (\gamma_{ij}) \in \Delta$ then a simple calculation yields

$$\gamma Z_{ij} = \sum_{1 \leq k \leq J(i)} \begin{vmatrix} \gamma_{k_1 j_1} & \cdots & \gamma_{k_1 j_i} \\ \vdots & \ddots & \vdots \\ \gamma_{k_i j_1} & \cdots & \gamma_{k_i j_i} \end{vmatrix} Z_{ik},$$

confirming the second statement in (i). Furthermore, for $j = j_i$, examining the definition of Δ confirms the remainder of (i).

The second statement in (ii) follows from $\beta_i \leq \sum_{k=1}^{i} \alpha_{j_k}$ as α admissible.
$\square$

Let us fix notation and denote the coefficients occurring in γZ_{ij} by $a_{ijk} = a_{ijk}(\gamma)$ thus

$$\gamma Z_{ij} = \sum_{1 \leq k \leq J(i)} a_{ijk} Z_{ik}.$$

2.5 Analytification

Define variables $z_{ij} = Z_{ij}/Z_{i\jmath_i}$, $j \neq \jmath_i$ and for simplicity of notation define $z_{i\jmath_i} = 1$. Similarly define $y_{ij} = Y_{ij}/Y_{i\jmath_i}$, $j \neq \jmath_i$ and define $y_{i\jmath_i} = 1$.

Form the affinoid algebras $A_B := \mathbb{Q}_p < z_{ij} >$ and $A_F := \mathbb{Q}_p < y_{ij} >$. The map $\phi : B \to F$ induces a map $\phi : A_B \to A_F$, ie. the map of affinoid algebras between A_B and A_F sending z_{ij} to y_{ij}. Clearly ϕ is surjective.

Let z^t denote $\prod_{j\neq\jmath_i} z_{ij}^{t_{ij}}$. Let $t \in \mathbb{Z}_{\geq 0}$ denote that $t_{ij} \geq 0, \forall i, j$. With respect to the standard multiplicative norm $|f| := max|a_t|$, $f = \sum_{t\geq 0} a_t z^t \in A_B$, A_B is a $\mathbb{Q}_p$-Banach algebra. Since the ideal $\ker\phi$ is closed in A_B, see [4], A_F inherits a complete quotient norm defined by $|f| = \inf_{\xi\in\phi^{-1}(f)} |\xi|$, $f \in A_F$.

Let A_B^0 denote the unit ball $\mathbb{Z}_p < z_{ij} >$ in A_B. Define A_F^0 to be the unit ball of A_F with respect to the norm given above.

Define $\mathbb{Q}_p$-algebra homomorphisms $\pi : \mathbb{Q}_p[Z_{ij}] \to \mathbb{Q}_p[z_{ij}] \subset \mathbb{Q}_p < z_{ij} >$ by

$$f(Z_{ij}) \mapsto f(z_{ij})$$

and define similarly a map, by abuse of notation also denoted π, from $\mathbb{Q}_p[Y_{ij}] \to \mathbb{Q}_p[y_{ij}] \subset \mathbb{Q}_p < y_{ij} >$ by

$$f(Y_{ij}) \mapsto f(y_{ij}).$$

Lemma 2.5.1. *The following, induced by the action of* GSp_{2n} *on* B*, defines a right action of* Δ *on* A_B*. For* $\gamma \in \Delta$ *define*

$$\gamma z_{ij} = \frac{\pi(\gamma Z_{ij})}{\pi(\gamma Z_{i\jmath_i})}$$

and extend linearly and multiplicatively to A_B*.*

Proof: The only thing that is nonimmediate is that this expression for γz_{ij} is in A_B. From Lemma 2.4.1 we have that $\gamma\pi(Z_{i\jmath_i}) \in p^m(\mathbb{Z}_p^* + p\sum_{j\neq\jmath_i} \mathbb{Z}_p z_{ij})$, with $m \geq 0$ and that $|\pi(\gamma Z_{ij})| \leq p^{-m}$. Thus, following dividing through numerator and denominator in the expression for γz_{ij} by p^m it is clear that the denominator is a unit in A_B and thus γz_{ij} lies in A_B. $\square$

Similarly we define an actions of Δ on A_F by replacing z_{ij}'s by y_{ij}'s. Then we have $\phi(\gamma z_{ij}) = \gamma\phi(z_{ij}) = \gamma y_{ij}, \forall\gamma \in \Delta$, and thus the map $\phi : A_B \to A_F$ is also Δ equivariant. Thus we have:

Corollary 2.5.2. *This action descends to an action of Δ on A_F.*

Proof: By the observation above, the action of Δ on A_B preserves $\ker\phi$ and thus descends to an action of Δ on A_F. $\square$

Corollary 2.5.3. *If $g \in \Delta$ then we can express*

$$gz_{ij} = \frac{a_{\jmath_i} + \sum_{k\neq \jmath_i} a_k z_{ik}}{\lambda + p(\sum_{k\neq\jmath_i} b_k z_{ik})}$$

where $a_k, b_k \in \mathbb{Z}_p, \lambda \in \mathbb{Z}_p^*$. Thus gz_{ij} is in A_B^0.
Similarly for gy_{ij}.

Proof: This is a restatement of what precedes. $\square$

Lemma 2.5.4. *(i) A_B and A_F are orthonormalisable over $\mathbb{Q}_p$.*
(ii)The monoid Δ acts by continuous linear operators of norm ≤ 1 on A_B and A_F.
(iii)For α admissible, u^α acts completely continuously on A_B and A_F $\Leftrightarrow$ $\alpha_n < \alpha_{n-1} < \ldots < \alpha_1 < \alpha_{n+1} < \ldots < \alpha_{2n}$.

Proof: (i) For A_B, clearly monic monomials in the z_{ij} provide an orthonormal basis whereas for A_F it is clear that property (N) from [16] is satisfied and thus Proposition 1 and Lemma 1 of [16] imply A_F orthonormalisable.

(ii) From Corollary 2.5.3 that for $\gamma z_{ij} \in A_B^0$, $\gamma \in \Delta$, and thus as γ acts as $\mathbb{Q}_p$-algebra homomorphism on A_B, which preserves the unit ball A_B^0 and thus acts as a linear operator of norm ≤ 1.

For the maps induced by Δ on A_F, we have $f \in A_F^0$ means that $f = \phi(\xi)$, some $f \in A_B^0$ under ϕ. Then $\gamma\xi \in A_B^0$, $\forall\gamma \in \Delta$, from the above, and since $\gamma f = \phi(\gamma\xi) \in A_F^0$ so Δ also acts by continuous linear operators of norm ≤ 1 on A_F.

(iii) Choose an ordering for the basis of A_B consisting of monic monomials in the z_{ij}. Let η_{mn} denote the coefficient of the m-th basis monomial in the expression of the n-th monomial under the action of u^α. Note that $\eta_{mn} = 0$ if $m \neq n$ as u^α acts diagonally. Then $\alpha_n < \ldots < \alpha_1 < \alpha_{n+1} < \ldots < \alpha_{2n} \Leftrightarrow u^\alpha(z_{ij}) = p^k z_{ij}, k > 0, \forall i, j \Leftrightarrow lim_{n\to\infty}|\eta_{nn}| = 0 \Leftrightarrow u^\alpha$ acts completely continuously on A_B.

To show that u^α, for strictly increasing α, acts completely continuously on A_F we let $\{u_n^\alpha\}$ be a sequence of linear operators on A_B of finite rank converging to u^α. We can write $A_B = \ker\phi \oplus W$ and ensure that $u_n^\alpha = u^\alpha$ as maps on $\ker\phi$ by replacing u_n^α with $\widetilde{u_n^\alpha} = u^\alpha \circ \pi_{\ker\phi} + u_n^\alpha \circ \pi_W$. So $\widetilde{u_n^\alpha}$ preserve $\ker\phi$ and thus induce finite rank operators on A_F which converge to $u^\alpha \in \mathrm{End}(A_F)$. Thus u^α acts completely continuously on A_F. If α is not strictly increasing $\exists y_{ij}$ such that $u^\alpha y_{ij}^n = y_{ij}^n, \forall n$ and since these y_{ij}^n span an infinite dimensional subspace of A_F, the map u^α on A_F is not completely continuous. $\square$

2.6 Twisted Representation Spaces

Let $t = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. We construct a Δ-module $N_t := e_t A_B$ with $e_t := \prod_{i=1}^n Z_{ij_i}^{m_i}$. Let addition in N_t be induced by that of A_B and make N_t a Banach space by $|e_t b| := |b|$, $b \in A_B$. Then $\psi_t : A_B \to N_t$, $b \mapsto e_t b$ is an isometric isomorphism of Banach spaces.

Similarly, we construct $S_t := f_t A_F$ with $f_t := \prod_{i=1}^n Y_{ij_i}^{m_i}$ which we make into a Banach space by $|f_t c| = |c|$, $c \in A_F$ and we have a map which we again denote by $\psi_t : A_F \to S_t$. We also have a map of Banach spaces induced by ϕ which we will again denote $\phi : N_t \to S_t$.

We let Δ act on N_t and S_t as follows: For $g \in \Delta$ let

$$g(e_t b) = e_t \cdot \pi(g e_t) \cdot g b$$

for $b \in A_B$.

Similarly, define

$$g(f_t c) = f_t \cdot \pi(g f_t) \cdot g c$$

for $c \in A_F$.

Lemma 2.6.1. *For all $t = (m_1, \dots, m_n) \in \mathbb{N}^n$, the maps of Banach spaces*

$$F_t \to S_t : c \mapsto f_t \pi(c)$$

and

$$B_t \to N_t : b \mapsto e_t \pi(b)$$

are injective. These maps respect the actions of Δ.

Proof: In the second case, it is evident that $\pi : B_t \to A_B$ is injective and hence so is the map $B_t \to N_t$.

In the first case, we notice that the open subgroup

$$\Gamma_0(p)_{\mathbb{Z}_p} = \{\gamma \in \mathrm{Sp}_{2n}(\mathbb{Z}_p) | \quad \gamma \equiv \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \bmod p,$$
$$A \text{ lower triangular, } D \text{ upper triangular}\}$$

of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$ also acts on F_t and S_t. The kernel of the map $F_t \to S_t$ must contain an irreducible subrepresentation of $sp_{2n}(\mathbb{Q}_p)$, which is also the Lie algebra of the open subgroup $\Gamma_0(p)_{\mathbb{Z}_p}$. Since F_t is an irreducible representation of $sp_{2n}(\mathbb{Q}_p)$, the kernel of the map $F_t \to S_t$ is either trivial or all of F_t. But the kernel is not all of F_t since $f_t \mapsto f_t \neq 0$. Thus the kernel is trivial.

A simple calculation verifies that these maps are Δ equivariant. $\square$

We confirm Lemma 2.5.4 for N_t and S_t.

Lemma 2.6.2. *(i) N_t and S_t are orthonormalisable over $\mathbb{Q}_p$.*
(ii)The monoid Δ acts by continuous linear operators of norm ≤ 1 on N_t and S_t.
(iii)For α admissible, u^α acts completely continuously on N_t and $S_t \Leftrightarrow \alpha_n < \alpha_{n-1} < \ldots < \alpha_1 < \alpha_{n+1} < \ldots < \alpha_{2n}$.

Proof: The proofs of (i) and (ii) are simple modifications of those in Lemma 2.5.4.

(iii) The actions of u^α on N_t and S_t differ only by a factor of $p^{\sum_i m_i \beta_i}$ from the actions on A_B and A_F via the isometric isomorphism ψ_t. $\square$

Alternatively, we may describe N_t as follows: Let N_t have the same underlying Banach space as A_B but with the action of Δ twisted by

$$\gamma f = \prod_{i=1}^{n} (j_i(\gamma))^{m_i} \cdot \gamma b$$

for $b \in N_t, \gamma \in \Delta$, where j_i is the 1-cocycle defined by

$$j_i(\gamma) = \pi(\gamma Z_{ij_i}), \quad 1 \leq i \leq n.$$

Remark: Straightforward computation confirms that the j_i are indeed 1-cocyles, that is

$$j_i(\gamma\gamma') = j_i(\gamma) \cdot \gamma j_i(\gamma')$$

for $\gamma, \gamma' \in \Delta$.

Similarly, we may define S_t analogously with, by abuse of notation, the j_i now defined by

$$j_i(\gamma) = \pi(gY_{ij_i}), \quad 1 \leq i \leq n.$$

For ease of notation it is these descriptions that we shall use henceforth.

Given an action of a monoid M on a K-vector space V, we denote by V^* be the dual space of K-valued continuous linear functionals on V equipped with the dual action of M induced by the trivial action of Δ on K. We cite [16] 8.14 which states that the dual of a completely continuous endomorphism on a Banach space is again completely continuous. Thus

Proposition 2.6.3. *(i) N_t^* and S_t^* are orthonormalisable over $\mathbb{Q}_p$.*
(ii)The monoid Δ acts by continuous linear operators of norm ≤ 1 on N_t^* and S_t^*.
(iii)For α admissible, with $\alpha_n < \alpha_{n-1} < \ldots < \alpha_1 < \alpha_{n+1} < \ldots < \alpha_{2n}$, u^α acts completely continuously on N_t^* and S_t^*. □

2.7 Finiteness Properties of $\Gamma_0(p)$

Let R be a ring. Recall the following definitions from [5].

Definition 2.7.1. Let M be an R-module. A *resolution* of M is an exact sequence of R-modules

$$\ldots F_2 \to F_1 \to F_0 \to M \to 0.$$

A *partial resolution* is a sequence of R-modules

$$F_n \to \ldots \to F_1 \to F_0 \to M \to 0$$

which is exact at each F_i, $i < n$, and at M.
A resolution or partial resolution is said to be *projective* (resp. *free*) if each F_i is a projective (resp. free) R-module.

Definition 2.7.2. If M is an R-module, then a projective resolution or partial projective resolution (P_i) is said to be of *finite type* if each P_i is finitely generated.

Definition 2.7.3. An R-module M is said to be *of type FP_n* if there is a partial projective resolution $P_n \to P_{n-1} \to \ldots \to P_0 \to M$ with each P_i finitely generated as R-modules. A module is said to be *of type FP_∞* if these equivalent conditions hold:
(a) M admits a free resolution of finite type.
(b) M admits a projective resolution of finite type.
(c) M is of type FP_n for all $n \geq 0$.

Definition 2.7.4. We say a group Γ is of *type FP_n* $(0 \leq n \leq \infty)$ if $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}\Gamma$-module.

Definition 2.7.5. A projective resolution is said to be *finite* if it is both of finite type and finite length. A group Γ is said to be of *type FP* if $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}\Gamma$. A group Γ is said to be of *type FL* if $\mathbb{Z}$ admits a finite free resolution over $\mathbb{Z}\Gamma$.

We also recall

Definition 2.7.6. A subgroup Γ of $\mathrm{Sp}_{2n}(\mathbb{Q})$ is said to be *arithmetic* if it is commensurable with $\mathrm{Sp}_{2n}(\mathbb{Z})$ (ie. $\Gamma \cap \mathrm{Sp}_{2n}(\mathbb{Z})$ has finite index in both Γ and $\mathrm{Sp}_{2n}(\mathbb{Z})$).

We chain together standard results to the following end:

Proposition 2.7.7. *The subgroup $\Gamma_0(p)$ of Sp_{2n} is of type FP_∞.*

Proof: Proposition VIII.5.1 of [5] states that if $\Gamma^{'} \subset \Gamma$ a subgroup of finite index then, for all $n \geq 0$, Γ is of type FP_n if and only if $\Gamma^{'}$ is of type FP_n.

In chapter VIII §9 of [5], Brown refers to Borel and Serre's result from [3] that torsion free arithmetic subgroups are of type FL (and thus of type FP_∞).

In [17], Serre proves that any arithmetic subgroup has a torsion free subgroup of finite index. Thus, $\Gamma_0(p)$ has a torsion free subgroup of finite index, which is therefore of type FP_∞. And so from [5] so is $\Gamma_0(p)$. $\square$

2.8 The map U_p on Group Cohomology

Let us denote $\Gamma := \Gamma_0(p)$. Choose once and for all a free resolution

$$\ldots \to F_k \xrightarrow{\delta_k} F_{k-1} \to \ldots \to F_0 \to \mathbb{Z}$$

of finite type for $\mathbb{Z}$ as a $\mathbb{Z}\Gamma$-module. Let $r(k)$ be the rank of free $\mathbb{Z}\Gamma$ - module F_k. Fix also generators $x_1^k, \ldots, x_{r(k)}^k$ of for F_k. Let V be a $\mathbb{Q}_p$-Banach space and $\mathbb{Z}\Gamma$-module. Then $C^k(\Gamma, V) \cong V^{r(k)}$ as $\mathbb{Z}\Gamma$-modules via $f \mapsto (f(x_1^k), f(x_2^k), \ldots, f(x_{r(k)}^k))$. Give $C^k(\Gamma, V)$ the structure of a Banach space induced by this isomorphism and the sup norm on $V^{r(k)}$, so that for $f \in C^k(\Gamma, V)$, $\parallel f \parallel = sup_{1 \leq i \leq r(k)} |f(x_i^k)|$.

Assume henceforth that each $\gamma \in \Gamma$ acts as a bounded linear operator on V. Then the boundary maps $\delta_k : C^{k-1}(\Gamma, V) \to C^k(\Gamma, V)$ are continous, and so for each $k \in \mathbb{Z}_{\geq 0}$, $Z^k := Z^k(\Gamma, V)$ is closed in $C^k := C^k(\Gamma, V)$ as the preimage of a closed point under a continuous map. However, $B^k := B^k(\Gamma, V)$ may not be closed in Z^k. We essentially follow [2], to prove the following result.

Lemma 2.8.1. *V a Banach space over a valued field K. Let R be the unit ball in K. Assume that R is compact. Let V_0^* be the unit ball in V^* with respect to the strong topology on V^* induced by the operator norm and assume Γ acts by continuous linear operators of norm ≤ 1 on V, thus* *making V_0^* a $\mathbb{Z}\Gamma$-module. If Γ is FP_∞ then for each k and with respect to any resolution of finite type, $B^k(\Gamma, V_0^*)$ is closed in $Z^k(\Gamma, V_0^*)$.*

Proof: Define the weak topology on V_0^* by saying that a sequence converges for the weak topology in V_0^* if and only if its images converge in K for any $v \in V$.

Then V_0^* is compact in the weak topology as follows: Let $\{f_i\}$ be a sequence in V_0^*. Let $\{e_i\}$ be a basis for V with $|e_i| \leq 1$ then any subsequence of $\{f_i(e_j)\}$, j fixed, has a convergent subsequence. Pick a subsequence $\{f_{1i}\}$ of $\{f_i\}$ such that $\{f_{1i}(e_1)\}$ converges to some $a_1 \in R$. Now pick a subsequence $\{f_{2i}\}$ of $\{f_{1i}\}$ such that $\{f_{2i}(e_2)\}$ converges to some $a_2 \in R$. Continue so that for each n we get a subsequence $\{f_{ni}\}$ of $\{f_i\}$ such that $\{f_{ni}(e_j)\}$ converges to $a_j \in R$ for all j fixed, $j \leq n$. Take $f_i^{'} = f_{ii}$. Then for each $N \in \mathbb{N}$, the tail of $\{f_i^{'}\}$ is in f_{Ni} so $f_i^{'}(e_N)$ converges to a_N. Thus $f_i^{'}$ converges to f as defined by $f(e_i) = a_i$ in the weak topology.

Now choose a projective resolution of finite type, $P = (P_i)$, of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and a basis of each P_i thus giving an isomorphism $C^k(\Gamma, V_0^*)$ with $(V_0^*)^r$ some integer r.

Now let $b_i \in B^k(\Gamma, V_0^*)$ converge to $c \in C^k(\Gamma, V_0^*)$. Then it also converges to c in the weak topology. Let $b_i = \delta a_i$. Passing to a subsequence a_i converges to some a in the weak topology. Since δ is continuous in both topologies a subsequence of the b_i converges to δa. Since the weak topology is Hausdorff we have $\delta a = c$. $\square$

Definition 2.8.2. Let A be normed ring in the sense of [4]. By a *Banach A-module* we shall mean a normed A-module in the sense of [4] that is in addition complete.

Corollary 2.8.3. *Lemma 2.8.1 gives $H^k(\Gamma_0(p), M_0^*)$ the structure of a Banach $\mathbb{Z}_p$-module where M_0^* is the unit ball in the dual of M, for $M = N_t, S_t$. Thus $H^k(\Gamma_0(p), M_0^*)\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Q}_p$ is a Banach space over $\mathbb{Q}_p$ for $M = N_t, S_t$.*

Proof: For each of the above choices of M, $C^k(\Gamma, M_0^*)$ is isometrically isomorphic to $(M_0^*)^{r(k)}$ equipped with the sup norm and is thus a Banach

$\mathbb{Z}_p$-module. Then $Z^k(\Gamma, M_0^*)$ is closed in $C^k(\Gamma, M_0^*)$ and complete and thus a Banach $\mathbb{Z}_p$-module.

As $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$ and Γ acts on M by linear operators of norm ≤ 1, the conditions of Lemma 2.8.1 are satisfied and $B^k(\Gamma, M_0^*)$ is closed in $Z^k(\Gamma, M_0^*)$. Thus, by [4] §2.1.2 Proposition 3, $H^k(\Gamma, M_0^*)$ equipped with the residue norm is complete and thus a Banach $\mathbb{Z}_p$-module. Clearly then, $H^k(\Gamma_0(p), M_0^*)\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Q}_p$ is a $\mathbb{Q}_p$-Banach space. $\square$

Fix admissible α such that $\alpha_n < \alpha_{n-1} < \ldots < \alpha_1 < \alpha_{n+1} < \ldots < \alpha_{2n}$ and let u denote u^α. Fix a coset decomposition for $\Gamma u \Gamma = \coprod u_i \Gamma$, then by [1] Lemma 3.1.2 we have a coset decomposition $\Gamma = \coprod \beta_i(\Gamma \cap u\Gamma u^{-1})$, with $\beta_i = u_i u^{-1}$. From [1] Lemma 3.3.1 we have that the commensurator of Γ is $\mathrm{GSp}_{2n}^+(\mathbb{Q})$ and thus these coset decompositions are finite, say $\Gamma u \Gamma = \coprod_{i=1}^N u_i \Gamma$.

Recall our resolution F_* of $\mathbb{Z}$ by free, finitely generated $\mathbb{Z}\Gamma$-modules. We may use this resolution to compute the cohomology of Γ and $u^{-1}\Gamma u \cap \Gamma$. Define $\rho : u\Gamma u^{-1} \to \Gamma$, $u\gamma u^{-1} \mapsto \gamma$. For the group $u\Gamma u^{-1} \cap \Gamma$ we may use the resolution $F^\bullet$ where the underlying groups are the same as F but the group action is defined by $gf^\bullet = (\rho(g)f)^\bullet$, $g \in u\Gamma u^{-1}$. Define $\tau : F^\bullet \to F$ to be the map sending $f^\bullet \mapsto f$, ie. the identity map on underlying spaces. Then τ is a chain map compatible with ρ and is a homotopy equivalence between the two $u\Gamma u^{-1} \cap \Gamma$ resolutions $F^\bullet$ and F.

Let M be Δ-module. By definition the Hecke operator U_p at p is $U_p := tr \circ \Phi \circ res$, where res, Φ and tr are defined as follows:

$res : H^*(\Gamma, M) \to H^*(u^{-1}\Gamma u \cap \Gamma, M)$ is the map induced by the restriction map $Hom_\Gamma(F, M) \to Hom_{u^{-1}\Gamma u \cap \Gamma}(F, M)$.

$\Phi : H^*(u^{-1}\Gamma u \cap \Gamma, M) \to H^*(u\Gamma u^{-1} \cap \Gamma, M)$, is the map induced by $\rho|_{u\Gamma u^{-1} \cap \Gamma}$ and $f : M \to M$, $m \mapsto um$. Then, using the notation from [5] § III.8 (ρ, f) is a pair in $\mathcal{D}$. So the map Φ on cocycles

$$\Phi : Hom_{u^{-1}\Gamma u \cap \Gamma}(F, M) \to Hom_{u\Gamma u^{-1} \cap \Gamma}(F^\bullet, M)$$

is given by

$$\Phi(\mu)(x) = u\mu(\tau(x)),$$

for $\mu \in Hom_{u^{-1}\Gamma u \cap \Gamma}(F, M)$ and $x \in F^\bullet$.

tr: The map on cochains:

$$tr : Hom_{u\Gamma u^{-1}\cap\Gamma}(F_k, M) \to Hom_\Gamma(F_k, M)$$

defined by

$$tr(\mu)(x) = \sum \beta_i \mu(\beta_i^{-1}x)$$

commutes with δ and thus induces a map on cohomology

$$H^k(u\Gamma u^{-1} \cap \Gamma, M) \to H^k(\Gamma, M).$$

The induced map on cohomology agrees with the map tr in [5] $\S III.9$ on H^0.

Consider the following two cohomological functors on $\mathbb{Z}\Gamma$ modules: $S = (S_k)$, $S_k(M) = H^k(u\Gamma u^{-1} \cap \Gamma, M)$, $T = (T_k), T_k(M) = H^k(\Gamma, M)$. There is a proof in [5] $\S III.6$ Proposition 6.1, that S and T are coeffacable in dimension $i > 0$ on the category of $\mathbb{Z}\Gamma$-modules.

We verify straightforwardly that tr commutes with connecting homomorphisms and is natural on H^0 so it is the unique map of δ-functors extending tr on H^0, see [5] $\S III.7$ Theorem 7.5, and is thus is the transfer map of [5] $III.9$.

Let the Δ-module M be also an orthonormalisable $\mathbb{Q}_p$-Banach space such that Γ acts on M by continuous operators and u acts completely continuously on M. Endow the cochains $Hom_\Gamma(F_k, M)$ with the Banach space structure as described above:

$$\| f \| = sup_{1\leq i\leq r(k)} |f((x_i^k))| \text{ for } f \in Hom_\Gamma(F_k, M).$$

As $(\Gamma : u^{-1}\Gamma u \cap \Gamma) < \infty$, F is also a resolution of finite type for $u^{-1}\Gamma u \cap \Gamma$ with F_k generated as a $\mathbb{Z}(u^{-1}\Gamma u \cap \Gamma)$-module by the finite set $\{\beta_j^{-1} x_i^k\}$. For convenience let us fix an order for the set $\{\beta_j^{-1} x_i^k\}_{i=1,\dots,r(k),j=1,\dots,N}$ and denote its elements by $\{y_i^k\}_{i=1\dots,Nr(k)}\}$. Define the Banach norm $\| f \| = sup_i |f(y_i^k)|$ on $Hom_{u^{-1}\Gamma u\cap\Gamma}(F_k, M)$ and similarly define the Banach norm $\| f \| = sup_i |f((y_i^k)^\bullet)|$ on $Hom_{u\Gamma u^{-1}\cap\Gamma}(F_k, M)$.

To simplify notation somewhat let us denote $Hom_{u^{-1}\Gamma u\cap\Gamma}(F_k, M)$ by $C^k(u^{-1}\Gamma u \cap \Gamma, M)$ and the cocycles and coboundaries in $C^k(u^{-1}\Gamma u \cap \Gamma, M)$ by $Z^k(u^{-1}\Gamma u \cap \Gamma, M)$ and $B^k(u^{-1}\Gamma u \cap \Gamma, M)$ respectively. Similarly let us denote $Hom_{u\Gamma u^{-1}\cap\Gamma}(F_k^\bullet, M)$ by $C^k(u\Gamma u^{-1}\cap\Gamma, M)$ and the cocycles and coboundaries in $C^k(u\Gamma u^{-1}\cap\Gamma, M)$ by $Z^k(u\Gamma u^{-1}\cap\Gamma, M)$ and $B^k(u\Gamma u^{-1}\cap\Gamma, M)$ respectively.

We now provide conditions sufficient for Φ to be completely continuous on cochains.

Proposition 2.8.4. *With M as above*

$$\Phi : C^k(u^{-1}\Gamma u\cap\Gamma, M) \to C^k(u\Gamma u^{-1}\cap\Gamma, M)$$

is completely continuous.

Proof: Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of M. Define orthonormal bases $\{\mu_i, m\}$ and $\{\mu_i^\bullet, m\}$, $i \in \{1, \ldots, Nr(k)\}$, $m \in \mathbb{N}$ of $C^k(u^{-1}\Gamma u\cap\Gamma, M)$ and $C^k(u\Gamma u^{-1}\cap\Gamma, M)$ respectively by

$$\mu_{i,m}(y_j^k) = \delta_{ij}e_m$$

and

$$\mu_{i,m}^\bullet((y_j^k)^\bullet) = \delta_{ij}e_m.$$

Order these bases as follows: $\mu_{1,1}, \ldots, \mu_{r(k),1}, \mu_{1,2}, \ldots, \mu_{r(k),2}, \ldots$ and similarly for the $\mu_{i,m}^\bullet$.

We compute $\Phi\mu_{i,m}$ by looking at $(\Phi\mu_{i,m})((y_j^k)^\bullet)$. We have

$$(\Phi\mu_{i,m})((y_j^k)^\bullet) = u\mu_{i,m}(\tau((y_j^k)^\bullet)) = u\mu_{i,m}(y_j^k) = \delta_{ij}ue_m.$$

If we write $ue_m = \sum_{n\in\mathbb{N}} a_{nm}e_n$ then, from [16], $lim_{n\to\infty}sup_m|a_{nm}| = 0$.

We have

$$(\Phi\mu_{i,m}((y_j^k)^\bullet) = \delta_{ij}\sum_{n\in\mathbb{N}} a_{nm}e_n.$$

So

$$\Phi\mu_{i,m} = \sum_{n\in\mathbb{N}} a_{nm}\mu_{i,n}^\bullet.$$

If we write

$$\Phi\mu_{i,m} = \sum_{n\in\mathbb{N}} A_{imn}\mu_{i,n}$$

then Φ is completely continuous if and only if

$$lim_{n\to\infty}sup_{m\in N, i\in\{1,\ldots,r(k)\}}|A_{imn}| = 0$$

but as we have just seen $A_{imn} = a_{mn}$ so

$$\Phi : C^k(u^{-1}\Gamma u \cap \Gamma, M) \to C^k(u\Gamma u^{-1} \cap \Gamma, M)$$

is completely continuous. $\square$

We define completely continuous maps of Banach modules: Let A be a normed ring and let M, N be orthonormalisable Banach A-modules.

Definition 2.8.5. For $L \in \mathcal{L}(M, N)$, if there exist orthonormal bases $\{e_i\}$ and $\{d_i\}$ of M and N respectively such that $L(e_i) = \sum_j a_{ji} d_j$ with

$$lim_{j\to\infty} sup_i |a_{ji}| = 0 \tag{$*$}$$

then we say L is a completely continuous map of Banach A-modules.

We now have the following corollary to Proposition 2.8.4.

Corollary 2.8.6. *The map Φ restricts to a completely continuous map of Banach $\mathbb{Z}_p$-modules*

$$C^k(u^{-1}\Gamma u \cap \Gamma, M_0) \to C^k(u\Gamma u^{-1} \cap \Gamma, M_0)$$

for $M = N_t^*, S_t^*$.

Proof: We note that $C^k(u^{-1}\Gamma u \cap \Gamma, M_0)$ and $C^k(u\Gamma u^{-1} \cap \Gamma, M_0)$ are the unit balls in $C^k(u^{-1}\Gamma u \cap \Gamma, M)$ and $C^k(u\Gamma u^{-1} \cap \Gamma, M)$ respectively. Since the action of Δ on M preserves the unit ball, so does Φ and thus we get a restricted map

$$\Phi : C^k(u^{-1}\Gamma u \cap \Gamma, M_0) \to C^k(u\Gamma u^{-1} \cap \Gamma, M_0)$$

of Banach $\mathbb{Z}_p$-modules.
Furthermore for any orthonormal basis $\{e_i\}$ of $C^k(u^{-1}\Gamma u \cap \Gamma, M)$, $\{e_i\}$ is also an orthonormal basis for the unit ball $C^k(u^{-1}\Gamma u \cap \Gamma, M_0)$. Similarly for $C^k(u\Gamma u^{-1} \cap \Gamma, M)$ and its unit ball. Thus choosing $\{e_i\}$ and $\{f_i\}$ such that

$$\Phi(e_i) = \sum_j a_{ji} e_j$$

with $a_{ji} \in \mathbb{Z}_p$ and $lim_{j \to \infty} sup_i |a_{ji}| = 0$ we confirm that

$$\Phi : C^k(u^{-1}\Gamma u \cap \Gamma, M_0) \to C^k(u\Gamma u^{-1} \cap \Gamma, M_0)$$

is completely continuous. $\square$

In what follows we need the following observation:

Lemma 2.8.7. *Let K be a complete, non-archimedean field and V a Banach space over K. Let W be a closed subspace of V. Then any orthonormal basis for W can be extended to an orthonormal basis of V.*

Proof: Let $\{e_i\}_{i \in I'}$ be an orthonormal basis for W. Then by [16] Lemma 1.1, the images $\overline{e_i}$ in $\overline{W}$, the reduction of W mod the maximal ideal of K, form an algebraic basis and are thus linearly independent in $\overline{V}$. Thus we can extend to a basis $\{\overline{e_i}\}_{i \in I}$, $I^{'} \subset I$ of $\overline{V}$. If we take e_i, $i \in I - I^{'}$ to be any lift of $\overline{e_i}$ then again [16] Lemma 1.1 says that $\{e_i\}$, $i \in I$ is an orthonormal basis of V, extending the basis $\{e_i\}_{i \in I'}$ of W. $\square$

The map Φ commutes with the maps δ_i and thus maps cocycles to cocycles and coboundaries to coboundaries. The above lemma allows us to prove the following.

Lemma 2.8.8. *The map Φ from $Z^k(u^{-1}\Gamma u \cap \Gamma, M_0)$ to $Z^k(u\Gamma u^{-1} \cap \Gamma, M_0)$ is completely continuous for $M = N_t^*, S_t^*$.*

Proof: Note that both $C^k(G, M)$ and $Z^k(G, M)$, $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$ satisfy condition (N) from [16].

We have $Z^k(u^{-1}\Gamma u \cap \Gamma, M)$ is closed in $C^k(u^{-1}\Gamma u \cap \Gamma, M)$ so choose an orthonormal basis $\{e_i\}_{i \in I'}$ of $Z^k(u^{-1}\Gamma u \cap \Gamma, M)$ and extend it to an orthonormal basis, $\{e_i\}_{i \in I}$, $I^{'} \subset I$, of $C^k(u^{-1}\Gamma u \cap \Gamma, M)$.

Similarly choose an orthonormal basis $\{f_j\}_{j \in J'}$ of $Z^k(u\Gamma u^{-1} \cap \Gamma, M)$ and extend it to an orthonormal basis, $\{f_j\}_{j \in J}$, $J^{'} \subset J$, of $C^k(u\Gamma u^{-1} \cap \Gamma, M)$.

As before $\{e_i\}_{i \in I}$ is also an orthonormal basis of $C^k(u^{-1}\Gamma u \cap \Gamma, M_0)$ and $\{e_i\}_{i \in I'}$ is also an orthonormal basis of $Z^k(u^{-1}\Gamma u \cap \Gamma, M_0)$. Similarly for $\{f_j\}_{j \in J}$ and $\{f_j\}_{j \in J'}$. Then if we write $\Phi(e_i) = \sum_j a_{ji} f_j$, Φ completely continuous from $C^k(u^{-1}\Gamma u \cap \Gamma, M_0)$ to $C^k(u\Gamma u^{-1} \cap \Gamma, M_0)$ implies the a_{ji},

$i \in I, j \in J$ satisfy condition $(*)$ and thus the a_{ji}, $i \in I^{'}, j \in J^{'}$ also satisfy condition $(*)$ and we are done. $\square$

We now prove some auxiliary results on the way to proving that U_p is completely continuous on $H^k(\Gamma, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $M = N_t^*, S_t^*$.

First we notice that as in Corollary 2.8.3 the groups $H^k(G, M_0)$ for $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$ and $M = N_t^*, S_t^*$ are Banach $\mathbb{Z}_p$-modules.

Let R be a normed ring. Define the following property for a normed R-module M:

$$\forall x, y \in M, |x| < |y| \Rightarrow |x+y| = |y|. \qquad (P1)$$

Lemma 2.8.9. *The $\mathbb{Q}_p$-Banach space N_t satisfies $(P1)$.*

Proof: The space N_t is isometrically isomorphic to $A_B = \mathbb{Q}_p < z_{ij} >$. For $g \in A_B$, $|g| = sup(a_t)$, where $g = \sum a_t z^t$. The result then follows as $(P1)$ holds in $\mathbb{Q}_p$. $\square$

Lemma 2.8.10. *The quotient of a $(P1)$ normed $\mathbb{Q}_p$-vector space or $\mathbb{Z}_p$-module, M, by a closed submodule N is again $(P1)$.*

Proof: Let $\bar{x}, \bar{y} \in M/N$, with $0 < |\bar{x}| < |\bar{y}|$, and $x, y \in M$ lifts of $\bar{x}, \bar{y} \in M/N$. Since the norm on $\mathbb{Q}_p$ is discrete away from 0, there is an $\tilde{x} \in M$, such that

$$|\bar{x}|_{M/N} = inf_{n \in N}|x+n|_M = |\tilde{x}|_M.$$

Then

$$|\tilde{x}|_M = |\bar{x}|_{M/N} < inf_{n \in N}|y+n|_M = |\bar{y}|_{M/N}$$

and thus

$$|\bar{x}+\bar{y}|_M = inf_{n \in N}|x+y+n|_M = inf_{n \in N}|\tilde{x}+y+n|_M = inf_{n \in N}|y+n|_M = |\bar{y}|_{M/N}$$

since M is $(P1)$. $\square$

Thus S_t is $(P1)$.

Lemma 2.8.11. *If R is a normed ring, M is a normed R-module and R satisfies ($P1$) then the dual M^* of continuous R linear maps from M to R also satisfies ($P1$).*

Proof: This follows straightforwardly from the definitions. $\square$

Clearly the unit ball in a vector space satisfying condition ($P1$) also satisfies ($P1$), thus M_0 and further $C^k(G, M_0)$, $M = S_t^*, N_t^*$, $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$ satisfy condition ($P1$).

Define the following property for a normed R-module M:

$$\forall a \in R, \forall x \in M, |ax| \neq |a||x| \Rightarrow |ax| = 0. \qquad (P2)$$

This property is clearly satisfied by unit balls in $\mathbb{Q}_p$-vector spaces and thus by M_0 and further $Z^k(G, M_0), B^k(G, M_0)$, $M = N_t^*, S_t^*$, $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$. It is also satisfied by $H^k(G, M_0)$ with M as above, with $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$:

Lemma 2.8.12. *The cohomology groups $H^k(G, M_0)$, $M = S_t^*, N_t^*$, $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$, satisfy condition ($P2$).*

Proof: Let $a \in \mathbb{Z}_p$, $\bar{x} \in H^k(G, M_0)$ be the image of $x \in Z^k(G, M_0)$. Then

$$|a\bar{x}| = inf_{n \in B^k(G,M_0)}|ax + n| = inf_{|n| \leq |ax|}|ax + n|,$$

since $Z^k(G, M_0)$ satisfies ($P1$). Then as $B^k(G, M_0)$ is isometrically isomorphic to a subspace of $M_0^{Nr(k)}$ and since M_0 is the unit ball in a vector space, for each $n \in B^k(G, M_0)$, with $|n| \leq |a|$, there exists $\tilde{n} \in B^k(G, M_0)$, with $n = a\tilde{n}$. Thus

$$|a\bar{x}| = inf_{n \in B^k(G,M_0)}|a(x + n)|$$

and since $Z^k(G, M_0)$ satisfies ($P2$), $|a\bar{x}| = |a||\bar{x}|$ or $|a\bar{x}| = 0$. $\square$

Lemma 2.8.13. *If M is a normed $\mathbb{Z}_p$-module satisfying condition ($P2$) then for any $g \in M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, there exist $m \in M, q \in \mathbb{Q}_p$ such that $g = m \otimes q$ and $|g| = |m||q|$.*

Proof: We have $|g| = inf\{max_i|m_i||q_i|\}$ where the infimum is over all possible representations of of $g = \sum m_i \otimes q_i$. Let $g = \sum_i m_i \otimes q_i$ be a representation of g. Choose an $A \in \mathbb{Q}_p$ with $|A|$ minimized such that $q_i = Az_i$ with $z_i \in \mathbb{Z}_p$, for all i. Then $g = (\sum z_i m_i) \otimes A$. Also

$$|\sum z_i m_i||A| \leq max|z_i m_i||A| \leq max|z_i||m_i||A| = max|m_i||z_i A| = max|m_i||q_i|.$$

Thus $|g| = inf|m||q|$ over all representations $g = m \otimes q$.

Since any two representations $g = m_1 \otimes q_1 = m_2 \otimes q_2$ differ only by a factor of $z \in \mathbb{Z}_p$, WLOG $m_1 = zm_2$ say, and since M satisfies $(P2)$ then either $|m_1||q_1| = |m_2||q_2|$ or $|m_1| = 0$. Thus $|m||q|$ can only be one of two values for $g = m \otimes q$ and thus there exist $m \in M, q \in \mathbb{Q}_p$ such that $g = m \otimes q$ and $|g| = |m||q|$. $\square$

Lemma 2.8.14. *For $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$ and $\mu \in H^k(G, M_0)$ such that $0 < |\mu| < 1$, with M_0 the unit ball in a $\mathbb{Q}_p$-vector space, there exists $a \in \mathbb{Z}_p$, $\mu_0 \in H^k(G, M_0)$ such that $|\mu_0| = 1$ and $a\mu_0 = \mu$.*

Proof: Let $|\mu| = p^{-m}$, $m \in \mathbb{N}$. Let $u^{'} \in Z^k(G, M_0)$ be a lift of μ such that $|\mu^{'}| = p^{-m}$. Then define $\mu_0^{'} = p^{-m}u^{'}$. Then we check easily that both $\mu_0^{'}$ and its reduction μ_0 to $H^k(G, M_0)$ are of norm 1 and $\mu_0 = p^{-m}u$. $\square$

Proposition 2.8.15. *For $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$, the space $H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a $\mathbb{Q}_p$-Banach space, for $M = S_t^*, N_t^*$, so*

$$H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^k(G, M_0) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof: Observe that Lemma 2.8.13 means that $g \in H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $|g| = 0$, implies $g = 0$.

Let $\{x_n\}$ be a Cauchy sequence in $H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and fix representations $x_n = m_n \otimes q_n$ such that $|x_n| = |m_n||q_n|$. If infinitely many $x_n = 0$ then the sequence converges to zero and thus we may assume WLOG that none of the x_n are zero.

As $\{x_n\}$ is Cauchy it is bounded. By Lemma 2.8.14 we may assume $|m_n| \geq 1$ for all $n \in \mathbb{N}$. Thus we can choose an n_0 such that $|q_{n_0}| = sup_n|q_n|$. Then write $q_n = a_n q_{n_0}$ with $a_n \in \mathbb{Z}_p$ for all $n \in \mathbb{N}$ and thus we have also $x_n = (a_n m_n) \otimes q_{n_0} =: \tilde{m_n} \otimes q_{n_0}$ also with $|x_n| = |\tilde{m_n}||q_{n_0}|$. Then the sequence $\{\tilde{m_n}\}$ is Cauchy in M and thus converges to $m \in M$ say. Then $x_n \to x := m \otimes q_{n_0}$. □

We also observe

Lemma 2.8.16. *For $G = u^{-1}\Gamma u \cap \Gamma$ or $u\Gamma u^{-1} \cap \Gamma$ and $M = S_t^*, N_t^*$, we have*

$$\frac{Z^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p}{B^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} = H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof: This follows from flatness of $\mathbb{Q}_p$ over $\mathbb{Z}_p$. We have

$$0 \to B^k(G, M_0) \xrightarrow{i} Z^k(G, M_0) \xrightarrow{\pi} H^k(G, M_0) \to 0$$

and thus by flatness have

$$0 \to B^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{i \otimes 1} Z^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\pi \otimes 1} H^k(G, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

as desired. □

We have

Proposition 2.8.17. *For $M = S_t^*, N_t^*$, there is a map*

$$\Phi_p \otimes 1 : Z^k(u^{-1}\Gamma u \cap \Gamma, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to Z^k(u\Gamma u^{-1} \cap \Gamma, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

such that

$$(\Phi \otimes 1)(m \otimes q) = \Phi(m) \otimes q.$$

This map is completely continuous as a map of $\mathbb{Q}_p$-Banach spaces. It commutes with the maps δ_k and reduces to a completely continuous map of $\mathbb{Q}_p$-Banach spaces from

$$H^k(u^{-1}\Gamma u \cap \Gamma, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^k(u\Gamma u^{-1} \cap \Gamma, M_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof: The first statement is merely [4] 2.1.7 Proposition 5. As in the proof of Lemma 2.8.8 we can choose an orthonormal basis $\{e_i\}_{i\in I'}$ of $Z^k(u^{-1}\Gamma u\cap\Gamma, M_0)$ and a basis $\{f_j\}_{j\in J'}$ of $Z^k(u\Gamma u^{-1}\cap\Gamma, M_0)$ such that $\Phi(e_i) = \sum_j a_{ji}f_j$ and a_{ji}, $i \in I^{'}, j \in J^{'}$ satisfy condition $(*)$. Then $\{e_i\otimes 1\}_{i\in I'}$ is an orthonormal basis of $Z^k(u^{-1}\Gamma u\cap\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\{f_j\otimes 1\}_{j\in J'}$ is an orthonormal basis of $Z^k(u\Gamma u^{-1}\cap\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\Phi(e_i\otimes 1) = \sum_j a_{ji}f_j\otimes 1$ which proves the second statement. The map Φ commutes with boundary maps and thus induces a map on cohomology. Arguments similar to those in Lemma 2.5.4(iii) confirm that the induced map on cohomology is completely continuous. $\square$

Finally we confirm

Proposition 2.8.18. *For $M = S_t^*, N_t^*$, the maps res and tr defined earlier in the section give rise to continuous maps*

$$res\otimes 1 : H^k(\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to H^k(u^{-1}\Gamma u\cap\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

defined by $(res\otimes 1)(m\otimes q) = res(m)\otimes q$ and

$$tr\otimes 1 : H^k(u\Gamma u^{-1}, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to H^k(\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

defined by $(tr\otimes 1)(m\otimes q) = tr(m)\otimes q$.
The composition

$$U_p\otimes 1 := (tr\otimes 1)\circ(\Phi\otimes 1)\circ(res\otimes 1) = (tr\circ\Phi\circ res)\otimes 1$$

on $H^k(\Gamma, M_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is completely continuous. By abuse of notation we will denote $U_p\otimes 1$ by U_p.

Proof: Recall from [16] that if E, V, W are Banach spaces and $u \in \mathcal{L}(E,V)$, $v \in \mathcal{L}(V,W)$ then $v\circ u \in \mathcal{L}(E,W)$ is completely continuous if and of if u or v is. This confirms that $U_p\otimes 1$ is completely continuous. Everything else is immediate. $\square$.

2.9 Forms of Small Slope

Let $t = (t_1, \dots, t_n) \in \mathbb{N}^n$. Define $Q_t = N_t/B_t$. Then we have an exact sequence of Δ-modules

$$0 \to B_t \xrightarrow{i} N_t \xrightarrow{j} Q_t \to 0.$$

Furthermore, if we define $N_{t,cl}$ to be the subspace of N_t with (finite) orthonormal basis $\{e_t \prod_{j \neq \jmath_i} z_{ij}^{m_{ij}} | \sum_{j \neq \jmath_i} m_{ij} \leq t_i, \forall i\}$ and $N_{t,ncl}$ to be the subspace of N_t with (infinite) orthonormal basis $\{e_t \prod_{j \neq \jmath_i} z_{ij}^{m_{ij}} | \sum_{j \neq \jmath_i} m_{ij} > t_i, \text{ for some } i\}$ then $N_t = N_{t,cl} \oplus N_{t,ncl}$ and $i(B_t) = N_{t,cl}$ isometrically. Note that $N_{t,ncl}$ is not preserved by the action of Δ. Furthermore, there is a continuous section of the map j which we will denote by θ, which maps and element $\bar{q} \in Q_t$ to the unique preimage $q \in N_t$ of $\bar{q}$ such that the coefficients of terms in the set $\{e_t \prod_{j \neq \jmath_i} z_{ij}^{m_{ij}} | \sum_{j \neq \jmath_i} m_{ij} \leq t_i, \forall i\}$ are zero. Then $\theta(Q_t) = N_{t,ncl}$ where θ is an isometry of Banach spaces but not Δ-equivariant.

We have an exact sequence of Δ-modules with an isometric section of j (considered as a map of Banach spaces)

$$0 \to B_t \xrightarrow{i} N_{t,cl} \oplus N_{t,ncl} \underset{\theta}{\overset{j}{\rightleftarrows}} Q_t \to 0$$

and so by taking continuous duals we get

$$0 \to Q_t^* \underset{\theta^*}{\overset{j^*}{\rightleftarrows}} N_{t,ncl}^* \oplus N_{t,cl}^* \xrightarrow{i^*} B_t^* \to 0$$

where again i^* and θ^* induce isometric isomorphisms of Banach spaces $N_{t,cl}^* \cong B_t^*$ and $N_{t,ncl}^* \cong Q_t^*$ respectively where i^* and j^* are Δ-equivariant.

If we restrict to the unit ball on the left and right we get

$$0 \to (Q_t^*)_0 \underset{\theta^*}{\overset{j^*}{\rightleftarrows}} (N_{t,ncl}^*)_0 \oplus (N_{t,cl}^*)_0 \xrightarrow{i^*} (B_t^*)_0 \to 0$$

with again $(Q_t^*)_0$ and $(B_t^*)_0$ isometrically isomorphic as Banach spaces to $(N_{t,ncl}^*)_0$ and $(N_{t,cl}^*)_0$ respectively.

Similarly, if we define $P_t = S_t/F_t$ then reducing

$$0 \to B_t \xrightarrow{i} N_{t,cl} \oplus N_{t,ncl} \underset{\theta}{\overset{j}{\rightleftarrows}} Q_t \to 0$$

modulo $ker(\phi : B_t \to S_t)$ we get

$$0 \to F_t \xrightarrow{i} S_{t,cl} \oplus S_{t,ncl} \underset{\theta}{\overset{j}{\rightleftarrows}} P_t \to 0$$

and proceed analogously to get

$$0 \to (P_t^*)_0 \underset{\theta^*}{\overset{j^*}{\rightleftarrows}} (S_{t,ncl}^*)_0 \oplus (S_{t,cl}^*)_0 \xrightarrow{i^*} (F_t^*)_0 \to 0$$

with $(P_t^*)_0$ and $(F_t^*)_0$ isometrically isomorphic as Banach spaces to $(S_{t,ncl}^*)_0$ and $(S_{t,cl}^*)_0$ respectively, where i^* and j^* are Δ-equivariant.

Definition 2.9.1. If V is a p-adic Banach space on which an operator U acts completely continuously, $f \in V$ and $h \in \mathbb{Q}$, we say U *acts with slope* h *on* f if there exists $P(X) \in \mathbb{Q}_p[X]$ such that $P(U)f = 0$ and all of the roots of $P(X)$ in $\mathbb{C}_p$ have p-adic valuation h.

We define V^h to be the subspace of V spanned by vectors on which U acts with slope h.

The slopes we will discuss in what follows will be slopes for U_p.

We also note that for the finite dimensional spaces $M = B_t, F_t$, we have $H^k(\Gamma, M_0^*) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^k(\Gamma, M^*)$ and furthermore remark that in [14] §31.6 it is established that $F_t^* \cong F_t$ as algebgraic representations of $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$.

Proposition 2.9.2. *Let* $h \in \mathbb{Q}$ *with* $h < \lambda := \sum_j (t_j (\sum_{k=1}^j \alpha_{n-k+1})) + min_{i=1}^n (t_i + 1)(\alpha_{n-i} - \alpha_{n-i+1})$ *where for the purposes of easing notation we let* α_0 *denote* α_{n+1}. *Then we have natural isomorphisms* $(H^k(\Gamma, (N_t^*)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^h \to H^k(\Gamma, B_t^*)^h$ *and* $(H^k(\Gamma, (S_t^*)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^h \to H^k(\Gamma, F_t^*)^h$.

Proof: We recall that $U_p = tr \circ \Phi \circ res$ as described above. We will prove that U_p/p^λ is of norm ≤ 1 on $H^k(\Gamma, (Q_t^*)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. To this end we will establish that

$$\Phi/p^\lambda : C^k(u^{-1}\Gamma u \cap \Gamma, Q_t^*) \to C^k(u\Gamma u^{-1} \cap \Gamma, Q_t^*)$$

is of norm ≤ 1.

Recall that $C^k(u^{-1}\Gamma u \cap \Gamma, Q_t^*) \cong (Q_t^*)^{Nr(k)}$, isometrically via

$$f \to (f(y_1^k), \ldots, f(y_{Nr(k)}^k))$$

where the y_i^k are as in the previous section. Similarly, $C^k(u\Gamma u^{-1} \cap \Gamma, Q_t^*) \cong (Q_t^*)^{Nr(k)}$, isometrically by

$$f \to (f((y_1^k)^\bullet), \ldots, f((y_{Nr(k)}^k)^\bullet)).$$

Via these isomorphisms Φ acts diagonally by u on $(Q_t^*)^{Nr(k)}$. Also we have

$$\theta^* : N_{t,ncl}^* \to Q_t^*$$

an isomorphism of Banach spaces which respects the action of u. Let $f \in N_{t,ncl}^*$ with $|f| \leq 1$. Recall that $N_{t,ncl}^*$ is spanned by monomials $\{e_t \prod_{j \neq \jmath_i} z_{ij}^{m_{ij}} | \sum_{j \neq \jmath_i} m_{ij} > t_i$, for some $i\}$. For a monomial in $z \in N_{t,ncl}^*$, $uz = p^N z$ with $N \geq \lambda$ and thus Φ/p^λ is of norm ≤ 1. Then

$$|(\Phi/p^\lambda)f| = \sup_{|x| \leq 1, x \in M} \left| \frac{f(ux)}{p^\lambda} \right| \leq |f| \leq 1$$

so

$$\Phi/p^\lambda : C^k(u^{-1}\Gamma u \cap \Gamma, Q_t^*) \to C^k(u\Gamma u^{-1} \cap \Gamma, Q_t^*)$$

is integral and thus we get a $\mathbb{Z}_p$-module map

$$\Phi/p^\lambda : C^k(u^{-1}\Gamma u \cap \Gamma, (Q_t^*)_0) \to C^k(u\Gamma u^{-1} \cap \Gamma, (Q_t^*)_0)$$

which induces a map on cohomology and composition with the maps res and tr gives a map U_p/p^λ of norm ≤ 1 on $H^k(\Gamma, (Q_t^*)_0)$. Thus the map $U_p \otimes 1$ is of norm $\leq p^\lambda$ on $H^k(\Gamma, (Q_t^*)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

We have the following portion of the long exact sequence in cohomology

$$\begin{aligned} \ldots \to H^k(\Gamma, (Q_t^*)_0) \otimes \mathbb{Q}_p \to H^k(\Gamma, (N_t^*)_0) \otimes \mathbb{Q}_p \to H^k(\Gamma, B_t^*) \\ \to H^{k+1}(\Gamma, (Q_t^*)_0) \otimes \mathbb{Q}_p \to \ldots \end{aligned}$$

where all the maps are U_p equivariant. Also we have the exact sequence of Banach spaces $0 \to Q_t^* \to N_t^* \to B_t^* \to 0$ and have maps in both directions between these spaces which commute with U_p and thus maps between their cohomology groups which also commute with U_p. Thus we deduce that the h-parts of the terms in this exact sequence correspond (ie the image of the h-part of one term lies in the h-part of the next and the preimage of the h-part of one term lies in the h-part of the previous term). Thus we can take the h-part of the exact sequence above, and it follows easily from the previous paragraph that the Q_t terms vanish so we arrive at

$$0 \to (H^k(\Gamma, (N_t^*)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^h \to H^k(\Gamma, B_t^*)^h \to 0.$$

and are done.

Everything proceeds similarly in the $0 \to F_t \to S_t \to P_t \to 0$ case to the desired conclusion. $\square$

Chapter 3

Theta maps

For this section we let $n = 2$.

In [10], Coleman proves the existence of a map θ^{k+1} from his space of overconvergent forms of weight $-k$ to his space of overconvergent forms of weight $k+2$, $k \geq 0$, that on q-expansions is qd/dq and commutes with the action of Hecke operators up to a certain power of twisting by the determinant. The existence of this map is predicted by the following heuristic: Associated to one of Coleman's Hecke eigenforms there is a Galois representation. The Hodge-Tate weights associated to this representation for an eigenform of weight $-k$ are 0 and $-k-1$. Following a twist by a $k+1$-st power of the determinant this is a representation with Hodge-Tate weights 0 and $k+1$ and thus looks plausibly like the representation associated to an overconvergent modular form of weight $k+2$. This turns out to be the case and Coleman's θ^{k+1} is the resulting map on forms.

In the Siegel case, even with $n = 2$, much less is known. One might hope for a sensible geometric definition of a overconvergent Siegel modular form of weight $[k_1, k_2]$ and denote the space of such forms $\boldsymbol{M}_{[k_1,k_2]}$. One might hope further that a Hecke eigenform in $\boldsymbol{M}_{[k_1,k_2]}$ would have a Galois representation associated to it. It is conjectured that the Hodge-Tate weights of the Galois representation associated to a classical Siegel eigenform of weight $[k_1, k_2]$ would be $0, k_1 - 2, k_1 + k_2 - 3, k_2 - 1$. There are 8 ways of twisting one of these weights to 0 and reassigning the weights k_1 and k_2. One may hope that if, as in the case for SL_2, the class of representations arising as Galois representations of overconvergent Siegel modular forms were well behaved under the appropriate twisting operations that this would yield 8 maps analogous to Coleman's θ^{k+1}. From the arithmetic of the Hodge-Tate weights we can see that these maps would arise:

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[k_2+1,k_1-1]}$ which commutes with Hecke operators,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[-k_1+4,k_2]}$ which commutes with Hecke operators up to a $-k_1+2$-nd power of the determinant,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[-k_2+3,k_1-1]}$ which commutes with Hecke operators up to a $-k_2+1$-st power of the determinant,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[k_2+1,-k_1+3]}$ which commutes with Hecke operators up to a $-k_1+2$-nd power of the determinant,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[k_1,-k_2+2]}$ which commutes with Hecke operators up to a $-k_2+1$-st power of the determinant,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[-k_2+3,-k_1+3]}$ which commutes with Hecke operators up to a $-k_1-k_2+3$-rd power of the determinant,

· $\boldsymbol{M}_{[k_1,k_2]} \to \boldsymbol{M}_{[-k_1+4,-k_2+2]}$ which commutes with Hecke operators up to a $-k_1-k_2+3$-rd power of the determinant,

and of course the identity map from $\boldsymbol{M}_{[k_1,k_2]}$ to itself.

We hope to find these maps in our cohomologically defined forms. Taking into account the change in indices caused by

$$S_t(\Gamma) \hookrightarrow H^{\frac{1}{2}n(n+1)}(\Gamma, V_{t-(n+1)t_0})$$

and being careful to recall the change dictionary $[k_1,k_2]=(k_2-k_1,k_1)$ the maps listed above translate to maps

$$H^3(\Gamma,(S^*_{t^i})_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to H^3(\Gamma,(S^*_t)_0)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

for $t=(k_1,k_2)$, $i=1,\dots,8$ with $t^1 := (-k_1-2, k_1+k_2+1)$, $t^2 := (k_1+2k_2+2, -k_2-2)$, $t^3 := (k_1+2k_2+2, -k_1-k_2-3)$, $t^4 := (-k_1-2k_2-4, k_1+k_2+1)$, $t^5 := (-k_1-2k_2-4, k_2)$, $t^6 := (k_1, -k_1-k_2-3)$, $t^7 := (-k_1-2, -k_2-2)$ and $t^8 = t$, where the maps commute with U_p up to a d_i-th power of the determinant where $d_1 = d_8 = 0$, $d_2 = d_4 = -k_2-1$, $d_3 = d_5 = -k_1-k_2-2$, $d_6 = d_7 = -k_1-2k_2-3$.

In what follows we exhibit maps δ_i, $i=1,\dots,8$, between the spaces S_t, for appropriate t, which induce the maps on cohomology listed above.

This provides evidence that the program to attach Galois representations to overconvergent Siegel Hecke eigenforms may bear fruit.

3.1 The Maps

We see from § 2.3 that the ideal of relations between the y_{ij} is generated by $y_{22}+y_{25}$, $y_{14}-y_{23}-y_{11}y_{22}$, $y_{26}+y_{13}y_{22}+y_{14}y_{24}$, $y_{11}y_{24}-y_{22}+y_{13}$, $y_{11}y_{26}-y_{13}y_{23}+y_{14}y_{22}$ and $y_{26}+y_{23}y_{24}+y_{22}^2$. However, having reduced from the Y_{ij}'s to the y_{ij}'s we see that the two relations $y_{26}+y_{13}y_{22}+y_{14}y_{24}$ and $y_{11}y_{26}-y_{13}y_{23}+y_{14}y_{22}$ are in fact generated by the other four.

We may define a continuous differential operator ∂_1 on A_F whose action on the variables y_{ij} is as follows:

$$\partial_1(y_{11})=1, \quad \partial_1(y_{13})=-y_{24}, \quad \partial_1(y_{14})=y_{22}, \quad \partial_1(y_{2j})=0$$

One checks that this operator preserves the ideal of relations between the y_{ij} and thus extends to a differential operator on A_F.

Let us denote the action of $g\in\Delta$ on $f\in S_t$ by $g|_t f$ to avoid confusion.

Lemma 3.1.1. *The operator ∂_1 induces a Δ-invariant map $\partial_1 : S_{(0,0)} \to S_{(-2,1)}$.*

Proof: For $g\in\Delta$ a straightforward calculation confirms that

$$\partial_1(g|_{(0,0)}y_{11}) = g|_{(-2,1)}1.$$

Similarly

$$\partial_1(g|_{(0,0)}y_{13}) = -g|_{(-2,1)}y_{24}$$

and

$$\partial_1(g|_{(0,0)}y_{14}) = g|_{(-2,1)}y_{22}.$$

From this we check that for $f\in S_{(0,0)}$ we have

$$\partial_1(g|_{(0,0)}f) = g|_{(-2,1)}(\partial_1 f)$$

as desired. $\square$

We also define another continuous differential operator ∂_2 on A_F such that

$$\partial_2(y_{1j})=0, \quad \partial_2(y_{22})=y_{11}, \quad \partial_2(y_{23})=-y_{11}^2, \quad \partial_2(y_{24})=1, \quad \partial_2(y_{25})=-y_{11},$$

and

$$\partial_2(y_{26}) = -y_{14} - y_{11}y_{13}.$$

Again this map preserves the ideal of relations between the y_{ij} and thus extends to A_F.

Recall that for $M \in \mathrm{GSp}_{2n}$, $c(M)$ denotes the multiplier satisfying $c(M)J = M^TJM$. More elaborate but similarly straightforward calculations confirm

Lemma 3.1.2. *The map ∂_2 induces a map $\partial_2 : S_{(0,0)} \to S_{(2,-2)}$ satisfying*

$$\partial_2(g|_{(0,0)}f) = c(g)g|_{(2,-2)}(\partial_2 f), \qquad f \in S_{(0,0)}, g \in \Delta. \qquad \square$$

These simple calculations give rise to maps on $S_{(k_1,k_2)}$ for general k_1, k_2.

Proposition 3.1.3. *Repeated application of the maps ∂_1 and ∂_2 give a Δ-invariant map*

$$\delta_1 := \partial_1^{k_1+1} : S_{(k_1,k_2)} \to S_{(-k_1-2,k_1+k_2+1)}, \qquad k_1 \geq 0$$

and a map

$$\delta_2 := \partial_2^{k_2+1} : S_{(k_1,k_2)} \to S_{(k_1+2k_2+2,-k_2-2)}, \qquad k_2 \geq 0$$

satisfying

$$\delta_2(g|_{(k_1,k_2)}f) = c(g)^{-d_2}g|_{(k_1+2k_2+2,-k_2-2)}(\delta_2 f), \qquad f \in S_{(k_1,k_2)}, g \in \Delta.$$

Proof: First, let us treat the map δ_1. Verifying that this map is Δ-invariant boils down to verifying the following identity in A_F

$$\delta_1 \left[j_1^{k_1} j_2^{k_2} f(gy_{ij}) \right] = j_1^{-k_1-2} j_2^{k_1+k_2+1} (\delta_1 f)(gy_{ij})$$

for $g \in \Delta$ where the $j_i := j_i(g)$ are the 1-cocycles defined previously.

We prove this by induction on k_1. Lemma 3.1.1 is the case $k_1 = 0, k_2 = 0$ and the case $k_1 = 0$, k_2 arbitrary follows immediately as $\partial_1(j_2) = 0$.

Fix $k_1 \geq 1$. Assume the identity holds for all (k, k_2), $k < k_1$, $k_2 \in \mathbb{N}$. We prove the identity holds for (k_1, k_2), $k_2 \in \mathbb{N}$.

$$
\begin{aligned}
\partial_1^{k_1+1}\left[j_1^{k_1}j_2^{k_2}f(gy_{ij})\right] &= j_2^{k_2}\partial_1\left[\partial_1^{k_1}\left(j_1^{k_1}f(gy_{ij})\right)\right] \\
&= j_2^{k_2}\partial_1\Big[k_1(\partial_1 j_1)\partial_1^{k_1-1}\Big(j_1^{k_1-1}f(gy_{ij})\Big) \\
&\quad + j_1^{-k_1}j_2^{k_1}(\partial_1^{k_1}f)(gy_{ij})\Big], \qquad \text{as } \partial_1^2(j_1)=0 \\
&= j_2^{k_2}\Big[k_1(\partial_1 j_1)\partial_1^{k_1}\left(j_1^{k_1-1}f(gy_{ij})\right) \\
&\quad - k_1 j_1^{-k_1-1}j_2^{k_1}(\partial_1 j_1)(\partial_1^{k_1}f)(gy_{ij}) \\
&\quad + j_1^{-k_1-2}j_2^{k_1+1}(\partial_1^{k_1+1}f)(gy_{ij})\Big] \\
&= j_1^{-k_1-2}j_2^{k_1+k_2+1}(\partial_1^{k_1+1}f)(gy_{ij})
\end{aligned}
$$

as desired.

A similar proof goes through for δ_2 as again $\partial_2^2 j_2 = 0$ and the base case was done above. □

Remark: Note that we may define analogous differential operators ∂_1 and ∂_2 on A_B, simply by replacing y_{ij} above by z_{ij}, which can therefore be seen as maps between the spaces N_t. By construction the maps δ_1 and δ_2 on N_t and S_t commute with ϕ. Although it may at some points be useful to think of δ_1 and δ_2 as maps on these spaces, it is however important to note that they are merely maps of vector spaces and do not satisfying the compatibility properties with the action of Δ that the maps on the spaces S_t do.

3.2 Subquotients

Given $t = (k_1, k_2)$, $k_1, k_2 \in \mathbb{N}$, we identify certain subquotients of S_t as Δ-invariant subspaces of $S_{t'}$, for other t' via compositions of the maps ∂_1^n and ∂_2^m as above for suitable $m, n \in \mathbb{N}$.

To do this we first need to identify some Δ-invariant subspaces of the spaces S_t. We have already discussed the finite dimensional subspace F_t, by which we mean, by abuse of notation the image of F_t in S_t under the injection $F_t \hookrightarrow S_t$.

Now consider the subspaces of N_t,

$$N_t^1 := (\text{polynomials of degree } \leq k_1 \text{ in } z_{1j}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p < z_{2j} > \subset N_t$$

and

$$N_t^2 := (\text{polynomials of degree } \leq k_2 \text{ in } z_{2j}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p < z_{1j} > \subset N_t.$$

We observe that both of these subspaces are Δ-invariant. Thus so are their images $S_t^1 := \phi(N_t^1)$ and $S_t^2 := \phi(N_t^2)$ under ϕ.

We now consider the maps δ_1 and δ_2 restricted to these subspaces and observe the following.

Lemma 3.2.1. *With t^1 and t^2 as above, the maps δ_1 and δ_2 restrict to maps*

$$\delta_1 : S_t^2 \to S_{t^1}^2$$

and

$$\delta_2 : S_t^1 \to S_{t^2}^1.$$

Proof: The fact that the images of these maps lie in $S_{t^1}^2$ and $S_{t^2}^1$ respectively is a restatement of the fact that the maps δ_1 and δ_2 commute with ϕ coupled with observation of what these maps do to the variables z_{ij}. $\square$

As $S_{t^1}^2$ and $S_{t^2}^1$ are Δ-invariant and the δ_i are Δ-invariant up to a constant, the subspaces

$$T_t^2 := (\delta_1)^{-1}(S_{t^1}^2), \qquad T_t^1 := (\delta_2)^{-1}(S_{t^2}^1)$$

in S_t are Δ-invariant. We will see shortly that $S_t^1 \subset T_t^2$ and $S_t^2 \subset T_t^1$ — see Lemma 3.2.2 and Lemma 3.2.3.

Thus, by construction we get

$$\delta_1 : T_t^2 \to S_{t^1}^2$$

and

$$\delta_2 : T_t^1 \to S_{t^2}^1.$$

Furthermore, composing $\partial_1^{k_1+1}$ and $\partial_2^{k_1+k_2+2}$ we get

$$\delta_3 := \partial_2^{k_1+k_2+2} \circ \partial_1^{k_1+1} : S_t \to S_{t^3}, \qquad k_1 \geq 0, k_1 + k_2 + 1 \geq 0$$

with t^3 as above which satisfies

$$\delta_3(g|_t f) = c(g)^{-d_3} g|_{t^3}(\delta_3 f), \qquad f \in S_t$$

and similarly

$$\delta_4 := \partial_1^{k_1+2k_2+3} \circ \partial_2^{k_2+1} : S_t \to S_{t^4}, \qquad k_2 \geq 0, k_1 + 2k_2 + 2 \geq 0$$

with t^4 as above which satisfies

$$\delta_4(g|_t f) = c(g)^{-d_4} g|_{t^4}(\delta_4 f), \qquad f \in S_t.$$

We follow these compositions one step further as follows and calculate:

$$\delta_5 := \partial_1^{k_1+2k_2+3} \circ \partial_2^{k_1+k_2+2} \circ \partial_1^{k_1+1} : S_t \to S_{t^5}, \qquad k_1 \geq 0, k_1 + 2k_2 + 2 \geq 0$$

with t^5 as above which satisfies

$$\delta_5(g|_t f) = c(g)^{-d_5} g|_{t^5}(\delta_5 f), \qquad f \in S_t$$

and similarly

$$\delta_6 := \partial_2^{k_1+k_2+2} \circ \partial_1^{k_1+2k_2+3} \circ \partial_2^{k_2+1} : S_t \to S_{t^6}, \qquad k_2 \geq 0, k_1 + k_2 + 1 \geq 0$$

with t^6 as above, which satisfies

$$\delta_6(g|_t f) = c(g)^{-d_6} g|_{t^6}(\delta_6 f), \qquad f \in S_t$$

and

$$\delta_7 := \partial_2^{k_2+1} \circ \partial_1^{k_1+2k_2+3} \circ \partial_2^{k_1+k_2+2} \circ \partial_1^{k_1+1} : S_t \to S_{t^7}, \qquad k_1, k_2 \geq 0$$

with t^7 as above which satisfies

$$\delta_7(g|_t f) = c(g)^{-d_7} g|_{t^7}(\delta_7 f), \qquad f \in S_t.$$

Alternatively we can compose $\partial_1^{k_1+1} \circ \delta_6$ to get

$$\delta_7' := \partial_1^{k_1+1} \circ \partial_2^{k_1+k_2+2} \circ \partial_1^{k_1+2k_2+3} \circ \partial_2^{k_2+1} : S_t \to S_{t^7}, \qquad k_1, k_2 \geq 0$$

with t^7 as above which satisfies

$$\delta'_7(g|_t f) = c(g)^{-d_7} g|_{t^7}(\delta'_7 f), \qquad f \in S_t.$$

In fact $\delta_7 = \delta'_7$. The proof of this reduces to a combinatorial identity which can be checked by a computational algorithm the details of which we omit. At the end of this chapter we present a heuristic explaining this equality.

Finally, let $\delta_8 := \mathrm{id}$, $t^8 := t = (k_1, k_2)$.
We define

$$U^1_t := \delta_1^{-1}(T^1_{t^1}), \qquad U^2_t := \delta_2^{-1}(T^2_{t^2})$$
$$V^2_t := \delta_1^{-1}(U^2_{t^1}).$$

We now examine the kernels of these maps.

First we observe that the $\mathbb{Q}_p$-algebra homomorphism

$$u : N_t \to \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t \subset N_t$$

defined by

$$u(z_{11}) = z_{11}, \quad u(z_{13}) = z_{22} - z_{11}z_{24}, \quad u(z_{14}) = z_{23} + z_{11}z_{22},$$

$$u(z_{2i}) = z_{2i}, \quad i = 2, 3, 4, \quad u(z_{25}) = -z_{22}, \quad u(z_{26}) = -z_{22}^2 - z_{23}z_{24},$$

gives an isomorphism of $\mathbb{Q}_p$-Banach algebras,

$$u : N_t/ker(\phi) \cong S_t \to \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t .$$

Since $N_t/ker(\phi)$ is canonically isomorphic to S_t, via replacing z's by y's we will consider u as a map from S_t.

The inverse to u is easily seen to be the map that takes $f(z_{11}, z_{22}, z_{23}, z_{24})$ to $f(y_{11}, y_{22}, y_{23}, y_{24}) + I \in S_t$. The maps δ_1 and δ_2 induced by this isomorphism are the restriction to $\mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t$ of the maps δ_1 and δ_2 previously defined on N_t.

Consider the usual degree in z_{11},

$$deg_1 : \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t \to \{-\infty, 0, 1, \ldots, \infty\}.$$

This allows us to prove the following:

Lemma 3.2.2. *The kernel of the map $\delta_1 : S_t \to S_{t^1}$ is S^1_t.*

Proof: This result follows from the following two observations:
(i) the map ∂_1 on $\mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t$ decreases deg_1 by 1, except on things of $deg_1 = 0$ which are the kernel of ∂_1.
(ii) the image of S_t^1 in $\mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >_t$ via the isomorphism above are exactly the things of $deg_1 \leq k_1$.
Assertion (i) is evident upon inspection of the definition of ∂_1. We now prove assertion (ii).
If $f \in S_t^1 \subset S_t$ if and only if $f = \phi(g)$ where $g \in N_t^1$ is of degree $\leq k_1$ in the z_{ij} variables. This is if and only if $g(z_{11}, z_{22} - z_{11}z_{24}, z_{23} + z_{11}z_{22}, z_{22}, z_{23}, z_{24}, -z_{22}, -z_{22}^2 - z_{23}z_{24}) = u(f)$ has $deg_1 \leq k_1$. $\square$

It follows from this lemma that:
$\cdot$ $ker(\delta_1 : S_t^2 \to S_{t^1}^2) = S_t^2 \cap S_t^1 = F_t$.
$\cdot$ $ker(\delta_1 : T_t^2 \to S_{t^1}^2) = S_t^1$.
$\cdot$ $ker(\delta_4 : S_t \to S_{t^4}) = T_t^1$.
$\cdot$ $ker(\delta_5 : S_t \to S_{t^5}) = U_t^1$.

We clarify the equality $S_t^2 \cap S_t^1 = F_t$. Under the isomorphism of S_t with $\mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >$, S_t^2 maps to the set of $f \in \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >$ such that f can be expressed in the form

$$f = \sum c_{a_i,b_i} z_{11}^{a_1}(z_{22} - z_{11}z_{24})^{a_2}(z_{23} + z_{11}z_{22})^{a_3} z_{22}^{b_1} z_{23}^{b_2} z_{24}^{b_3}(z_{23}z_{24} + z_{22}^2)^{b_4}$$

with $c_{a_i,b_i} = 0$ if $\sum_i b_i > k_2$.
The image of S_t^1 under this isomorphism is the set of $f \in \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >$ such that f has an expression of the form

$$f = \sum c_{d,e_i} z_{11}^d z_{22}^{e_1} z_{23}^{e_2} z_{24}^{e_3}$$

with $c_{d,e_i} = 0$ if $d > k_1$.

So the image $S_t^1 \cap S_t^2$ under this isomorphism is the set of $f \in \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >$ such that f has an expression of the form

$$f = \sum c_{a_i,b_i} z_{11}^{a_1}(z_{22} - z_{11}z_{24})^{a_2}(z_{23} + z_{11}z_{22})^{a_3} z_{22}^{b_1} z_{23}^{b_2} z_{24}^{b_3}(z_{23}z_{24} + z_{22}^2)^{b_4}$$

with $c_{a_i,b_i} = 0$ if $\sum_i a_i > k_1$ or $\sum_i b_i > k_2$.
This is clearly also the image of F_t.

Now we can analyze the kernels of the remaining maps by exploiting another isomorphism of S_t with a disc. We observe that the $\mathbb{Q}_p$-algebra homomorphism

$$u : N_t \to \mathbb{Q}_p < z_{11}, z_{13}, z_{14}, z_{24} >_t \subset N_t$$

defined by

$$u(z_{1i}) = z_{1i}, i = 1, 3, 4, \quad u(z_{22}) = z_{13}+z_{11}z_{24}, \quad u(z_{23}) = z_{14}-z_{11}z_{13}-z_{11}^2 z_{24},$$

$$u(z_{24}) = z_{24}, \quad u(z_{25}) = -z_{13} - z_{11}z_{24}, \quad u(z_{26}) = z_{13}^2 + z_{14}z_{24} + z_{11}z_{13}z_{24},$$

gives an isomorphism of $\mathbb{Q}_p$-Banach algebras,

$$u : N_t/ker(\phi) \cong S_t \to \mathbb{Q}_p < z_{11}, z_{13}, z_{14}, z_{24} >_t .$$

Since $N_t/ker(\phi)$ is canonically isomorphic to S_t, via replacing z's by y's we will consider u as a map from S_t. As before, the inverse of this map takes $f(z_{11}, z_{13}, z_{14}, z_{24})$ to $f(y_{11}, y_{13}, y_{14}, y_{24}) + I \in S_t$.

Again the maps δ_1 and δ_2 induced by this isomorphism are just the restrictions of the previously discussed analogous maps defined on N_t.

Consider now the usual degree in z_{24},

$$deg_2 : \mathbb{Q}_p < z_{11}, z_{13}, z_{14}, z_{24} >_t \to \{-\infty, 0, 1, \ldots, \infty\}.$$

As with ∂_1, this allows to prove the following:

Lemma 3.2.3. *The kernel of the map $\delta_2 : S_t \to S_{t^2}$ is S_t^2.*

Proof: Analogous to the proof of Lemma 3.2.2. □

It follows from this lemma that:

· $ker(\delta_2 : S_t^1 \to S_{t^2}^1) = S_t^1 \cap S_t^2 = F_t$.

· $ker(\delta_2 : T_t^1 \to S_{t^2}^1) = S_t^2$.

· $ker(\delta_3 : S_t \to S_{t^3}) = T_t^2$.

· $ker(\delta_6 : S_t \to S_{t^6}) = U_t^2$.

· $ker(\delta_7 : S_t \to S_{t^7}) = V_t^2$.

Observing

$$S_t \xrightarrow{\partial_1^{k_1+1}} S_{t^1} \xrightarrow{\partial_2^{k_1+k_2+2}} S_{t^3} \xrightarrow{\partial_1^{k_1+2k_2+3}} S_{t^5} \xrightarrow{\partial_2^{k_2+1}} S_{t^7}$$

and

$$S_t \xrightarrow{\partial_2^{k_2+1}} S_{t^2} \xrightarrow{\partial_1^{k_1+2k_2+3}} S_{t^4} \xrightarrow{\partial_2^{k_1+k_2+2}} S_{t^6}$$

it is clear that

$$0 \subset F_t \subset S_t^1 \subset T_t^2 \subset U_t^1 \subset V_t^2 \subset S_t$$

and

$$0 \subset F_t \subset S_t^2 \subset T_t^1 \subset U_t^2 \subset S_t$$

and furthermore one concludes easily that the inclusions are strict.

Note: These maps aren't surjective. For example,

$$pz_{11}^{p-1} + p^2 z_{11}^{p^2-1} + p^3 z_{11}^{p^3-1} + \ldots \in \mathbb{Q}_p < z_{11}, z_{22}, z_{23}, z_{24} >$$

is not in the image of δ_1. However, using the isomorphisms of S_t with discs discussed above we see that $\delta_1 : S_t \to S_t^1$ has image

$$\left\{ f \in S_t^1 \cong \mathbb{Q}_p < z_{2j} > [z_{11}], f = \sum_{i=0}^{\infty} f_i(z_{2j}) z_{11}^i \,\text{s. t.} \lim_{i\to\infty} \frac{|f_i(z_{2j})|}{|i+1|} = 0 \right\}.$$

Similarly, $\delta_2 : S_t \to S_t^2$ has image

$$\left\{ f \in S_t^2 \cong \mathbb{Q}_p < z_{1j} > [z_{24}], f = \sum_{i=0}^{\infty} f_i(z_{1j}) z_{24}^i \,\text{s. t.} \lim_{i\to\infty} \frac{|f_i(z_{2j})|}{|i+1|} = 0 \right\}.$$

3.3 Maps on Cohomology

Let $t = (k_1, k_2) \geq 0$. From the Δ-invariant map

$$\delta_1 : S_t \to S_{t^1}$$

we get

$$(\delta_1)^* : S_{t^1}^* \to S_t^*$$

and since δ_1 is of norm ≤ 1 we then get

$$(\delta_1)^* : (S_{t^1}^*)_0 \to (S_t^*)_0.$$

Thus, see [5], we get a map on cohomology,

$$H^3(\Gamma, (S^*_{t^1})_0) \to H^3(\Gamma, (S^*_t)_0)$$

and finally a map which we will denote Θ_1,

$$\Theta_1 : H^3(\Gamma, (S^*_{t^1})_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^3(\Gamma, (S^*_t)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proceeding similarly we have, as desired,

Theorem 3.3.1. *For $1 \le i \le 8$, δ_i induces a map*

$$\Theta_i : H^3(\Gamma, (S^*_{t^i})_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^3(\Gamma, (S^*_t)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

such that

$$\Theta_i(g|_{t^i} f) = c^{d_i} g|_t \Theta_i(f),$$

for $f \in H^3(\Gamma, (S^*_{t^i})_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $g \in \Delta$. $\square$

3.4 Further Remarks

Finally, we outline a correspondence between the maps Θ_i and the Weyl group of Sp_4. As we have seen previously, the positive roots corresponding to our choice of Borel subgroup U are $[2,0]$, $[1,1]$, $[0,2]$ and $[-1,1]$ where the simple roots are $[2,0]$ and $[-1,1]$. The Weyl group W of Sp_4 is generated by reflections in weight space perpendicular to the positive roots. We will denote these reflections by $w_{[2,0]}$, $w_{[1,1]}$, $w_{[0,2]}$ and $w_{[-1,1]}$ respectively. The elements of $W \cong D_8$, the corresponding elements of $N_{\mathrm{Sp}_4}(T)/T$, and the maps they induce on weight space are listed below:

$$e \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : [a,b] \to [a,b],$$

$$w_{[2,0]} \quad \rightsquigarrow \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : [a,b] \to [-a,b],$$

$$\begin{aligned}
w_{[-1,1]} &\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} : [a,b] \to [b,a], \\
w_{[0,2]} = w_{[-1,1]}w_{[2,0]}w_{[-1,1]} &\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} : [a,b] \to [a,-b], \\
w_{[1,1]} = w_{[2,0]}w_{[-1,1]}w_{[2,0]} &\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} : [a,b] \to [-b,-a], \\
\varepsilon := w_{[2,0]}w_{[-1,1]} &\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} : [a,b] \to [-b,a], \\
\varepsilon^3 = w_{[-1,1]}w_{[2,0]} &\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} : [a,b] \to [b,-a], \\
\varepsilon^2 = w_{[2,0]}w_{[-1,1]}w_{[2,0]}w_{[-1,1]} &\rightsquigarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : [a,b] \to [-a,-b]
\end{aligned}$$

where we can also write ε^2 in terms of the generators $w_{[2,0]}$ and $w_{[-1,1]}$ as

$$\varepsilon^2 = w_{[-1,1]}w_{[2,0]}w_{[-1,1]}w_{[2,0]}.$$

It is easily seen that the expressions above for the eight elements are of minimal length in the generators $w_{[2,0]}$ and $w_{[-1,1]}$ and hence $\ell(w_{[2,0]}) = \ell(w_{[-1,1]}) = 1$, $\ell(\varepsilon) = \ell(\varepsilon^3) = 2$, $\ell(w_{[1,1]}) = \ell(w_{[0,2]}) = 3$ and ε^2 is the long element of the Weyl group with $\ell(\varepsilon^2) = 4$.

We denote $H^3(\Gamma, (S^*_{[a,b]})_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by $H_{[a,b]}$ and recall the maps Θ_i as follows:

$$\begin{aligned}
\Theta_1 : H_{[a,b]} &\to H_{[b+1,a-1]},\\
\Theta_2 : H_{[a,b]} &\to H_{[-a-2,b]},\\
\Theta_3 : H_{[a,b]} &\to H_{[-b-3,a-1]},\\
\Theta_4 : H_{[a,b]} &\to H_{[b+1,a-3]},\\
\Theta_5 : H_{[a,b]} &\to H_{[a,-b-4]},\\
\Theta_6 : H_{[a,b]} &\to H_{[-b-3,-a-3]},\\
\Theta_7 : H_{[a,b]} &\to H_{[-a-2,-b-4]}.
\end{aligned}$$

We renormalize the indices by setting $\widetilde{H}_{[a,b]} = H_{[a-1,b-2]}$ and note that the following correspondence between $\{\Theta_i\}$ and W respects multiplication (ie. it turns composition, where defined, into multiplication in W):

$$\begin{aligned}
\Theta_1 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[b,a]} &\rightsquigarrow w_{[-1,1]} : [a,b] \to [b,a],\\
\Theta_2 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[-a,b]} &\rightsquigarrow w_{[2,0]} : [a,b] \to [-a,b],\\
\Theta_3 = \Theta_2 \circ \Theta_1 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[-b,a]} &\rightsquigarrow \varepsilon = w_{[2,0]}w_{[-1,1]} : [a,b] \to [-b,a],\\
\Theta_4 = \Theta_1 \circ \Theta_2 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[b,-a]} &\rightsquigarrow \varepsilon^3 = w_{[-1,1]}w_{[2,0]} : [a,b] \to [b,-a],\\
\Theta_5 = \Theta_1 \circ \Theta_3 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[a,-b]} &\rightsquigarrow w_{[0,2]} = w_{[-1,1]}\varepsilon : [a,b] \to [a,-b],\\
\Theta_6 = \Theta_2 \circ \Theta_4 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[-b,-a]} &\rightsquigarrow w_{[1,1]} = w_{[-1,1]}\varepsilon^3 : [a,b] \to [-b,-a],\\
\Theta_7 = \Theta_2 \circ \Theta_5 : \widetilde{H}_{[a,b]} \to \widetilde{H}_{[-a,-b]} &\rightsquigarrow \varepsilon^2 = w_{[2,0]}w_{[0,2]} : [a,b] \to [-a,-b]
\end{aligned}$$

and of course $id \rightsquigarrow e$.

This correspondence and the fact that

$$\varepsilon^2 = w_{[2,0]}w_{[-1,1]}w_{[2,0]}w_{[-1,1]} = w_{[-1,1]}w_{[2,0]}w_{[-1,1]}w_{[2,0]}$$

predicts the not otherwise obvious equality $\delta_7 = \delta_7'$.

To conclude we note that one might expect to generalize Chapter 2 to define overconvergent forms cohomologically for any connected classical group G and to find a corresponding Θ_i map for each element of the Weyl group of G, where the correspondence respects multiplication and the maps corresponding to simple roots are induced by differentiation with respect to variables of distinct left weights.

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