Notes on inner twists.

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There’s a weight 2 level 243 trivial character cuspidal normalised eigenform $f$ whose $q$-expansion looks like

$$q + aq^3 + 4q^4 -aq^5 + 2q^7 + 2aq^8 - 6q^{10} + aq^{11} - q^{13} + 2aq^{14} + 4q^{16} - \cdots$$

with $a$ a square root of 6. It looks from the $q$-expansion that every coefficient is either an integer, or an integer multiple of $a$, and this is indeed the case. Moreover, the integer multiples look like they’re happening for $q^n$ with $n=1\mod 3$, and the multiples of $a$ look like they’re happening for $n=2\mod 3$, and this is also correct. We can see this by considering the twist of the form by the Dirichlet character $\chi$ of conductor 3—we get another eigenform $f \otimes \chi$, of level at most $243 \times 9$, and by checking all possibilities we see that the only eigenform whose $q$-expansion agrees with the twist of $f \otimes \chi$ for the first few terms is $f^\sigma$, the Galois conjugate of $f$, where $1 \neq \sigma \in \Gal(Q(\sqrt{6}/Q))$.

This is an example of a form with an “inner twist”. But this is a rather “generic” form too. For there is no CM involved (it is not the case that 50 percent of the $a_p$ vanish—indeed $p=3$ and $p=89$ are the only primes less than 100 for which $a_p$ vanishes) and furthermore there is no $Q$-curve involved either: the 2-dimensional abelian variety $A/Q$ associated to $f$ does not have the property that over $k := Q(\sqrt{3})$, the field corresponding to $\chi$, $A$ splits (up to isogeny) as the product of two elliptic curves. What is happening, I think, is that $A$ has an interesting endomorphism ring.

Some general result of Shimura shows that $\End_{Q}(A)$ will be $E := Q(\sqrt{6})$, the coefficient field of the modular form, so we get 2-dimensional $\lambda$-adic Galois representations attached to $f$, for $\lambda$ running through the primes of $E$. But over $k$ there are more endomorphisms: indeed, Cremona shows that $\End_{Q}(A) = \End_{Q}(A)$ is a quaternion algebra $(-3, 6/Q)$, and 6 is not a norm for $Q(\sqrt{-3})$, because the conic $A^2 + 3B^2 - 6C^2$ has no $Q$-points, so this quaternion algebra does not split and an easy calculation (check all possibilities) shows that $A$ must hence be absolutely simple. I think that the quaternion algebra must be the one of discriminant 6: if I’ve understood correctly it will be split by $Q(\sqrt{6})$ and hence must be indefinite, at any rate.

Now here’s a funny thing. Let $\lambda$ be a prime of $E$ and let $p \neq 3$ be a prime not dividing the norm of $\lambda$. Let’s consider the $\lambda$-adic representation attached to $f$. If $p = 1 \mod 3$ then the char poly of Frobp will be $x^3 - tX + p$ with $t$ an integer, and if $p = 2 \mod 3$ then it will be $x^3 - t\sqrt{6}x + p$ again with $t$ an integer. If the eigenvalues of this latter matrix are $\alpha$ and $\beta$, then $\alpha + \beta = t\sqrt{6}$ and $\alpha\beta = p$, so $\alpha^2 + \beta^2 = 6t^2 - 2p \in Z$, and we see that the Galois representation restricted to $G_k$, the absolute Galois group of $k$, has trace in $Z_{24}$, where $\lambda|\ell$. Note that if $\ell$ splits in $E$ then the full Galois representation attached to the abelian variety is taking values in $GL_2(Q_{24})$ and hence the restriction to $G_k$ is too. But if $\ell$ is inert in $E$ then the Galois representation is taking values in $GL_2(Z_{24})$ and it’s not clear to me whether one can tease it into $GL_2(Z_{24})$. Andrei Yafaev told me that the Mumford-Tate group of the abelian variety will be $D^\times$. This sounds very right but I can’t prove it. Is the Mumford-Tate group the centralizer of $D^\times$ in $GSp_{24}$?

Here’s my guess: the centralizer of $D^\times$ is just something isomorphic to $D^\times$. I think this because I’m pretty sure that $D \otimes D = M_4(Q)$ and this gives two commuting actions of $D^\times$ on a 4-dimensional vector space. I can’t find a pairing preserved by this though.
So I am guessing that the Mumford-Tate group is $D^\times$, so my guess is that the Galois representation attached to the modular form over the im quad field has image commensurable with $(\mathcal{O}_D \otimes \mathbb{Z}_\ell)^\times$, meaning that there will only be problems at the primes 2 and 3.

1 Remarks on the mod $p$ Galois representations.

I just noticed that if the Mumford-Tate group is $B^\times$ then this forces the mod $p$ representation of the absolute Galois group of $k$ to be reducible if $B$ ramifies at $p$. This is because $\mathcal{O}_B$, when tensored up to the integers $\mathcal{O}$ in a quadratic extension of $\mathbb{Q}_p$, doesn’t become the full maximal order in $M_2(\mathcal{O})$. Hence one expects the mod 2 and mod 3 representations attached to the form to be reducible when restricted to the imaginary quadratic field.

The mod 2 representation attached to $f$ takes values in $\text{GL}_2(\mathbb{Z}/2\mathbb{Z})$ and a bit of experimenting shows that the splitting field of the representation is the Galois closure of $\mathbb{Q}(6^{1/3})$ (modulo the bad primes 2 and 3, the coefficient of $q^p$ should be 1 mod 2 if $p$ is 1 mod 3 and doesn’t split completely in $\mathbb{Q}(6^{1/3})$, that is, iff 6 has no cube root mod $p$. This representation is irreducible but when restricted to $k$ of course becomes reducible.

The mod 3 representation attached to the form over $\mathbb{Q}$ is just 1 plus cyclo, so is certainly reducible over $k$ as well.