What the Harish-Chandra homomorphism looks like.

Kevin Buzzard

February 9, 2012

Last modified 02/09/2010.

1 Introduction/summary.

Written 17/11/07. Tinkered with a little (fixed typos) in 2010. I think my source was Knapp’s book on rep theory of semisimple groups.

The Harish-Chandra homomorphism is a way of completely identifying what the centre of the universal enveloping algebra of a complex reductive Lie algebra is. The centre is in fact always isomorphic to a polynomial ring in \(d\) variables, where \(d\) is the dimension of a Cartan subalgebra.

The definition of the homomorphism depends on the choice of a notion of positivity but it has been cleverly normalised so that the homomorphism itself does not depend on this choice!

2 The homomorphism.

If \(g\) is a complex reductive Lie algebra and \(h\) is a Cartan subalgebra (i.e. a “maximal torus” on the Lie algebra level) then \(h\) is abelian so its universal enveloping algebra \(H := U(h)\) is just a polynomial algebra. Harish-Chandra observed that if we choose an ordering and hence get positive roots \(E_1, E_2, \ldots\), and let \(I\) denote the ideal \(U(g)E_1 + U(g)E_2 + \ldots\), then \(H + I\) (within \(U(g)\)) is a direct sum, the centre \(Z(U(g))\) of \(U(g)\) is contained within \(H + I\), and that the projection \(Z(U(g)) \rightarrow H\) along \(I\) was an injection. But much much better: if you then compose this projection with the algebra automorphism of \(H\) induced by sending \(h \in h\) to \(h - \delta(h)1 \in H\) (with \(\delta\) half the sum of the positive roots) then the resulting map \(Z(U(g)) \rightarrow H\) was an injective ring homomorphism, with image precisely \(H^W\), the things fixed by the Weyl group. It’s a general fact that \(H^W\) is isomorphic to a polynomial ring in \(\text{dim}(h)\) variables.

Note that if \(z\) is the centre of \(g\) then \(z \subseteq h\) and the induced map \(U(z) \rightarrow Z(U(g)) \rightarrow H\) is the obvious one; the projection \(Z(U(g)) \rightarrow H\) induces the identity on \(U(z)\), and the twist by \(\delta\) doesn’t change anything because \(\delta\) doesn’t move \(z\). Standard algebra arguments show that the algebra maps \(H^W \rightarrow \mathbb{C}\) all come via restriction from algebra maps \(H \rightarrow \mathbb{C}\), and the maximal ideals of \(H^W\) are just the \(W\)-orbits of the maximal ideals of \(H\).

3 An example.

Let \(g\) be \(\mathfrak{gl}_2(\mathbb{C})\) with basis \(E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\) and \(Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\). Let the Cartan subalgebra be the span of \(H\) and \(Z\). Now the root spaces are spanned by \(E\) and \(F\), the roots are the characters of \(h\) sending \(H\) to \(\pm 2\) and \(Z\) to zero, with \(E\) corresponding to the number \(+2\). The un-normalised H-C homomorphism sends \(Z\) to \(Z\). The adjoint representation of \(\mathfrak{gl}_2\) (wrt the basis \(E, F, H, Z\) and with the algebra acting on the left) is this:

\[
E \mapsto \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
\( F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

\( H \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

\( Z \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

and the associated Killing form (the one whose \((i,j)\)th entry is the trace of \(\rho(e_i)\rho(e_j)\)) is

\[ \begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

which isn’t non-degenerate, so I don’t think reductive Lie algebras have Casimir elements! Had we worked with \(\mathfrak{sl}_2\) one checks that we would have just thrown away the last row and column for both the adjoint representation and the Killing form, which is now non-degenerate with inverse

\[ \begin{pmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \]

and so if \(X_1 = E, X_2 = F\) and \(X_3 = H\) then the dual basis is \(X^1 = F/4, X^2 = E/4\) and \(X^3 = H/8\) and the Casimir element is \(\sum g_{i,j} X^i X^j\) (with \(g_{i,j}\) the Killing form) is \(4X^1 X^2 + 4X^2 X^1 + 8(X^3)^2 = EF/4 + FE/4 + H^2/8\). Well, that’s what I made it! Another way of writing it is \(\sum X_i X^i\) and again I get \(EF/4 + FE/4 + H^2/8\). Why is everyone else out by factors of 4 or 8 or whatever? Let’s multiply by 8 just to clear the denominators, and set \(C = 2EF + 2FE + H^2\) even though as far as I can see \(C\) is in fact eight times the Casimir element. The un-normalised Harish-Chandra homomorphism is easily evaluated on \(C\): one has \(2EF = 2H + 2FE\) so \(C = H^2 + 2H + 4FE\) and by definition this gets sent to \(H^2 + 2H\). Now to normalise it we have to compose with the algebra automorphism of the polynomial ring \(C[Z, H]\) sending \(Z\) to \(Z\) and \(H\) to \(H - \delta(H) = H - 1\) (the positive root is \(E\) and the associated linear map on the Cartan sends \(H\) to 2 and \(Z\) to zero). So we get \((H - 1)^2 + 2(H - 1) = H^2 - 1\). Hence the normalised Harish-Chandra homomorphism sends \(Z\) to \(Z\) and \(C\) to \(H^2 - 1\), and the theorem is that the map is injective with image the subring of \(C[Z, H]\) fixed by the Weyl group, and the Weyl group sends \(H\) to \(-H\) and fixes \(Z\) so the image is \(C[Z, H^2]\) and lo and behold we’ve proved that the centre of the universal enveloping algebra is generated freely as a polynomial ring by \(Z\) and \(C\).