Imperial College London

M1F

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)

May-June 2018

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Foundations of Analysis

Date: Monday, 14 May 2018

Time: 10:00 AM - 12:00 PM

Time Allowed: 2 hours

This paper has 4 questions.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

All required additional material will be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Each question carries equal weight.
- Calculators may not be used.

In this exam, you may assume any theorems from the course without proof, unless you are specifically asked to prove them.

1. (a) Let $n \geq 1$ be an integer.

   (i) Using the binomial theorem, or otherwise, compute the sum $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

   (ii) Now compute $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}$.

   (b) (i) Using de Moivre's theorem and its consequences, or otherwise, prove that for all $n \in \mathbf{Z}_{\geq 1}$ there is a polynomial $P_n(X)$, with real coefficients, such that for all θ we have $\cos(n\theta) = P_n(\cos(\theta))$.

   (ii) Compute $P_4(X)$.

   (iii) The *leading coefficient* of a polynomial $f(X)$ of degree d is the coefficient of X^d in $f(X)$. What is the leading coefficient of $P_n(X)$? Hint: you might find part (a) useful.

   (iv) Compute $P_n(1)$ for general n.

2. In this question you can assume any standard results about inequalities, rationals and irrationals, least upper bounds and so on, unless you are explicitly asked to prove them.

   (a) Let S be a set of real numbers. Define the following terms.

   (i) $x \in \mathbf{R}$ is an *upper bound* for S;

   (ii) S is *bounded above*;

   (iii) $b \in \mathbf{R}$ is a *least upper bound* for S.

   (b) Prove that if $S \subseteq \mathbf{R}$ has a least upper bound, then S is non-empty and bounded above.

   (c) (i) Prove that $S = \{x \in \mathbf{R} : x < 59\}$ has a least upper bound, and compute it explicitly (with proof).

   (ii) Prove that if S is the subset of the open interval $(0.7, 0.9)$ consisting of real numbers which have no 8's in their decimal expansion, then S has a least upper bound.

   (d) For the below statements about a set S of real numbers, either give a proof, or a counterexample (with justification):

   (i) If b is an upper bound for S and $b \in S$ then b is a least upper bound for S.

   (ii) If S and T are subsets of $\mathbf{R}$, and b is a least upper bound for S, and c is a least upper bound for T, then $b + c$ is a least upper bound for $S + T := \{s + t : s \in S, t \in T\}$.

3. Let S be a set.

(a) (i) What is a *binary relation* $\sim$ on a set S?

(ii) What does it mean for this binary relation $\sim$ to be an *equivalence relation*?

(iii) If $\sim$ is an equivalence relation on S, define the *equivalence class* $\mathrm{cl}(a)$ of an element $a \in S$.

(b) Prove that if $a, b \in S$ then either $\mathrm{cl}(a) = \mathrm{cl}(b)$ or $\mathrm{cl}(a) \cap \mathrm{cl}(b) = \emptyset$.

(c) Let $\star$ be a transitive binary relation on $S = \mathbf{Z}$ such that for all $x \in \mathbf{Z}$ we have $x \star (x+3)$ and $x \star (x-5)$. Prove that $\star$ is reflexive.

(d) Now let S be a set with two elements, and let $\star$ be a binary relation on S which is reflexive. Prove that $\star$ is transitive.

4. Let X and Y be sets, and $f : X \to Y$ be a function.

(a) What does it mean for f to be

(i) injective;

(ii) surjective;

(iii) bijective.

(b) For each situation below, either give (with proof) an explicit example of a function f with these properties, or prove that no function f has these properties. Here $\mathbf{N} = \{1, 2, 3, \ldots\}$ denotes the positive integers.

(i) $f : \mathbf{N} \to \mathbf{N}$ injective but not surjective.

(ii) $f : \mathbf{N} \to \mathbf{N}$ surjective but not injective.

(iii) $f : \mathbf{N} \to \mathbf{N}$ bijective but not injective.

(iv) $f : \mathbf{N} \to P(\mathbf{N})$ bijective (here $P(\mathbf{N})$ denotes the power set of $\mathbf{N}$, that is, the set of subsets of $\mathbf{N}$).

Now go back to the general case of X and Y sets, and $f : X \to Y$ a function. We define the *graph* of f to be subset $G := \{(x, f(x)) : x \in X\}$ of $X \times Y$.

(c) Define $p_1 : G \to X$ by $p_1(x, y) = x$. Prove that p_1 is a bijection.

(d) Define $p_2 : G \to Y$ by $p_2(x, y) = y$. Prove that if p_2 is a bijection then f is a bijection.