The Langlands Programme: An Instructional Course on Automorphic Forms

0: Introduction

Dr. Richard Taylor

Welcome to everybody. Richards going to chat for ~10-15 mins, then things will get going at 10 with Martin.

This is supposed to be an introduction to automorphic forms for number theorists.

An automorphic form is an analytic map on GL_n. Related to reps of Gal(Q/Q) & also to arithmetic varieties.

Examples:
- L-functions
- Elliptic modular forms
- Already very tricky

The aim of the course is to teach number theorists the flashy automorphic forms language.

No attempt at an overview will be given. They decided to stick to GL_2 & do that here. Eisenstein series, Theta series (Weak rep) will hardly be mentioned, either.

In Week 1 we'll set up automorphic forms on GL_2 with elliptic modular forms, to show why they're the same thing.

In Week 2 we'll take the trace formula technique from automorphic forms & use it to understand L-function change for GL_2.

Heuristics: don't evaluate lattice change non-

If we have aut rep \pi \rightarrow Gal rep \rho then we can do S^\pi(p), \rho_{i\sigma} p_{r} etc.

If \Gamma \rightarrow \Gamma_{L} then we can do \rho_{Gd}(\Gamma_{L})

If \Gamma \rightarrow \Gamma_{L} really exists we should be able to do comparable things at automorphic rep level.

The easy things on RNS usually turn not to be quite tricky on LHS.

E.g. \pi_{Gd}(\Gamma_{L}) \rightarrow \pi_{r} \rightarrow \pi_{L}, base change.

If Gal(L/K) \cong \Gamma_{L}, then \pi_{L} exists.

\pi_{L} \cong \pi_{Gd}(\Gamma_{L})

Somebody says R of Gal(R/K) comes in this way \Rightarrow \pi_{L} \cong \pi_{R} \cdot \pi_{R}(\sigma \tau \sigma^{-1})

In the case of GL_2 we can do this latter base change once we have CFT.

x_{L} = x_{K} N_{L/K} ". Tony Scholl will tell us more next week.
An underlying assumption is that we knew more algebra than analysis.

Another underlying assumption is that we are not experts, even though Jean-

Benoit Laborie at the back in fact is. So we'll spend lots of time doing stuff
that experts would regard as standard. & we'll have never seen before.

Tea & coffee & snacks above. Lots of snacks today & say if we want
more in the future. (Presumably you're not that bothered about this bit, John).
I. Background & Goals

Martin Taylor

He'd like to start with a few announcements:
- Bursaries
- Regular!!
- First at lunchtime

He doesn't want to begin with Haar measure, but peer pressure is forcing him.

SO Haar measure.

Principal object of study is $G$ a locally compact group.

Let $C_c(G) := \{ f : G \to \mathbb{R} \text{cts, spc support} \}$

A measure is a do (def below) linear form $m : C_c(G) \to \mathbb{R}$

For $K \subseteq G$, cpts $K \subseteq C_c \quad m(f) \leq C_K \sup_{x \in K} |f(x)|$ (this maybe def of $c_K$)

Write $m[f] : f \to \int f(x) \, dm(x)$

John has nudged him into a def of the measure.

Call $m$ sub if $f \geq 0 \Rightarrow m(f) \geq 0$

For $f \in C_c(G)$ & $s \in G$, define $f$, or $f_s$ by

$f^s(x) = f(x^s) \quad (sf)(y) = f(y^s)$

Def: Measure $m$ is called left invariant if $m(f) = m(f_s)$

Def: A non-zero, left invariant, the measure on $C_c(G)$ is called a Haar measure.

Then $O \quad$ No Haar measure on $G$. Moreover, such a measure is unique up to a positive multiplicative constant.

He's summoned Brian Burch to move a strange wooden thing which appears to be connected to the ground.
Möbius function of $G$

$f \in C_c(G)$, $se G$. Define $f^{\text{conj}(s)}(x) = f-xs^{-1}$

$f \rightarrow m(f^{\text{conj}(s)})$ is also a measure. (Here $m$ is Haar measure)

$m(f^{\text{conj}(s)}) = m(f, xs^{-1})$. So this is also Haar measure.

So by the theorem, $m(f^{\text{conj}(s)}) = \Delta_G(s) \cdot m(f)$.

$\Delta_G : G \rightarrow \mathbb{R}^*$ is the modular function. It's a 1 to 1 HM

It somehow measures how $m$ fails to be right-invariant.

If $m(f) > 0$, $\Delta_G(s) = \frac{\int g(x) \, dm(x)}{\int f(x) \, dm(x)}$.

Note: if $s \in Z_G$ then $\Delta_G(s) = 1$.

Prop 0.2: The modular function $\Delta_G$ is identically 1 if $G$ is connected and $G/Z$ is simple & non-abelian.

Def: If $\Delta_G = 1$ we say $G$ is unimodular.

Prop 0.3: Let $H$ be a closed normal subgroup of $G$. Then $\Delta_G|_H = \Delta_H$.

Def: By standard theory, $G/H$ is locally compact. So we have Haar measure on $G, H, G/H$. Say $f \in C_c(G)$.

Here define a gadget $n(f) = \int \int f(xh) \, dh \, dm(x)$.

Here $dh$ is a Haar measure for $H$.

$n(f)$ is a Haar measure for $G/H$.

It's easy to check that $n$ is a Haar measure for $G$.

Now, say $se H$. Then $n(f) = \int \int f(xs^{-1}) \, dh \, dm(x)$.

$\Delta_H(s) = \int \Delta_H(s) \, dh \, dm(x)$.

$= \Delta_H(s) \cdot n(f)$

$= \Delta_G(s) \cdot \Delta_H(s)$. □
He briefly wants to talk about going from $G$ to $G/H$.

Say now $H \triangleleft G$, Hope it. We have $G \to G/H$ and this induces $C_c(G/H) \to C_c(G)$.

And hence Haar measure on $G$ induces one on $G/H$.

He'll now give us (hopefully lots of relevant) examples. Sticking things on Haar measures is a worthwhile exercise. I'd not just lecture on behalf of the lecturer.

Examples:

1. $G$ discretely. Then $m_G = 1$ for $G$.

2. $G$ profinite, if $H \triangleleft G$, Hope it. Then $m(H) = (G:H)^{-1}$.

3. $G = \mathbb{R}/e^\mathbb{Z}$. A $\mathbb{R}$ is Lebesgue measure.

4. (Perhaps the most interesting yet)

$G = SO_n(\mathbb{R}) = \{ G \in (\mathbb{R}, +) \mid A^T = A^{-1} \}$, $dG = \frac{dA}{a}$ I think this should be.

5. $K/\mathbb{Q}$ finite. 0 integral closure of $\mathbb{Z}$ in $K$, $p$ not ideal.

1: $K^* \to \mathbb{R}$, $x \to (0, xO)^{-1}$, $m(\mathbb{R}) = \mathbb{R}$, $(\mathbb{R}^*)^{-1}$, 

$m(1) = m(0)$, $d(1) = d(0)$.

So $G = K^*$ has Haar measure $dx/|x|$.  

In fact this is case not of $Gln(K)$ which has Haar measure $\frac{d\gamma}{\gamma}$.

$Gln(K)$ is unimodular. By prop 0.3, $SLn(K)$ is also unimodular.

If $x$ were at the end of the course then he would have already defined $A_K$, Kaas field $K$ a $\mathbb{R}$-field.

$G = A_K$: choose Haar measure. Im $\gamma$, $\gamma^{-1}m(0) = 1$ a.e. $\gamma^{-1}m(0) = 1$ a.e.

Then $m$: Time is a Haar measure.

$G = J_K = A_K$: pick $m^2$ s.t. $m^2(0) = 1$ a.e. almost everywhere

& $m$: Time.
For some group G, you may want to know that all we're interested in is the unimodular group, but this is false.

(7) Non-unimodular: \( G = \{ (a, b) \in \text{GL}_2(\mathbb{R}) \} \)

Check: \( a \) is left- and right-invariant. Then \( \Delta_G(a, b) = a^2 \)

That ends the Haar measure bit.

Now, begin the course proper.

- Little bit on local CFT
- Substantial bit on global CFT
- Late, more interesting stuff

### 3.1 Local Class Field Theory

\( K/Q_p \) is a finite extension. \( \mathcal{O} \) the integral closure of \( \mathbb{Z} \); \( \mathcal{O}_p \) any ideal

1.1: \( K^* \to \mathbb{R}_+ \), \( \mathcal{N}_p = (O_p) \)

If \( N/K \) is a field ext., Galois & \( G = G(N/K) \) def. \( H = \text{max}^2 \text{ n.c. ext.} \) of \( K \) in \( N \), & \( I = G(N/M) \) is the inertia g.p., \( F = E \text{Frob. automorphism} \)

\[
\begin{pmatrix}
N \\
M \\
K
\end{pmatrix}
\]

\( G/K \) \( \cong H \leq F \). Here \( E(k) = x^{NP \mod p^m} \forall x \in \mathcal{O}_m \).

Recall the local reciprocity map

\( \hat{\mathbb{H}}^2(G.N)^2 \cong \mathbb{G} = \mathbb{G}(N/K) \otimes \frac{1}{\mathbb{Z}} \mathbb{Z}_{(p)} \mod \mathbb{Z} \)

\( \mathbb{H} \) def. \( \mathbb{H}^2(G.N)^2 = H^2(G.N) \otimes \mathbb{M}_2^{\text{sym.}} = \hat{\mathbb{H}}^2 \)

\( \hat{\mathbb{H}}^2(G.N^2) = K^2/N_{K/K}(N^2) \)
The obvious generator \( \frac{1}{2} \) of \( \mathbb{Z} \) can be pulled back to an elt of 
\( \mathcal{D}^*(G,M) \) which is called \( c_{VR} \), the canonical class.

The cup product \( \cup_{VR} : G^* \to K^*/N(N) \) is an IM.

We're really interested in its inverse, which we'll call \( \theta = \theta_{VR} \).

Because \( K^* \to K^*/N(N) \) we'll call the composition \( K^* \to \mathcal{D}^* \).

Fact: For simplicity, assume Galois \( \theta \) has the following properties.

- (1) \( \theta(0) = 1 \)
- (2) \( \theta(1+p) = P \) - with respect to \( p \)
- (3) \( \theta(p) = \theta(p) \) - with respect to \( p \)

Other people may need \( \theta(p) = F \) but I'll stick with \( F \).

Functionality properties:

If \( N \supset K \), then

\[
\begin{align*}
\mathbb{L} & \xrightarrow{\theta_{VR}} \mathcal{D}(N/K)^{ab} \\
& \xrightarrow{\text{Ver}} \mathcal{D}(K)^{ab}
\end{align*}
\]

Both commute.

This only avoids \( N/K \) Galois. Suppose now \( L/K \) is Galois.

\[
\begin{align*}
\mathcal{D}(N/K)^{ab} & \xrightarrow{\text{Ver}} \mathcal{D}(K)^{ab} \\
& \xrightarrow{\text{Gal}(L/K)^{ab}} \text{Gal}(L/K)^{ab}
\end{align*}
\]

So by taking limits we get \( \text{Gal}(K) \to \text{Gal}(K(K)^{ab}/K) \to \text{Gal}(L(K)^{ab}/K) \)

which is \( \text{cts} \) (RHS has Kan topology), \( \text{cts} \) (finite topology) and dense (\( \text{cts} \) is surjective at finite levels).
Let's have a quick look at the functionality of the new gadget.

If $v$ is connected in $\mathbb{N}_K$, then $F = \mathbb{N}_K$, so $\mathbb{N}_K$ is a group.

The group $G$ is:

Let $v = x_k$, then $x_k = x_{k+1}$.

The group $G$ is:

Recall: Tanya burl at 3:15

52. Global Group Theory

If $v = \phi$, then $K = \mathbb{N}_K$.

K has a number field $v = \phi$.

If $u = \psi$, then $L = \mathbb{N}_K$.

$\phi, \psi, \mathbb{N}_K$. We have $G = \mathbb{N}_K$.

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The Frobenius \( \phi \) satisfies

\[ (1) \; \phi : \sigma N \rightarrow \sigma N ; \quad F(\sigma w, \sigma N / \sigma K) = \sigma \cdot F(w, N/K), \sigma^{-1} \]

\[ (2) \; \forall \lambda < \infty \quad N - \lambda \quad \text{N/K Galois} \quad \lambda \quad \text{N/L} \quad F(w, N/L) = F(w, N/K)^{\sigma_{l(L/K)}} \]

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\[ F(w, N/K) = F(w, L/K) \]

Fix \( v \): \( E(v) = \{ F(w, N/K) \mid w \mid v \} \). This is a def: it is a conjugacy class in \( G \).

**Passage to infinite case:**

\[ \Sigma = N \leq K^c = \text{alg closure of } K. \text{ Set } \Sigma_K = \text{ set of places of } K \]

\[ \Sigma_{K_{\text{lf}}} \cup \Sigma_K \]

**Put:** Here \( K \) is a number field, & \( N/K \) may be infinitely ramified.

Put: \( \Sigma_N = \lim_{\downarrow \infty} \Sigma_L \) where \( K \leq L \leq N \) & the limit is taken w.r.t. restriction of places.

So technically we \( \Sigma_N \) is an infinite coherent vector of places.

\[ w^* = (u_L), \; u_L \in \Sigma_L \]

Define \( G_w(N/K) = \lim_{L} G_u(L/K) \)

Say \( w \) is unramified if \( u_L \) is unramified \( \forall L \).

If \( w \) is unramified then we have \( F(w) = F(u_L, L/K) \) (there are elements by property(3) above).

\[ \{ F(w) \}, \; w/v, \text{ form a conjugacy class in } G(N/K). \text{ Call this class } F(v). \]

So everything we did in the finite case goes through happily to the infinite case.
Let's now talk about

**Density**

Set $\Sigma = \Sigma_K$. $N/K$ finite again, Galois gp $G$.

For $S \in \Sigma$ define $\delta_n(S) = \# \{ \sigma \in S \mid N \sigma = n \}$

We say that $S$ has density $\delta \in [0,1]$ if $\lim_{n \to \infty} \frac{\delta_n(S)}{\delta_n(S)} = \delta$.

**NB** this is natural density. Thus, if $S$ has natural density $\delta$, then $S$ has Dirichlet density $\delta$, but not conversely.

**Thm 2.1 (Chebotarev)** Say $N/K$ is an ext. of number fields with Galois gp $G$.

Let $X \subseteq G$ be stable under conjugation.

Let $P_X = \{ v \in \Sigma_K \mid F(v) \subseteq X \}$

Then $P_X$ has density $|X|/|G|$. ☐

**Cor 2.2.** Allow $N/K$ to be infinite (inside $K$).

Suppose only a finite no. of places of $K$ ramify in $N$.

Then the $\{ F(w) \} \subseteq \Sigma_N$, non-ramified $w$ is dense in $G(N/K)$. ☐

**N > L**: $K$ finite. Density follows as Frob of $L/K$ cover Gal($L/K$) by Thm 2.1.

**Ideles & Adeles**

Given $G$, a locally cplt group, $\exists H_v$ a cplt open subgp.

$S \subseteq \Sigma$ s.t. $|S| < \infty$. **NB** from now on $S$ will always be finite.

Then $G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$

Set $G = \bigcup_{S} G_S$. It is a locally cplt gp, a restricted direct product.

$G$ is topologized by a fundamental system of

Note that here we have liberty not to define $H_v$ for finitely many $v$ because we can just enlarge $S$ to include these $v$. 
Add: Take $G_v = K_v^*$, & for $v < w$ set $H_v = O_v$.

The restricted product $A = \prod_v A_v$ is the adele ring.

Check: $A$ is actually a topological ring.

Ideles $G_v = K_v^*$, & for $v < w$ set $H_v = O_v^*$.

The restricted product $J_k = \prod_v J_v$ is the ideles.

NB $A^*_k = J_k$.

We have a diagonal map $K^* \rightarrow A$, $x \mapsto \prod_v x_v$ where $x_v : K \rightarrow K_v$

Prop 2.3 $K$ has discrete image in $A$, and the quotient group $A/K$ is compact for the quotient topology. $\square$

The same recipe gives us $K^* \rightarrow J$, $x \mapsto \prod_v x_v$ ($x_v$ is a unit almost everywhere).

Def: $C_k = J/K^*$, the idele class group.

Def: $\| \cdot \| : J \rightarrow \mathbb{R}_+$, almost additive so the product makes sense.

$$\|x\| = \prod_{v < w} \|x_v\| \times \prod_{v \lor w} \|x_v\|$$

By the product formula we have $K^* \subseteq \ker \| \cdot \|$, so we can view $\| \cdot \|$ as a map from $C_k$ to $\mathbb{R}_+$.

Def: $J_k^0 = \ker (\| \cdot \| : J_k \rightarrow \mathbb{R}_+)$

Prop 2.4: $K^*$ is discrete in $J$, and $J/k^0$ is compact. $\square$

Functoriality

$N/K$ number fields, $v$ a place of $K$, $w$ a place of $N$ above $v$, so $K_v \hookrightarrow N_w$

(1) Inclusion: $J_k \hookrightarrow J_N$,

$$ \prod_v J_v \hookrightarrow \prod_w J_w$$

where $y_w = x_v$ for $v | w$

(2) Norm: $N_{w/k} : J_N \rightarrow J_k$

$$ \prod_w y_w \mapsto \prod_v x_v$$

where $x_v = \prod_w N_{w/k}(y_w)$

Thus is etc.
Also, for \( A \), we get \( A_K \hookrightarrow A_N \) \& \( T_{N/K} : A_N \longrightarrow A_K \). Check the details.

To relate ideals to ideals we use the

**Content map** \( I : I_K \), the gp of fractional \( O^* \)-ideals

The content map \( (x) : J \hookrightarrow I \)

\[
(\omega) = \prod_{v|\omega} \mathcal{P}_v^w(\omega)
\]

Here \( v : K_v \rightarrow \mathbb{Z} \) is the valuation associated to the place \( v \).

By inspection, \( K (\omega) = \prod_{v|\omega} O_v^* \times \prod_{v|\omega} K_v^* = U_K \), the unit ideals

\( U_K \) is an open set in \( J \).

\( (x) \) is the \( I \) has the discrete topology.

He's now coming up to the concept of admissibility. This is distinct from the concept of an admissible rep that Tony is talking about.

**Admissibility** part 1. Say \( S_0 \in S \subseteq \Sigma \)

\( I_S = \text{group of fractional } O^* \)-ideals \"prime to } S \"

\[ (S)^* : J \rightarrow I_S \], defined by \( (S)^* = \prod_{v|S} P_v^{w(S)} \)

Related to this is the group of \( S \)-ideals \( J_S = \{ j \in J | jv = 1 \forall v \in S \} \)

**Def.** Let \( G \) be a topological abelian group. For a HM \( \phi : I_S \rightarrow G \), we call the pair \( (S, \phi) \) admissible if for each open \( \omega \) in \( G \)

\[ \exists \varepsilon > 0 \text{ s.t. } \phi((a)^S) \in \mathcal{V} \text{ for } a \in K^* \text{ s.t. } |a|_\omega \leq \varepsilon \forall v \in S \]

Note that if \( G \) is discrete then we can take \( \mathcal{V} = \{ 1 \} \) & then get \( (a)^S \in \ker \phi \).

**Lecture 3**

Feb 16, '93

2:00 pm

He finished off yesterday with the def of admissibility, in the non-Tony Scholl case.

Recall \( N/K \) an ext of number fields, \( \Sigma \subseteq S \subseteq \Sigma \), \( |S| < \infty \)

\( G \) a top. ab. gp; \( \phi : I_S \rightarrow G \); \( (S, \phi) \) is admissible if \( \forall \text{ nheld } \mathcal{N} \text{ of } 1 \text{ in } G \)

\[ \exists \varepsilon > 0 \text{ s.t. } \phi((a)^S) \in \mathcal{V} \text{ for } a \in K^* \text{ s.t. } |a|_\omega \leq \varepsilon \forall v \in S \text{ & } |a|_\omega \leq \varepsilon \]
Prop 25 (Glorified weak approximation)

(i) Suppose $G$ is complete, & $(\psi, s)$ is admissible. Then $\exists! \, HM \, \eta: J \rightarrow G$

(a) $\psi$ cts
(b) $\eta(K^s) = 1$
(c) $\eta(x) = \psi((s)) \quad \forall x \in J^s$

(ii) Suppose now $E$ is an open nhd of $G$ in which $1^G$ is the only sylb ("the no small subgp hypothesis") & Then, given cts $HM, \eta: J \rightarrow G$ s.t. $\eta(K^s) = 1$, $\eta$ comes from $(\psi, s)$ admissible as in (i).

Sketch proof (i) $(\psi, s)$ given. Given $x \in J$, choose $(a_n \in K^s)^\infty_0$ to which converge to $x^s$ in $K^s, \forall s \in S$.

Define $\eta(x) = \lim_{n \to \infty} \psi(a_n)^s(s)

A key part is $\frac{\psi((a_n)^s(s))}{\psi((a_n)^s)} \rightarrow 1$

It's clear that $\eta(K^s) = 1$.

(ii) No $\psi, \eta$ given. Then $U \cap J^s = U^s$ (this is def. of $U^s$; recall $U_K = U \cap \ker(\psi)$).

We have $U_T$ for $T \subset S$. Then lie in arbitrarily small nhds of $1$ in $J$ and are groups. Hence by "no small subgps", we have $\eta(U_T) = 1$ for large $T$.

Then $\eta : J^s = \frac{J^s}{U_T} \xrightarrow{\eta} G$.

The Artin map

Say $N/K$ an ab. ext. of no. fields, & $G = G(K/N)$, & $\Sigma_\infty \leq S \leq \Sigma$.

Assume that $S$ contains all ramified primes of $N/K$.

Define $F = F_{K/N} : I_S \rightarrow G$ by $F(v) = F(v, N/K)$.

$F$ is onto. Here's a pf: $f = \text{Im} F$ then we get an ext $N^H/K$ & by Chebotarev we see $N^H = K$, so $H = G$.

This is one of the few bits of global classfield theory that he can prove. He'll just sketch the rest, I'll tell you about the rest of it in as attractive a manner as possible.
**Theorem 2.6 (Main theorem of CFT)**

1. $F$ is admissible (i.e., $(F, S)$ is admissible).
2. By (2.5) $F$ corresponds to $\theta : G_{N/K} : \mathfrak{J} / K^* \rightarrow G$, with $\theta(x) = F(x') \forall x' \in \mathfrak{J}$.
3. $\ker \theta = K^* N_{w/K}(J_0) / K^*$.

(d) Given any open subgroup $X \leq C_K$ with finite index, find $\ker \theta_{N/K} = \ker \theta_{W/K} = X$.

A Galois-like arrangement here (it is killed by its dual)

If $N \trianglelefteq L \trianglelefteq K$ then

$$
\begin{align*}
C_K & \xrightarrow{\theta_{N/K}} G(N/K) \\
& \downarrow \text{res} \\
G_L & \xrightarrow{\theta_{L/K}} G(L/K)
\end{align*}
$$

Taking limits gives us $\theta_K : C_K \rightarrow \text{Gal}(K^{\text{ab}} / K) \cdot G_{K}^{\text{ab}}$.

Recall that in the local case $\theta$ was injective with dense image. In the global case it's essentially the opposite.

**Theorem 2.7** $\theta_K$ is surjective, & $\ker \theta_K$ is the connected component of the identity in $C_K$.

More functionality:

1. If $\sigma \in G_w$, then

$$
\begin{align*}
C_K & \xrightarrow{\theta_{w/K}} G_w^{\text{ab}} \\
& \downarrow \text{conj}(\sigma) \\
C_w & \xrightarrow{\theta_{w/K}} G_w^{\text{ab}}
\end{align*}
$$

(check for Frobenius & then patch up).

2. If $N/K$ is a not necessarily abelian ext., then

$$
\begin{align*}
C_N & \xrightarrow{\theta_N} G_N^{\text{ab}} \\
& \downarrow \text{ner} \\
N_{w/K} & \xrightarrow{\theta_{w/K}} G_{w/K}^{\text{ab}} \\
C_K & \xrightarrow{\theta_K} G_K^{\text{ab}} \\
& \downarrow \text{ner}
\end{align*}
$$

We'll next talk about local/global compatibility.
Compatibility. A/K abelian ext. of number fields, v a place of K, w a place of N, wv.

Recall $G_v: K_v \to G(N_v/K_v) \to G$.

**Theorem 2.8** For $v \in J$, $\Theta_v(a) = \prod \Theta_v(a_v)$. Note that $\Theta_v(a_v) = 1$ for almost all $v$.

**S. Groessencharaktere**

This is far too long as are so many Germanic words, be an Anglo-Saxon like Master, so he'll call them $G_v$.

Say $K$ is a number field & $F$ an $\Theta$-ideal.

$$U(F) := \{ u \in U_K \mid u \equiv 1 \text{ mod } F, vK_{v < \infty}, u_v = 1 \forall \nu \}$$

**Def.** $G_v$ of $K$ is a cts hom. $K \to \mathbb{C}^*$, or $K_v \to \mathbb{C}^*$.

Note $G_v$ has the 'no small subgps' property.

**I.** Let $N$ be a small sub of $G_v$. Then $G_v(N)$ is open, so contains some $U(F)$, & thus $K(U(F)) = I$.

Call the largest such $f$ the conductor of $K$, write $f(K)$.

**II.** Can apply prop 2.5 to obtain an admissible pair $(\varphi, S)$. (It seems that we can arrange $S$ s.t. it is $\Sigma_{v \in U(F)}$)

$$\varphi(\langle x \rangle) = \chi(x) \quad \forall x \in J^S.$$  (3.1)

**Def.** Set $K_p := \{ x \in K_v \mid x > 0 \text{ & } x \equiv 1 \text{ mod } F \}$

(i.e. $v(<x>) \geq v(F) \forall v$)

**Def.** For $x \in J$, $x_S \in J$ is the $S$-part of $x$.

Then $\langle x \rangle_S := \{ x_v \mid x_v \in S \}$

If $f = f(K_v)$, $x \in K_p$, then $\chi(x, x_2^s) = \varphi((x, x_2^s))$ by (3.1). Here $S := \{ \nu \} \cup \Sigma_{v < \infty}$

**Def.** $X_v = \prod_{v \in S} \chi_v(x_v(x_2^s))$.

(3.2) Hence $\varphi(\langle x \rangle) = \prod_{v \in S} \chi_v(x_v(x_2^s))$. Now $\chi_v: K_v \to \mathbb{C}^*$ & in the case $v \not\in S$ we have $K_v = K_v^s$ or $K_v^s$. New $\chi_v: K_v^s \to \mathbb{C}^*$ & in the case.
Lemma 3.3. Viewing \( K_v \), \( v \in \Omega \), as a subfield of \( \mathbb{C} \), then any \( \chi_v : K_v \to \mathbb{C}^\times \) may be written (non-uniquely) in the form

\[
\chi_v(x) = x^{a_v} |x|^{b_v}, \quad a_v \in \mathbb{Z}, \quad b_v \in \mathbb{C}
\]

Idea of proof: \( R^x = R, x \{ \pm 1 \} \) & \( C^x = S^x \times R \).

Definition: \( \chi \) is of type \( A \) if \( \forall v \in \Omega \)

& of type \( A_0 \) if \( \exists \{ v \in \mathbb{Z} \mid v \text{ real} \} \forall v \in \Omega \).

For \( v \in \Omega \), we have \( \sigma_v : K \to K_v \).

For \( x \in K \), (3.2) becomes

\[
\phi((x))^{-1} = \prod_{v \in \Omega} \chi_v(\sigma_v(x)) = \prod_{v \in \Omega} |\sigma_v(x)|^{a_v} |\sigma_v(x)|^{b_v}
\]

So if \( \chi \) has type \( A_0 \), we have \( |\sigma_v(x)|^{a_v} = \sigma_v(x) \rho_v(\sigma_v(x)), v \text{ complex} \)

\( |\sigma_v(x)| = \sigma_v(x), v \text{ real} \)

(3.4) So we obtain \( \phi(x) = \prod_{v \in \Omega} \sigma_v(x)^{\alpha_v} \), where \( \alpha_v = \text{Hom}(K, C) \)

& \( \alpha_v \) are integers.

We still don't have a feel as to why type \( A \) & type \( A_0 \) are of importance. It's to do with algebraic...

Algebraicity of values

Set \( E = \text{compositum of } aK, \ P_v = \text{primes ideals with a } K_v \text{-generator} \)

Proposition 3.5: \( a \). If \( \chi \) is of type \( A \), then for \( a \in \mathbb{Z} \), we have \( \phi(x) \text{ is alg./R} \).

b) If \( \chi \) is of type \( A_0 \), then \( \phi(\chi(x)) \text{ is a number field} \).

Proof: Similar ideas as a) & b). Hence b). From (3.4) we have \( \phi(P_v) \subseteq E \).

But \( (I_v : P_v) < \infty \). □
It's grossencharacter part 2 today.

The story so far: \( \chi: \mathbb{J}/\mathbb{K}^* \rightarrow \mathbb{C}^*; \quad f_\chi(x) = f(x) \cdot \text{supp}(f) \cdot \Xi(x) \uparrow \)

\((f, s) \text{ admissible} \quad \phi: \mathbb{I}_s \rightarrow \mathbb{C}^* \)

We had \( a, b, c \) for \( v_{100} \); \( \chi \) could be of type \( A \) or \( A_o \).

If \( \omega \in K \), then \( \phi(\omega) = \prod \sigma(\omega)^{r_i} \).

Set \( T = \sum_{c} c \in \mathbb{Z}\Gamma \)

\( T \) is the type.

Now we'll do

Purity of values

Clearly \( T \) can't be any old elt of \( \mathbb{Z}\Gamma \). E.g., \( \phi(\omega) \) only depends on \( \omega \), not \( \omega \), so we immediately see that \( T \) must annihilate \( Y_F = \{ x \in \mathbb{O}^* \mid x \gg 0 \text{ & } x \equiv 1 \text{ mod } \mathbb{F} \} \)

Note \( \langle \mathbb{O}^*, Y_F \rangle < \infty \)

Prop 3.6: If \( T \) is a type then \( \forall \omega \in \Gamma \) \( n_o + n_{p, \omega} = w \), a constant, indpt of \( \sigma \).

Proof idea: all the bits you need to assemble a proof.

Define \( \chi': \mathbb{K}^* \rightarrow \text{Map}(\Gamma, \mathbb{R}) \)

\( \chi'(x)(\sigma) = \log |\sigma(x)| \quad \text{Note } \chi(x)(\rho \sigma) = \chi(x)(\sigma) \)

Set \( \chi = \chi' \bigg|_{\mathbb{O}^*} \)

\( \text{Ker } \chi = \mu_{\mathbb{K}} \) & \( \chi(\mathbb{O}^*) \) is a lattice in the vector space

\( \mathbb{H} = \{ \text{maps } s \mid s(\rho \sigma) = s(\sigma), \quad s(\sigma) = 0 \} \)

Define \( \langle, \rangle : \text{Map}(\Gamma, \mathbb{R}) \times \mathbb{Z}\Gamma \rightarrow \mathbb{R} \) in the obvious way.

\( \{ 1 \} \subseteq (\mathbb{O}^*/\mu_{\mathbb{K}})^T \iff \langle \chi(\mathbb{O}^*), T \rangle = 0 \iff \langle H, T \rangle = 0 \iff \sum_{\omega} \sigma(\omega) n_{\omega} = 0 \quad \forall T \in H \)

\( n_{\omega} \)

get \( \prod_{\omega} n_{\omega} \in \mu_{\mathbb{K}} \)

That's all we need.
Def: An alg no. $w\in\mathbb{M}$ say is called pure if $\ker w$ is an stalk $\mathcal{O}_E$.

Prop 3.7: The values of $\varphi$ of type $A_\alpha$ & $N^\alpha$ this is where of notation: $\varphi^*_\alpha L$ & $L \sim (\varphi,\pi)$ are pure, &

$$\varphi(\alpha)(\varphi(\alpha)^\alpha = N^{\alpha^w} \quad V(\alpha,\pi) \leq 1.$$ Moreover, the number field $\Omega(\varphi(\alpha))$ is either $\mathbb{Q}$ or a CM field.

Since $L \sim (\varphi,\pi) < \infty$. So now it suffices to prove the result for $\alpha = \varphi_0$, $\varphi_0 = \mathbb{K}_P$

$$\varphi(\alpha)(\varphi(\alpha)^\alpha = N^{\alpha^w} = N \varphi^w$$

(up to sign, but)

That's the end of grosscharacter. He wants to talk about $l$-adic reps and Weil-Deligne reps, & that'll be about it.

§4. $l$-adic reps

$K$ a number field or local field, $G_K = G(K^s/K)$, $l$ a prime no., $E$ a number field, $\lambda$ a place of $E$, $\lambda | l$.

Def: A $l$-adic rep is a ch HM $\rho: G_K \rightarrow GL(V)$, $V$ f.d. $E \mathbb{K}_l$ v.s.

We know (1) $\ker \rho$ is open & closed.

(2) Say $\rho$ is abelian if $\ker \rho$ is abelian. Then $\rho$ factors through

$G_K^0$

(3) $\rho$ is semi-simple or ss if $V$ is ss as a $G_K$-module, i.e. all $G_K$-submodules have complements.

(4) If $E = \mathbb{Q}$ & $l | \ell$, then $\rho$ is called an $l$-adic rep.

Example: (1) $T_{l}(\mu) = \lim_{\leftarrow} \mu_{l^n}$, a $Z_{l}$-module

$V_{l}(\mu) = T_{l}(\mu) \otimes Z_{l}$, This affords $k_{l}, G_{K} \rightarrow O_{K}$, an abelian rep.

(2) Because we're using $E$ well let $E'$, finite $E$, be an elliptic curve $/K$.

Set $E_{m} = V_{l}(E_{m}: E \rightarrow E)$

$$T_{l}(E) = \lim_{\leftarrow} E_{m}$$

$$V_{l}(E) = T_{l}(E) \otimes Z_{l} \mathbb{Q}_{l}$, a $Z_{l}$-dual rep.

I think he said that if $E$ had CM then the rep is abelian.
Definition: A $\lambda$-adic rep $\rho : G_k \to GL(V)$ is called unramified at $v \in \Sigma_k$ if

$$\rho(I_v(K^c/K)) = 1 \quad \forall \nu \in \Sigma_k \text{ over } v$$

or just $1$ if because they're all conjugate.

In this case, let $w$ be a place of $(K^c)^{\text{der}}$ above $v$ & define the char poly

$$P_{v,\rho}(T) := \det (1 - \rho(v^w) T) \quad \text{Bernoulli } w \text{ not in set of } w.$$ 

John notes that traditionally this def is uses the geometric Frobenius, but Martin will stick with his arithmetic one.

Definition: Let $K$ be a number field. Then $\rho$ is called rational (over $E$) if $\exists$ a finite set $S \subseteq \Sigma_k$ s.t.

1. $\rho$ is unramified outside $S$
2. $\forall \nu \in S$ we have $P_{v,\rho} \in E[T].$

(examples above)

Exercise (1) $S = \{ v \in \Sigma, v | \nu \} \Rightarrow \rho$ rational / $E$

(2) $S = \{ v \in \Sigma, v | i\nu, v \text{ prime } i \} \Rightarrow \rho$ rational / $E$. (Well)

Compatibility: Suppose $\rho_1, \rho_2$ are $\lambda$-adic $\lambda$-adic reps respectively with $\Sigma_k$.

We say $\rho_1$ & $\rho_2$ are compatible if $\exists$ finite set $S \subseteq \Sigma$ containing all places where either rep ramifies & st. $P_{v,\rho_1}(T) = P_{v,\rho_2}(T) \forall v \in S.$

A system $\{\rho_\lambda\}_{\lambda \in \Lambda}$ is called compatible if all are pairwise compatible.

We call this system strictly compatible if $\exists$ a finite set $S$ s.t.

1. $P_{v,\rho_1} \in E[T] \quad \forall v \in S \cup S_k$ (zero rationality condition)
2. Each pair $\lambda, \lambda'$ has $P_{v,\rho_1} = P_{v,\rho_2} \forall v \in (S \cup S_\lambda \cup S_{\lambda'})$

Note that in the compatible def, $S$ depended on $\lambda, \lambda'$. Error may catch up in the compatible case. In the strictly compatible case $\exists$ finite set $S$ which deals with the lot. A minimal such $S$ is called the exceptional set. For $\{\rho_\lambda\}$ but he won't be using this.
A new gadget coming up.

**Locally algebraic reps**

Now $K/Q_p$. Consider $T$, the torus restricting $G_m/K$ to $Q_p$

$$T = \text{Re}_{Q_p}(G_m)$$

we have

For $L/Q_p$ define $T(L) = (K \otimes_{Q_p} L)^*$

**Def.** $\rho : g \mapsto GL(V)$ is a $p$-adic rep, we call $\rho$ **locally algebraic** if 

algebraic $H^1$ (**HM of alg gps**)

$$r : T \to GL(V)$$

s.t. $\rho \circ \Theta(x) = r(x^*)$ for some lift of $1$ in $K^*$. Here $\Theta$ is the local reciprocity map.

\[
\begin{array}{ccc}
K^* & \xrightarrow{\rho} & G_{m, k} \\
\uparrow & \exists \Theta & \downarrow \\
GL(V) & \text{(Here } r(x) = r(x^*) \text{)} & \\
\end{array}
\]

**Note:** $r$ exists uniquely because the lift of $1$ is a Zariski-dense in $K^*$ something.

**Ex.** If $F$ is a Lubin-Tate formal group associated to unipotent $G$ of $K$

$$T(F) = \lim \text{Ker}[\pi^n] \ , \ V(F) = T(F) \otimes_{Z_p} Q_p$$

This has $Q_p$-dimension $(K^*Q_p)$.

$$\forall v \in O^*, \quad \Theta(w)v = [w^*]v$$

$r : K^* \to GL(V)$ defined by $K$-module structure of $V$

& I guess the point is that $\rho$ is locally algebraic in this case, although I'm quite lost here.
Prop. 4.1. $\rho$ locally algebraic $\Rightarrow \rho|_{I=I(K^s/K)}$ is ss.

Sketch of Proof:
1. Any open nhbd of $K^s$ is Zariski dense in $K^s$.
2. Any rep of torus is ss.

Identify $I=O^s$ by $\theta$.

Have $U \subseteq O^s_{\text{open}}$ such that $\rho(x) = r(x^{-1})$ for all $x \in U$.

Let $W$ denote any $\rho(O^s)$-submodule of $V$. We need to find a complement:

- $W$ is $r(U)$-stable
- $W$ is $r(K^s)$-stable by (1)

So by (2) $W \subseteq V$ where $\tau$ respects the $r(K^s)$-action & so it clearly respects the $r(U)$ action.

So $\tau$ respects the $\rho(U)$ action, as $\rho(x) r(x^{-1})$ for all $x \in U$.

We're trying to respect the $\rho(I)$ action but this is easy now, we just use the standard averaging trick:

$$\tau' = \frac{1}{(O^s:U)} \sum_{s \in O^s/U} \rho(s) \tau \rho(s)^{-1}$$

Lecture 5:

The last lecture we talked about locally algebraic reps.

$$\rho : G^\text{ab}_K \to GL(V)$$

was locally algebraic if

$$T(\rho) = K^s \times U \xrightarrow{\theta} G^\text{ab}_K$$

$$\xrightarrow{\tau'} \downarrow^p$$

$$GL(V)$$

for $U$ some small nhbd of 1.

How do you recognize when a rep is locally algebraic? We'll come to that.

Anyway, we must talk a little about characters.

Observe $T(\rho) = \left(K \otimes \mathbb{Q}_p\right)^X$. Now $K \otimes \mathbb{Q}_p \cong T \otimes \mathbb{Q}_p$ where $[\sigma](x,y) = x^\sigma y$.

(Here $\sigma$ is acting on $x$.)
Now any character \( \chi = \prod_{i} \chi_{i}^{N_{i}} \) for \( N_{i} \in \mathbb{Z} \).

Say \( \rho : G_{k} \to GL(V) \) is locally algebraic.

By (4.1), \( \rho|_{I} \) is s.s. By abuse of notation \( \rho|_{I} = \rho|_{G_{x}} \).

Hence \( \rho|_{U} \) is still semisimple & algebraic (i.e., \( r_{\pi} = 1 \)) & abelian.

So extending from \( GL(V) \) to \( GL(V \otimes E_{r}) \) for some suitable finite ext \( E_{r}/E_{p} \),
we can diagonalizing \( \rho|_{U} \) & hence \( \rho|_{U} = \sum \chi_{i} \cdot \chi_{i} \) & dim.

Here the \( \chi_{i} \) are algebraic, the \( \chi_{i} \) "come from" the torus \( T \).

Prop. 4.2 \( \rho \) is locally algebraic \( \iff \rho|_{U} = \sum \chi_{i} \cdot \chi_{i} \) & \( \chi_{i} \) a ndt of 1,
where \( \chi_{i}(U) = \prod_{0}^{n_{i}(s)} \chi_{i}(u) \).

He hopes it's a bit more comprehensible, now.

The main business of the day is now coming up:

\[ \text{ladic reps \& GCs} \]

\( E, K \) number fields, \( \rho \) a \( l \)-adic rep \( G_{k} \) on \( V \).

CFT \( \theta : C_{k} \to G_{k}^{ab} \), \( \ker \theta = C_{k} \).

Also, given \( C_{k} \to GL(V) \), it must factor thru \( G_{k}^{ab} \); for
some trivial reason, I think he said.

It's because \( GL(V) \) is totally disconnected.

This is all fine if \( V < \infty \).

If \( V | \infty \), \( E_{r} \), \( V | E_{r} \) a.v.s. \( GL(V) \) is not totally disconnected.

So we have \( \rho : C_{k} \to C^{*} \).

\( \chi : G_{k}^{ab} \to C^{*} \& \chi \to \rho \) but not the other way.

That was the preamble. Here's the business.
Let $X : J/K^x \to C^x$ be a GC of type $A_0$.

Let $\mathcal{F} = \mathcal{F}(K)$.

Set $T = \text{Supp}(\mathcal{F}) \cup \Sigma_m$.

We get $(\varphi, T)$ associated to $X$ by 2.5 as usual.

Say $\varphi : I_T \to E^x$. Say $\lambda$ is a prime of $E$.

Say $\lambda | \ell$. Set $S = T \cup \{\ell\}$.

Choose $h : E \to \mathbb{E}_\lambda$.

Then $\varphi_h : I_S \to E^x \to E^x_{\lambda}$

We want to pull $\varphi_h$ back to a map on $C_K$, or something.

$$C_K \cong \left. J^5/k^x \to J^5 \to I_S \to E^x_{\lambda} \right|_{\varphi_h}$$

Prop 4.3 $\varphi$ extends to a $\lambda$-adic rep $\varphi : C_K \to E^x_{\lambda}$ which is locally algebraic.

(Note that because $E_{\lambda}$ is $\lambda$-adic, we get $\chi_{\lambda} \Rightarrow \rho_{\lambda} : G_{K^n} \to E^x_{\lambda}$)

**Sketch proof** (Sutherland p158 for topology bit!)

Given a small nhd $Y$ of 1 in $E^x_{\lambda}$, need to show $\exists X \subseteq J^5/k^x$ with

If $n > 0$, we $K_{np^n}$, then

$$\phi_{\lambda}(\omega^n) = h(\phi(\omega^n)) = h(\omega^n)$$

where here $T$ is the type of $X$.

But $h(\omega^n)$ is $\lambda$-adically small so by making $n$ large we get $\chi_{\lambda} \subseteq Y$.

This is the basic observation that makes everything work.

So $X \to \mathcal{P}_{np^n} \to Y$ small nhd, & now we have to show $\chi_{\lambda}$ is locally algebraic.
Now pick $v \in \mathcal{O}_{K_E}^* \rightarrow J$. Here $K_E$ is the semilocalization of $K$ at $E$.

Pursue the def's & find that $\chi_\lambda(v) = \alpha_\lambda^*, T$.

The def' of locally algebraic, $\Lambda(1,2)$, shows $\chi_\lambda$ to be locally algebraic.

Remark($\Lambda$) $\chi_\lambda$ vanishes on $C_K$. So $\text{Im } \chi_\lambda$ must be central, as $C_K/C_K$ is.

Hence $\text{Im } \chi_\lambda \subseteq \mathcal{O}_{E,E}$.

(2) (exercise) (although like Richard Taylor he'll give us the key hint)

The system $\{ \chi_\lambda \}_{\lambda \in \Sigma^1}$, varying $\lambda$, gives us a family of strictly compatible reps.

Key point: $v \in S$, $v$ a uniformizing parameter for $v$.

Then $\chi_\lambda (\tau_v) = h(\psi(\tau_v))$ indpt of $\lambda$.

There's a thing that Martin calls "functoriality" although Richard appears to refer to it as "Base Change for $GL_2$". He was going to talk about it tomorrow, but he's so ahead of time he'll start now.

Functoriality

Keeping previous notation, $\chi_\lambda : C_K \rightarrow E^*_\mathbb{L}$ has kernel $\cong C_K$.

ie $\chi_\lambda^* : \rho_\lambda : G_K \rightarrow E^*_\mathbb{L}$.

Prop 4.4 Suppose now $N/K$ is any extension. Then $\rho_\lambda |_{G_N}$ comes from

$$\chi_{\lambda, N/K} : J_N \rightarrow C^*$$

$$\begin{array}{c}
\rho_\lambda |_{G_N} \\
\downarrow
\end{array}
\quad \begin{array}{c}
\rho_\lambda |_{G_K} \\
\downarrow
\end{array}
\quad \begin{array}{c}
\chi_{\lambda, N/K} \\
\downarrow
\end{array}
$$

$$h \mapsto g = inc^h(\lambda)$$

$g \in \text{inc}^h(\lambda) \in \text{inc}^h(\theta_w(y)) \in \text{inc}^h(x) \cong \theta_w N_{N/K} Y$

$\rho_\lambda(g) \cdot \rho_\lambda \cdot \theta_w N_{N/K} Y = \chi_{\lambda, N/K} Y$

Hence stop now, make next step for next time.
Whilst this isn’t his swan-song, and John was reminding him, we are now on the final lap.

**55. Weil-Deligne reps**

\[ K / \mathbb{Q} \] a local field.

**Prop 5.1** Let \( \rho \) be an irreducible rep, \( \rho : G_k \to GL_n(\mathbb{C}) \). If \( p > 2 \), then \( \rho \) is monomial.

**Proof**. In fact we’ll do considerably more: we’ll get a handle on how \( \chi \) character induces \( \rho \).

**Note**: \( GL_n(\mathbb{C}) \) does not have arbitrarily small subgroups. This is because of a gadget that Tony mentioned: \( \exp \).

\[ \exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C}) \]

\[
\begin{array}{ccc}
0 & \to & x \\
\downarrow & & \downarrow \\
2 & \mapsto & \exp(x)
\end{array}
\]

small nhds

\[ \exp(nx) = (\exp x)^n \quad n \chi \text{ burst out of the small nhds,} \]

so \( \exp(nx) \text{ burst out of the small nhds,} \)

For \( n \), proof by picture.

**By proof** Replace \( G_k \) by \( G_k / \ker \rho = G = Gal(N/k) \)

\[ \rho : G \to GL_n(\mathbb{C}), \quad 1 \leq p \leq I \leq G \]

\[ p \rho \text{ cyclic cyclic.} \]

\[
\rho |_I = \text{sum of } 2 \text{ abelian char}, \text{ as } 2 + p. 
\]

\( I \) is abelian cyclic (this is what they seem to call it in Manchester anyway—sometimes \( \mathbb{C} \) cyclic-by-abelian!)

\[ \left\{ \begin{array}{l}
\text{all irreps of } I \text{ are monomial, induced from abelian char of some subgroup } \mathbb{Z}_p. \\
\text{See eg. Serre’s book on } \mathfrak{p} \text{ rep theory 8.2.}
\end{array} \right. \]

\[ \text{Case 1} \quad \rho |_I \text{ is reducible. Then } \rho |_I = \chi_2 \oplus \chi_2 \]

\[ \text{Gaudron char of } (I, \chi - x^2) \]

\[ \Delta_2 = \text{Stab}_G(\chi_2). \quad \Delta_2 / I \text{ is cyclic.} \]

Thus \( \chi_2 \text{ extends to } \left[ K_i \right] \).

\[ \text{By Frob rec. } \rho \text{ must occur in } \text{Ind}_{\Delta_1}^G \chi_2, \text{ which is}\]

\[ \text{used by Mackey’s criterion.} \]

\[ \text{Hence } \rho : \text{Ind}_{\Delta_1}^G \chi_2. \]
Case 2. \( \beta \big| _I \) is irreducible. Apply \( \mathfrak{m} \). Then \( \beta \big| _I = \text{Ind}_{\Delta}^G \chi_3 \).

Here \( (I^* : I^2) = 2 \), \( I^2 \equiv \mathfrak{m} \) (\( I^1 \) isn't \( I^2 \), it's just notation).

Set \( \Delta_3 = \text{Stab}_G(\chi_3) \). Then he claims \( \Delta_3 / I^2 \) is cyclic.

\[ \Delta_3 / I = \Delta_3 / \Delta_3 n I = \Delta_3 / I^2 \]

Apply ext- to get ext's \( \{ \chi_3^3 \} \) of \( \chi_3 \) to \( \Delta_3 \).

The endgame's the same: \( \rho \) occurs in some \( \text{Ind}_{\Delta}^G \chi_3 \), which is killed by Mazur. □

The Wedgbp

\[ R : G_K \rightarrow G(K^\infty / K) = \lim \mathbb{Z}/n \mathbb{Z} = \hat{\mathbb{Z}} \supset \mathbb{Z} \]

\[ \text{Ker } R = I. \ (\text{As I will never be some ideal thing like } I^2, \text{so we're always local for this } \delta) \]

**Def:** The **Wedgbp of** \( K \), \( W_K = R^I(\hat{\mathbb{Z}}) \), topologized by declaring \( I \) open

(\( I \) has a profinite topology)

\[ W_K = G_K \] Pick \( \beta \in W_K \) s.t. \( R(\beta) \neq 1 \)

Then

\[ 1 \rightarrow I(K^{ab} / K) \rightarrow G_K^\text{ab} \overset{R}{\rightarrow} \hat{\mathbb{Z}} \rightarrow 0 \]

\[ \text{and } \]

\[ 1 \rightarrow \hat{\mathbb{Z}} \overset{\delta}{\rightarrow} K^\times \overset{\nu}{\rightarrow} \mathbb{Z} \rightarrow 0 \]

Easily check that \( W_K = G(K^\times) \).

**Def:** By a **rep** of \( W_K \), we mean a char HM \( \rho : W_K \rightarrow GL_n(\mathbb{C}) \)

If \( n = 1 \), we get \( \chi : W_K \rightarrow \mathbb{C}^\times \). Call \( \chi \) a **character**. (\( \chi \) is usually called \( \chi \) a **quadratic character**)

Call \( \chi \) **unramified** if \( \chi(I) = 1 \)

Note that a char HM \( G_K \rightarrow \mathbb{C}^\times \) must have finite image, as \( \mathbb{C}^\times \) has no small subgps. However, \( W_K \) char's are more exotic.
Example Fix $\mathfrak{C}$. Then $W_{\mathfrak{C}}$ is the unramified char.

$$W_{\mathfrak{C}} : W_K \rightarrow W_{ \mathfrak{C} } \xrightarrow{\theta^{-1}} \mathbf{K}^\times \xrightarrow{\lambda_{/\mathfrak{C}}} \mathbf{C}^\times$$

Set $W_{\mathfrak{A}} = W_{\mathfrak{C}}$ (The $\mathfrak{A}$ is def'ed)

Note this is already lots of reps of $W_{\mathfrak{K}}$ . Lots more then there are reps of $G_{\mathfrak{K}}$.

Call a rep $\rho$ of $W_{\mathfrak{K}}$ of Galois type if it is the restriction of a rep of $G_{\mathfrak{K}}$.

$\rho$ is of Galois type if $\rho(W_{\mathfrak{K}})$ is finite.

**Prop 5.2** An irreducible rep $\rho : W_{\mathfrak{K}} \rightarrow \mathbf{G}(\mathbf{C})$ can be written in the form $\sigma \otimes W_{\mathfrak{C}}$ where $\sigma$ is of Galois type.

**Proof** Use the fact that $W_{\mathfrak{K}}$ is very interesting if $\mathfrak{I}$ (profinite)

Firstly, factor out $\ker \rho \cap \mathfrak{I}$

$I \rightarrow I/\ker \rho \cap \mathfrak{I} \rightarrow W/\ker \rho \cap \mathfrak{I} \rightarrow \mathbf{Z} \rightarrow 0$

We have conj $\mathbf{Z} \rightarrow \text{Aut}(I/\ker \rho \cap \mathfrak{I})$ has kernel $\mathbf{Z}$

So $\rho(\mathbf{Z})$ central $= \sigma$ diag $(\lambda)$

Choose $\sigma \in C$ s.t. $\omega_\mathfrak{C}(\mathbf{Z}) = \lambda$

Check that $\sigma \otimes W_{\mathfrak{C}}$ has finite image & is hence Galois.

So we've had Galois reps & Galois rep's ; Weil-Galois & Weil reps.

Lastly, Weil-Deligne reps & their reps.

**Weil-Deligne reps**

The bad news: they're not reps but gp schemes. Fortunately after an initial fray we'll just be reduced to looking at the reps.

If $G$ is a finite gp, the constant gp scheme $G$ is $\text{Spec}(\mathbb{N} / \mathfrak{G})$.

Have to be careful as $W_{\mathfrak{C}}$ isn't finite.
The constant group scheme $W_k$ over $\mathbb{Q}$ is $\text{Spec}\left( \text{Map}_{\text{cont}}(W_k, \mathbb{Q}) \right)$

Locally at $f$ has $f^{-1}(f)$ closed & open locally.
I think this is a def of loc int.

**Def.** The Weil-Deligne gp is the $\mathbb{Q}$-group scheme

$$\text{WD}_k = G_0 / \mathbb{Q} \times W_k,$$
with action $1:1$.

$$\text{WD}_k(E) = G_0(E) \times W_k$$

$$\left( a_z, w_z \right)(a_t, w_t) = (a_z + \|w_z\| a_t, w_z w_t)$$

**Def.** Let $V$ be a $d$-v.s. Then a rep $\rho$ of $\text{WD}_k/\mathbb{Q}$ is a H.M of group schemes

$$\rho : \text{WD}_k/\mathbb{Q} \rightarrow \text{GL}_E(V)$$

(User-friendly translation)

(a) (Restrict to $W_k$) \[ \rho : W_k \rightarrow \text{GL}_E(V) \]

Ker $\rho$ (as pts) is open in $W_k$.

We have $\rho$'s $xij$ on $\text{GL}_E(V)$. Then $(\rho^{-1})(xij)$ is $\rho$ on $W_k$.

$$(\rho^{-1})(xij)^{-1}(S_{ij})$$

are all open, so the kernel is open.

(b) (much easier) (restrict to additive gp) $N \in \text{End}_E(V)$ with

$$\rho'(w)N\rho'(w)^{-1} = N \text{ for } N \in W_k$$

Here's how you get $N$. $\rho'$ gives an alg H.M $G_0(E) \rightarrow \text{GL}_E(V)$

Such an $\rho'$ is of the form $\rho'(e) = \exp(eN)$, $\exp(eN)$ is algebraic or $N$ is nilpotent.

This gives us $N$. We now use the composition law above.

We get

By (b), all eigenvalues of $N$ are stable under mult by $e$.

$$\Rightarrow \nu \text{ is only eigenvalue.}$$

$$\Rightarrow \nu \text{ is only eigenvalue.}$$

What's going on up here in these rather badly-taken notes. Is that given the complicated thing $\rho'$ we pull out a rep $\rho''$ of $W_k$ with open kernel & a nilpotent matrix $N$ satisfying (b). So this gives us a better handle on $\rho'$.

In fact:
This is a bijection because given $\rho''$ & $N$ we see can define

$$\rho'(a; w) = \exp(aN) \rho''(w)$$

Hell, start this lecture with a brief recap.

$$R: G_k \to \mathbb{Z} ; W_k = R^t(\mathbb{Z}) ; W_{2k} = G_k \times W_k.$$ For $E$ char 0, a rep $\rho^' : W_k \times E \to GL_E(V)$.

$$\rho^' \mapsto (\rho'', N) , \rho^'' : W_k \to GL_E(V) \text{ open kernel}$$

$N$ nilpotent

$s.t.$ $\otimes \rho''(w) N \rho''(w)^{-1} = 1 \text{ for } N$

where $||E|| = (\text{char } q_k : G_k) \otimes (R(\mathbb{R}) = 1)$

The reason we have $\rho' \& \rho''$ is that there's a $\rho$ coming.

**Def**: Call $\rho'$ ss if $\rho''$ is ss

Rk.(1) $\rho''$ is ss if $\rho''(\mathbb{R})$ is, because $\rho''(I)$ is finite, & so

$$\otimes (\text{Im } \rho'' : \rho''(\mathbb{R})) < \infty.$$ So use the fact that in char 0 a rep is ss $\Rightarrow$ it is ss on subgps of finite index.

(2) Ker $N$ in $W_k$ -stable by (1) - in fact its $W_{2k}$ -stable.

$\rho' = (\rho'', N)$ is irreducible $\Rightarrow N = 0$ (or $G_k$ ker $N$ is stable) & $\rho''$ is irreducible,

ie rep $\rho''$ is lifted from $W_{2k}$.

Clearly $\Leftarrow$ is true too. So $\rho'$ irreducible $\Rightarrow N = 0 \& \rho''$ irreducible.

We've kicked the defn 'round but haven't seen many examples yet.

We will get a handle on indecomposable reps of $W_{2k}$. 

$Lecture 7$

Sat 30th June 03
Example \( \text{Sp}(n) \), the special rep, is the rep \( (\rho^*, N) \) of \( \mathbb{W}_K \) over \( \mathbb{Q} \)

\[
V = \mathbb{Q} e_0 \oplus \mathbb{Q} e_{n-1}
\]

\[
\rho^*(w)e_i = \omega_i(w)e_i \quad \text{(ss action)}
\]

(Recall) \( \omega_i : \mathbb{W}_K \rightarrow \mathbb{W}_K \overset{\rho}{\rightarrow} K \overset{\rho^*}{\rightarrow} \mathbb{C}^* \)

& \quad \text{Ne}_e = e_{n-1}, \quad \text{Ne}_{n-1} = 0

\text{Sp}(n) \) is semisimple & indecomposable.

Prop 5.3 Every ss indec rep of \( \mathbb{W}_K \) is of the form \( \sigma^* \otimes \text{Sp}(n) \),
where \( \sigma^* \) is indecomposable irreducible. \( \square \) (Deligne Rep-fact)

[Note: Here \( \rho^* \otimes \sigma^* = (\rho^*, N) \otimes (\sigma^*, M) = (\rho^* \otimes \sigma^*, N \otimes M) \).]

Conjecture (Langlands) There is a "natural" bijection between IM classes of
dimensions \( n \) ss \( \mathbb{W}_K \) reps & of admissible reps of \( \text{GL}_n(K) \).

There are also L-series on e factors which match up. ("natural" \( \leftrightarrow \) supercuspidal, etc.

\( n - 1 \) is "just" classfield theory. \( (n-1) \otimes \mathbb{N} = \text{rep of } \mathbb{W}_K \overset{\rho}{\rightarrow} \text{rep of } K^\times : \text{GL}_n(K) \)

L-adic reps of \( sp(n) \)

\( K/\mathbb{Q}_p \); we also have \( k \) and \( q = |k|_p \)

\( K_n \) the unique ext of \( k \) of degree \( n \) (in \( K \)).

\( K_n \) is the non-ramified ext. of \( K \) of degree \( n \) (in \( K^\vee \)).

(see pg 182) \( T_n \) is the nodal totally tamely ramified ext. of \( K_n \) (in \( K^\vee \)).

\[
\mathbb{G}(T_n/K_n) \cong k^* , \quad \text{a } K^* - \text{IM.}
\]

\[
\mathbb{I}_k/P_k = \lim_{\rightarrow} \mathbb{G}(T_n/K_n) \cong \lim_{\leftarrow} k^* = \prod_{q \mid p} \mathbb{Z}_q
\]

\[
(G(T_n/K_n) \rightarrow G(T_n/K_n/K) \cong G(T_n/K_n))
\]

\[
\text{Def: } \text{Tr}_e : \mathbb{I}_k \rightarrow \mathbb{Z}_l \quad \text{is the tame character associated to } l
\]

\[
\text{We put canonical because } \lim_{\rightarrow} k^* = \prod_{q \mid p} \mathbb{Z}_q \text{ is int.}
\]
All chains $I_{\ell} \to \mathbb{Z}_{\ell}$ are multiples of $t_{\ell}$.

**Note:** We have $t_{\ell}(\text{w} \cdot \text{w}^{-1}) = \text{w}(t_{\ell}(\text{w})) = \text{w} \cdot t_{\ell}(\text{w})$

\[
\text{w}(\text{w}) = \text{w} \cdot t_{\ell}(\text{w}) \quad \text{and} \quad t_{\ell}(\text{w}) = t_{\ell}(\text{w}) \cdot \text{w}.
\]

The Haupsatz of the day:

Then $\exists$ nilpotent $\text{Ne}	ext{End}(V)$ st. $\rho(\sigma) = \exp(t_{\ell}(\sigma)N)$ for $\sigma$ open nbhd of $1$ in $I_{\ell}$ st.

\[\rho(\sigma) \cdot N \cdot \rho(\sigma)^{-1} = \text{null}N.\]

**Sketch of Proof:**

1. $\rho(1)$ is in $\text{GL}(V)$.
2. $\rho(1)$ stabilizes some $\mathbb{Z}_{\ell}$ lattice in $V$.

\[
\rho(I) \to \text{GL}(\mathbb{Z}_{\ell})
\]

\[
\begin{align*}
U & \to 1 + t_{\ell}M_{n}(\mathbb{Z}_{\ell}) \\
& \to \exp \left( t_{\ell}M_{n}(\mathbb{Z}_{\ell}) \right)
\end{align*}
\]

& replacing $K$ by a finite ext.

Get $\rho: I \to 1 + t_{\ell}M_{n}(\mathbb{Z}_{\ell})$.

Then $\rho(1) = \exp N$.

Say $\rho(1) = \exp(N) \cdot \exp(N)$.

Then $\rho(2) = \rho(1) \cdot \rho(1) = \exp(N) \cdot \exp(2N)$.

$\rho(2) = \exp(N)$.

So we get $\rho(\sigma) = \exp(t_{\ell}(\sigma)N) \cdot \text{null}N$. We're not shown $\rho$ holds yet.

This is true slightly shrunken $I$.

Now note $\rho(\text{w} \cdot \text{w}^{-1}) = \exp(t_{\ell}(\text{w} \cdot \text{w}^{-1})N)$.

\[
\rho(\text{w}) \cdot \rho(\text{w})^{-1} = \rho(\text{w}) \cdot \exp(t_{\ell}(\sigma)N) \rho(\text{w})^{-1}.
\]

Both these are $\equiv 1 \mod \ell$, so we can take logs.

$\rho(\sigma) \cdot t_{\ell}(\sigma)N \cdot \rho(\text{w})^{-1} = t_{\ell}(\sigma) \cdot \text{null}N$. Canceling $\sigma$ $t_{\ell}(\sigma)$ gives $\diamondsuit$. 
Finally we have to show $N$ is nilpotent.

But the eigenvalues of $N$ are stable under multiplication by $q^2$, by the 
dodge we used yesterday, so we're done. □

**Cor 1.** If new $p$ is ss, then ker $p$ is open (in $W_k$)

**Proof.** $p$ is ss = @ irreducible. So Wlog we can take $p$ irreducible.

Certainly ker $N$ is $W_k$-stable by @ $p(u)Np(u) = N(u)$

$N$ is nilpotent $\Rightarrow$ ker $N$ is open

$p(o) = \exp(1, o)N \forall o \in I$ by thm.

So $p(o) = 1$ is kernel is open

$\forall o \in I$ (hmm...)

**Cor 2.** If $p$ is ss & irreducible, then it can be written $\sigma \cdot \chi$ where $\chi$ is a char-

(is what he calls = quasichar), $\sigma$ is irreducible Galois type.

In fact, if $p \not\equiv 2, l, \sigma = I$-dim, then $p$ is monomial.

**Proof.** Use Cor 1 & now just apply (5.1) & (5.2) □

The last of our results is

**Theorem (Deligne) (1+P)**

$\Phi \in W_k, R(\Phi) = 1$

Then for $\sigma \in I$, $n \in \mathbb{Z}$, the equality

\[ \sigma(\Phi^n) = \rho^n(1) \exp(1, o)N \] 

sets up a bijection between

\[ \mathbb{Q}_l\text{-adic reps of } W_k \rightarrow \mathbb{Q}_l\text{-rep of WD}_k \text{ over } \mathbb{Q}_l \]

**Explanation of $\Phi$**

$x.$ Given $p$, apply (5.4) & thus produce a nilpotent $N$

$\cdot$ now define $\rho^n$ - check its a rep.

Then by (5.4), ker $\rho^n$ is open

$x.$ Given WD$_k$-rep $(\rho^n, N)$, then (5.4) defines $\rho$ □
II. GL over a local field

Recall that the lecture this afternoon at 2:15...

John Coates has asked me to ask us to ask questions.

He'll be talking about reps of GL(n), GL(a) & what these have to do with classical modular forms.

Classical theory: E.d. vs. S_k(1) with operators (Tn) etc which have a lot with the Fourier series of f ∈ S_k(1)

Adelic theory: ss. dim. rep of GL(n) = Ti GL(a) × GL(b)

with certain distinguished subspaces & algebra of operators also called Hecke operators, although Hecke would have known...

- explicit module (Kirillov) of local reps

Why? Well, adele setting gives us a ability to split stuff from global to local, especially helpful in case of GL(2) & understanding ramification of primes a bit better.

The plan: 1) GL(n)/adeles eg principal series. This is more difficult than GL(n), but much more number theoretic, & less analytical.

II) GL(n) GL(b)

III) p-adic Kirillov model, Atkin-Lehner theory

IV) L & E-factors, local Langlands Correspondence for GL(n)

References: Jacquet-Langlands, Lecture notes on L-series.

Greenberg, "Notes on J-L theory."

Articles by Deligne, Carayol in Antwerp vol II (modular forms, p-adic)

Corvallis proceedings: eg. Cartier (generalising adelic p-adic reps of GL(n)).

E.g. Shalika beyond a reference to Shalika by Lang.
3.2. If we go

I GL(\mathbb{Q}_p - \text{prime field})

3.1. F: a finite set of \( p_i, q, \{m_i\} \) are ideal.

\[ \pi = q^{\langle m_i \rangle} \quad \text{if} \quad q = \# (O/\mathfrak{m}) \]

\[ v(x) \text{ normalized:} \quad v(\pi x) = n \quad \text{if} \quad x \in \mathfrak{m} \]

\[ \text{ord}(x) \quad (\text{as } v \text{ is a weight lattice elsewhere}) \]

\[ G = GL_2(\mathbb{Q}_p), \quad \text{a topological group} \]

\[ g, h \in G \quad \text{are congruent modulo some high power of } \pi \]

\[ \Leftrightarrow \quad gh^{-1} = T \quad \text{mod some high power of } \pi \]

- topology on \( G \) is generated by the open subgroups

\[ K_n = \{ g \in GL_2(\mathbb{Q}_p) \mid g = T \mod \pi^n \} \]

& their translates

\[ K_0 \text{ is compact, as its profinite} \]

Prop. Any open subgroup \( G \) is conjugate to a subgroup of \( GL_2(\mathbb{Q}_p) \). In particular, \( GL_2(\mathbb{Q}_p) \) is a profinite group. (verify the last induction step, use \( m(g^{-1} H g) \cdot m(H) \)?)

(No! look at normalizer of \( GL_2(\mathbb{Q}_p) \) \( \cdot \) a corollary)

Proof. \( K \) a open subgroup of \( G \). Then \( K \) leaves fixed some lattice \( \Lambda_0 \subset \mathfrak{m} \mathfrak{m} \subset \mathbb{Q}_p \) (take \( \Lambda = \mathbb{Z}^2 \); then \( K \cdot \Lambda \) is open, so generates a lattice, int by \( K \))

\( \mathbb{Q}_p \) (e.g., standard basis, get \( K \subset GL_2(\mathbb{Q}_p) \) \( \square \))

\( K_n \): Every open nbhd of \( e \in G \) contains a compact open subgroup

Always \( K \): some open subgroup of \( G \). It's often not important which, since \( K \cdot K \) is of finite index in \( K \).

Haar measure. Normalize the measure so that \( \text{meas}(K) = 1 \) if \( K \subset GL_2(\mathbb{Q}_p) \)

Extend to \( \text{ext} gK \) by invariance.

\[ \varphi \mapsto \varphi \circ g \]

We can extend this to the functional \( m: C_c^\infty(G) \rightarrow \mathbb{C} \)

\[ \{ \text{locally cts} \} \quad \text{finite linear combinations}\]

\[ \{ f: \text{on } G \} \quad \{ \text{char. } f \text{ of } gK \} \]

(Nonlinear)
\[ \int_G \text{ch}(xg) \, dg = \text{meas}(K) \]

and
\[ \int_G \phi(g) \, dg = \int_G \phi(g) \, d(\chi(g)) = \int_G \phi(g) \, dg \]

\( G \) is unimodular so \( \text{d}g = d(\chi(g)) \)

If \( H \subset G \) is an algebraic subgroup we need
\[ N = \{(1, 0)\}, \quad A = \{(0, \lambda)\}, \quad B = \{(e, 0)\} = \text{NA} \]

We can normalise the Haar measure on \( H \) by \( \text{meas}(H \cap \text{GL}(1, \mathbb{Q})) = 1 \)

Measure: \( N = \{(1, 0)\} \) in \( da \) (additive Haar measure on \( F \))

\[ A = \{(0, \lambda)\} : d^* \chi_{-a}, \quad \text{where } d^* \chi \text{ is multiplicative Haar measure on } F, \quad \text{meas}(F^*) = 1 \]

\[ \left( \text{So } d^* \chi_{-a} = \frac{d a}{a} \right) \]

\[ B = \{(1, 0)\}, \quad \left[ \frac{d a}{a} \right] d^* \chi_{-a}, \quad \text{is the measure} \]

\[ N_8 \{(5, 0)(1, 0)\} = \{(1, 0)(5, 0)\} = \{(1, 0)(5, 0)\} = \{(1, 0)(5, 0)\} \quad \text{so we're not unimodular} \]

\( (N_8: \text{the all works for } f^* \text{ with values in any field of char zero.}) \)

Prop. 2: \( \text{Iwahori decomposition } \). Let \( K = \text{GL}(2) \)

Then \( G = \text{NAK} \cdot \text{KAN} = KB \)

Moreover,
\[ \int_G \phi(g) \, dg = \int_G \left( \prod_{i=1}^{n} ((5, 0)(1, 0)K) \right) \frac{d a}{a} d^* \chi_{-a}, \, dK \]

where \( dK \) is restriction of \( d(gK) \) to \( K \)

Prop. \( G/\mathbb{R} \cong \mathbb{P}^1(\mathbb{F}) \) because \( G \) acts transitively on \( \mathbb{P}^1(\mathbb{F}) \) by fractional linear transformation, \( B \) is stabiliser of \( 0 \). Also, \( GL(1, \mathbb{F}) \) acts transitively on \( \mathbb{P}^1(\mathbb{F}) \)

RH measure is left-invariant under \( R \), right-invariant under \( K \), so is multiple of Haar measure.

Finally, if \( \text{char } \mathbb{F} = \text{char } \mathbb{F}^* \) of \( GL(1, \mathbb{F}) \) then LHS = RHS = 1
Prop 3 (Cartan decomposition) \( K = G \Delta (O) \)

\[
G = \text{KAK} = \bigcup_{\text{min}} \text{K}(\mathfrak{a}, \mathfrak{a}^*) \text{K}
\]

Proof: amounts to showing that if \( \Lambda, \Lambda' \) are lattices in \( F^k \), then there is \( \mu, \nu \) such that \( e_i e_j + e_j e_i = \mu e_i + \nu e_j \) for some \( \mu, \nu \). \( \square \)

Prop 4 (Bruhat decomposition) If \( w \in G(\mathbb{F}_r) \) then \( G = BuBuB = BuBuSwBuSwB = BuSwBuSwB \)

where \( BuB = \{ g \in G(\mathbb{F}_r) \mid r \text{ is fixed} \} \).

Proof: N. \( G/S = B(\mathbb{F}_s) \) has 3 orbits, \( S \Delta \mathbb{F}_s \). \( \square \)

BuB is the "big cell."

Def: The Hecke algebra \( H(G) \) is the space \( C^\infty_c(G) \) of locally \( c \)-compact support on \( G \), with convolution product

\[
(f * g)(x) = \int g(y)f(x - y)dy
\]

an associative algebra (NB we have normalized the Haar measure\( \), \( \omega \) normalizes \( * \), \( \omega \) the measure of the \( \text{max} \) set \( \omega/2 \). \( \text{It has no unit (would be \( \delta_x \).) } \)

Sect. 2. Representations of \( G \)

Let \( \pi : G \to GL(V) \) be a homomorphism, \( V \) a \( \mathbb{C} \)-vector space.

Then \( V \) will almost always be \text{irreducible}. The thing is, even though \( V \) is \( \text{irreducible} \), there are some \( \text{way} \) some cases where \( V \) is sort of like a \text{1-dim thing}.

However, there are also some \text{very nasty cases of} \( G \).

Def: \( v \in V \) is \text{smooth} if the stabilizer is an open \text{subgroup} of \( G \).

Thus is quite a strong condition.

Say \( (\pi, V) \) is a smooth rep if every \( v \in V \) is smooth.

Prop 5: \( (\pi, V) \) is smooth \( \Rightarrow V = \bigcup_k V^k \), \( K \) running over all \text{open \text{subgroups}} of \( G \).

(Here \( V^k : K \)-invariants of \( V \))

(iii) \( v \in V \) is a \text{smooth vector} \Rightarrow \text{span} \{ \pi(k)v \mid k \in K \} \text{is fin. (for any} K). \text{Thus} v \text{is \text{finite}}.


Prop 5: Every smooth irreducible rep of $G$ of finite dimension is of the form $V \to \mathbb{C}$

Prop 6: Let $(n, V)$ be a finite dim smooth rep of $G$. The kernel $H = \ker(n)$ is an open normal subgroup of $G$. The image $\tilde{n}$ is a finite group (as the $\tilde{n}$ of stabilizers of all of the basis for $V$).

So $H > (1, I)$ for $1 \neq 1, 1 \neq 0$, for all $(1, I), x \neq 0$, are conjugate via $G$. Also $H \supset (1, I)$.

So $\ker(n) \leq (1, I)$, i.e. $\ker(n) = \text{SL}(V)$ (trivial).

So $V$ factors through $\tilde{n}$, so $\text{Im } V$ is abelian. Hence if $V$ is irreducible, $V = (1, I)$.

Example of irreducible smooth rep:

Let $V$ be locally const $f: G \to [0, \infty)$, not in the left $G$.

$\{ \text{locally const } f: \mathbb{R} \to \mathbb{R}, \text{int on the left } G \}$

We have $u(lg) = u(g), \forall g \in G, \forall \sigma \in G$.

$(\sigma(g), g)(x) = (g, \sigma g)$. It's a smooth rep.

It's one of a large family of reps that will look at in lecture 4.

The only irreducible subspace is $\{ \text{loc const } f: \}$ (we'll prove this later).

& so the quotient is an irreducible smooth rep of $G$. 
Recall this morning we looked at \((\pi, V)\) smooth rep of \(G = \text{GL}(F)\).

This will give us a certain rep of an algebra & this rep will often be f.d.

Say \(\pi : \mathcal{H}(G) \rightarrow \mathcal{C}_r(G)\) under \(\ast\) = convolution.

Define \(\pi(\varphi) : V \rightarrow V\) by \(\pi(\varphi)v = \int \varphi(g) \pi(g)v \, dg\)

\[\pi(g) v = \int_k \delta(g) \, \pi(\varphi) v \, dg\]

\(\pi(\varphi \cdot \psi) = \pi(\varphi) \cdot \pi(\psi)\)

Note: so we have a HM \(\mathcal{H}(G) \rightarrow \text{End}(V)\).

Now for \(K\). Define \(e_K = \text{char}_K / \text{meas}(K)\).

Note: \(e_K^2 = e_K\).

\(\pi(e_K) v \in V^K\) as \(\pi(k) \pi(e_K)v = \int_k \pi(k) \pi(e_K)v \, dk\)

\(\pi(e_K) v \in V^K\) (change of variables) \(\pi(e_K) v \in V^K\)

So \(\pi(e_K) : V \rightarrow V^K\) in a projector.

Remark: If \(\varphi \in \mathcal{H}(G)\), then \(\mathcal{H}(G) \ast (\varphi(g)) = \varphi(g)\) for all \(g \in K\), & hence \(e_K \ast \varphi = \varphi\).

Given the action of \(\mathcal{H}(G)\) on \(V\), we can recover the action of \(G\) thus:

If \(v \in V\), then \(v \in V^K\) for some \(K\), & \(\pi(g)v = \pi(e_K) \pi(g) v\)

\((\text{change of variables}) / \text{meas}(K)\)

\(\pi(g)v \in \mathcal{H}(G)\)

\(\pi(g)v \in \mathcal{H}(G)\)

Thus: The above construction gives a bijection:

\{
\text{smooth reps} \} \leftrightarrow \{
\text{rep. } \mathcal{H}(G) \rightarrow \text{EndV which are non-degenerate i.e. st. } v \in V^K
\}

\(\text{End } \mathcal{H}(G)\text{ with } \varphi(v) = v\)

Thus in fact gives an equivalence of categories.
Now define $\mathcal{H}(G,K) = e_K \mathcal{H}(G)e_K$.

$\mathcal{H}(G,K)$ is a subalgebra of $\mathcal{H}(G)$, with $e_K$ as unit.

If $\phi \in \mathcal{H}(G,K)$, $\text{re} \in V$, then $\tau_{G}(\phi) \text{re} \in V^K$, so we get a $\text{HM}$

$\mathcal{H}(G,K) \to \text{End}(V^K)$.

**Then 2.** Let $(\tau, V)$, $\lambda = 2$, be smooth reps of $G$. Assume:

(i) $V_1$ is generated as $\mathcal{H}(G)$ module by $V_1^K$.

(ii) Every $G$-invariant subspace of $V_2$ contains a non-zero vector fixed by $K$.

Then $\text{Hom}_G(V_1, V_2) = \text{Hom}_{\mathcal{H}(G,K)}(V_1^K, V_2^K) \quad \Box$

**Cor.** If $(\tau, V_1)$ are irreducible, & $V_1^K, V_2^K \subset [0,1]$, then $V_i^K$ are indecomposable $\mathcal{H}(G,K)$-modules, & $V_i^K \cong V_i^K$ ($\Rightarrow V_i \cong V_i$).

**Proof.** Let $F : V_1 \to V_2$ be a $G$-HM (NB: we'll never mention the field $F$ so there's no notational problem). Then $F|_{V_1^K} : V_1^K \to V_2^K$ is a $\mathcal{H}(G,K)$-HM.

Suppose now that $F : V_i^K \to V_i^K$ is an $\mathcal{H}(G,K)$-module HM. We want to extend $F$ to a $G$-HM $\tilde{F} : V_i \to V_i$.

Since $V_i - \tau_i(\mathcal{H}(G)) V_i^K$, we try to define

$\tilde{F}(\sum_j \tau_i(\phi_j)v_j) = \sum_j \tau_i(\phi_j)F(v_j), \quad v_j \in V_i^K, \quad \phi_j \in \mathcal{H}(G)$ (NB: he put $\mathcal{H}(G,K)$)

We need to check that this def. is unambiguous, & then we're done because $F$ is clearly unique.

So $\text{STP}$ $\sum_j \tau_i(\phi_j)v_j = 0$ $\Rightarrow$ $\sum_j \tau_i(\phi_j)F(v_j) = 0$.

But $\text{LHS:} 0$ $\Rightarrow$ $\sum_j \tau_i(e_K)\tau_i(\phi_j)v_j = 0$ $\forall \phi_j \in G$

$\Rightarrow$ $\sum_j \tau_i(e_K)\tau_i(\phi_j)v_j = 0$ $\Rightarrow$ $\sum_j \tau_i(e_K)\tau_i(\phi_j)F(v_j) = 0$

because $\tau_i(e_K)\tau_i(\phi_j) \in \mathcal{H}(G,K)$

$\Rightarrow$ $G$-module spanned by $\sum_j \tau_i(\phi_j)F(v_j)$ has no $K$-invts

$\Rightarrow$ $\sum_j \tau_i(\phi_j)F(v_j) = 0$ $\quad \Box$
Recall that $K$ is always a compact open subgroup of $G$.

**Def.** Let $(\pi, V)$ be a rep. of $G$. It is **admissible** if

1. it is smooth
2. $\dim(V^K) < \infty$ for all $K$. (This is a finiteness cond.)

Note that if $(\pi, V)$ is admissible and $V^K = \{0\}$, then we get a full rep. of $G$ a rep. of $G/K$. Then determines $\pi$ up to isom.

**Prop.** $(\pi, V)$ is admissible $\iff$ it is smooth and $\pi(\text{ad})$ has finite rank for all $\text{ad}$ of $G$

$\iff$ for some (any) $K$, $V$ is a direct sum of reps of $K$ (finite, by $\pi$ uniquely determined only a finite no. of times).

Proof:

1st equivalence: $\dim V^K < \infty \iff \text{rank } \pi(\text{ad}) < \infty$.

Now taking $K = \text{ad}$ gives (1), and taking $K$ s.t. $K \subseteq \text{ad}$ gives (2).

2nd equivalence: $V$ admissible, then for all normal, open $K \subseteq K$ we have that $V^K$ is a f.d. rep. of $K/k$; so is completely reducible $\iff V = \oplus_i$ of irreps of $K$. Multiplicities finite, as if $K = \text{ker}(\rho) K \subseteq GL_n(C)$ then $\dim V^K = N \times (\text{multiplicity of } \rho) < \infty$. 

Now a def. as the admissible reps of $G$, and this is what we'll be doing.

The **contragredient** of a smooth rep. $(\pi, V)$:

Let $V^* = \text{Hom}(V, C)$. We have $\times : V^* \otimes V \to C$.

Let $V^*(K) = \{ v^* \in V^* \mid \langle v^*, \pi(k)v \rangle = \langle v^*, v \rangle \forall v \in V \}$

= annihilator of $(\pi(\text{ad}) - 1) V \cong (V^K)^*$

Let $\hat{V} = \bigcup_{K} V^*(K)$, and let $G$ act by $\langle mg, v^* \rangle = \langle v^*, mg^*v \rangle$. Then $\hat{V}$ is a smooth rep. of $G$, called the **contragredient** of $G$.

If $\pi$ is admissible, then $\hat{V}^* = (V^K)^*$ is f.d., so $\pi$ is admissible & $\hat{\pi} = \pi$.

(Note $V = V^K \oplus \langle e_K - 1 \rangle V$)
Schur's lemma

Let \((\pi, V)\) be an irreducible rep of \(G\), & let \(f: V \rightarrow V^*\), linear, commute with \(\pi(g)\) for all \(g \in G\). Then \(f\) is a scalar.

**Proof.** Take \(K\), \(\lambda\) commutes with \(e_K\) & with \(\pi(G, K)\); so \(\lambda|_{V_K}\) is an endomorphism commuting with \(\pi(K, K)\). Take an eigenvalue \(\lambda\); then ker\((\lambda - \lambda|_{V_K})\) is a subspace of \(V_K\) invariant under \(\pi(K, K)\). So by the Corollary to thin 2, \(f = \lambda\) on \(V^*\). True for \(f = \lambda\) on \(V\). \(\square\)

**Cor.** Let \(\pi\) be an irreducible rep of \(G\). Then \(\exists \text{ HM } w: \omega_K: F^* \rightarrow C^*, \text{ s.t. } \pi(w\alpha) = \omega_K(\alpha) v \forall \alpha \in F^*, v \in V\). \(\omega_K\) is called the central sheaf of \(\pi\).

**Remark.** \(\chi: F^* \rightarrow C^*\) is a character (ie a ch HM) (ie \(\chi = 1\) on an open subgp of \(G^*\)) then we can define, given \((\pi, V)\), a rep \((\pi \otimes \chi, V)\) via 

\[
(\pi \otimes \chi)(g) = \pi(g) \chi(g) \delta(g)
\]

**Theorem.** Let \(\pi\) be irreducible. Then \(\pi = \pi \otimes \omega^0_K\). NB this is deeper than you think. \(\square\)

In this lecture he'll define a particular class of representations.

**§3 Unramified representations**

Say \((\pi, V)\) a rep of \(G\), assumed irreducible, admissible.

\[
\begin{array}{ccc}
\text{no non-trivial} & \Rightarrow & \text{rep}\in\text{admissible} \\
\text{irred. subrep} & \Leftrightarrow & \text{rep}\in\text{irred. subrep} \\
\text{end. V} & \Rightarrow & \text{rep}\in\text{end. subrep} \\
\end{array}
\]

Last time we showed that if \(K < G\) is open & cpt then we get a (fd.) rep

\[
\left\{ \text{functions on } G \text{ with } \\
\text{compacts support, left & right } K\text{-int.} \right\} \xrightarrow{\pi(G, K)} \text{End}(V^K)
\]

& this determines \((\pi, V)\) up to isomorphism. (If we know it for all some \(K\) with various special properties)

For this, \$ we will set \(K = \text{GL}_1(O)\), i.e. a maximal cpt.

**Def.** \((\pi, V)\) irreducible is unramified if \(V^{\text{GL}_1(O)} = \{0\}\).

We will completely determine all unramified \(\pi\)'s.
Theorem 3 (a classical thm.) $H(G, K)$ is commutative for this $K$.

Moreover, $H(G, K) = C[T, \pi, S, S^*].$ Here $\pi$ is a uniform rep., not the rep. & so we may well drop the $\pi$'s later.

Here $T = T_\pi = \text{char} K(\mathbb{F}_p)$ & $S = S_\pi = \text{char} K(\mathbb{F}_p)$.

$T_\pi$ & $S_\pi$ will be basically the classical Hecke operators, modified by suitable powers of $p$.

Case: If $(m, \nu)$ is unramified, then $\dim \nu^k = 1$ & it is determined by

$\pi(T) \in C$
$\pi(S) \in C^*$ (up to isomorphism)

\[ G = \prod_{m \in \mathbb{Z}} K(T_{\nu^m}, \pi) \quad \text{(Cartan)} \]

& so $H(G, K)$ has for a base the functions $\mathbb{C}[T_{\nu^m}, \pi^l].$ \text{char } K(T_{\nu^m}, \pi^l).

We can now proceed classically by explicitly computing everything, see e.g. chapter 3 of Shimura's book, where he does it for $Gln$. Alternatively, there's a more modern approach which generalizes well:

Let $A = \{ (z, y) \in A^0 \times \mathbb{C}^* \mid 0 \neq y \} \subset A \times \mathbb{C}^*.$

Form the Hecke algebra $H(A, A^0) = \{ \text{functions } f \text{ of finite support on } A/A^0 \subset \mathbb{C}^* \}$

as $A$ is abelian

\[ = \prod_{r \in \mathbb{Z}} C \lambda_{r,s} \]

with $\lambda_{r,s}$ the char of $(T_{\nu^s} O^* \cap O^*)$.

It's easy to see that $\lambda_{r,s} \ast \lambda_{r', s'} = \lambda_{r+r', s+s'}.$ (Here we have normalized the Haar measure $st. m_n(A) \times (A')$)

& hence $H(A, A^0) = C \times \times (xy)^*.$ with $x = \lambda_{r, 0}, y = \lambda_{0, s}.$
Now define the **Satake Transform**

\[ \Sigma : \mathcal{H}(G,K) \rightarrow \mathcal{H}(A,A^o) \]

by \((\Sigma \varphi)(a) = \delta(a)^{\frac{1}{k}} \int_F \varphi(a(a^o)) \, da\)

where \(\delta(a) = \left| \frac{a}{a_1} \right| \), this is the modular character for upper-triangular matrices, & it'll become clear why it appears, later.

\(A^o \in K \Rightarrow \Sigma \varphi \) is indeed invariant by \(A^o\), so we have a well-defined map.

Let \(G_2 = \text{Group } S_2\) be the symmetric group of degree 2, acting on \(A^o\) by letting the non-trivial element send \((1,0)\) to \((0,1)\).

\((G_2 \text{ is the Weyl group of } GL(2)). \) Then \(\Sigma \varphi\) is invariant under \(G_2\) ( & hence the reason \(\delta(a)^{\frac{1}{k}}\)).

**Thm.** \(\Sigma\) is an isomorphism of algebras \(\mathcal{H}(G,K) \rightarrow \mathcal{H}(A,A^o)^{G_2} \)

This is the **Satake Isomorphism** & it generalizes to a wide class of groups.

So we have shown \(\Sigma : \mathcal{H}(G,K) \cong \mathbb{C}[x,y,(xy)^{-1}]^{G_2} = \mathbb{C}[x,y,xy,(xy)^{-1}]\).

We will show that \(\Sigma : T \rightarrow q^s(xy)\) & \(T \rightarrow xy\), thus proving them 3.

The proof: 1. Algebra HM

2. Calculate \(\Sigma \varphi \in \mathbb{C}[x,y]^{G_2}\)

1. \(1 \, \text{ is tedious: if } \varphi, \psi \in \mathcal{H}(G,K), \text{ then } (\Sigma(\varphi \ast \psi))(a) = \delta(a)^{\frac{1}{k}} \int_N \varphi(g)\psi(g^{-1}an) \, dg \, dn\)

where \(N = \{c \in G \mid \text{ unipotent} \}\)

But \(G = BK\) by Iwasawa:

\[ \delta(a)^{\frac{1}{k}} \int_{B \cdot K \cdot N} \varphi(bk)\psi(k^{-1}b^{-1}an) \, db \, dk \, dn \]

\[ \delta(a)^{\frac{1}{k}} \int_{B \cdot N} \varphi(b)\psi(b^{-1}an) \, db \, dn \]

\(\psi, \varphi\) are \(K\)-int.
So \((\Sigma (\varphi \times \varphi))(a) = \delta(a)^{N} \int B_{N} \varphi(b) \varphi(b^{-1}an) d\lambda d\gamma\).

Now \((\Sigma \varphi)(\Sigma \varphi'))(d_{a}) = \int \Sigma \varphi(a_{1}) \Sigma \varphi'(a_{2}) da_{2}

= \int \delta(a_{1})^{N} \int \varphi(a_{1} \lambda_{1}) d\lambda_{1} \delta(a_{2})^{N} \int \varphi'(a_{2} \lambda_{2}) d\lambda_{2} d\gamma

= \delta(a)^{N} \int B_{N} \varphi(b) \varphi(b^{-1}an) d\gamma db

Here \(n_{1} = a^{-1}b, n_{2} = b^{-1}a\)

There seems to be a typo here.

= \delta(a)^{N} \int B_{N} \varphi(b) \varphi(b^{-1}an) d\gamma db, say, \(n = \lambda n_{1} \lambda n_{2}\)

= \delta(a)^{N} \int B_{N} \varphi(b) \varphi(b^{-1}an) d\gamma db

\(\Sigma \Sigma[\pi^{m}, \pi^{m}] = \text{char}_{K(\pi^{m})} \chi, \quad m \geq n\)

= \sum_{k=0}^{m-n} \sum_{b \equiv k} \text{char}_{K(\pi^{m+k} \pi^{m-k})}

So \(\Sigma \Sigma[\pi^{m}, \pi^{m}](\pi^{m}) = 0\) unless for some \(k\) we have \(m-k=r\), \(m-k-s\) in which case we have

= \(q^{-(r-1)/2} \sum_{b \equiv k} \int_{F} \text{char}_{K(\pi^{m+k} \pi^{m-k})}(x, y) dx\)

= \(0, \quad x \not \equiv 0 \mod (\pi^{m+k} \pi^{m-k}), \quad 1\) otherwise (for some \(b\)),

= \(q^{-(r-1)/2} \text{meas}(\pi^{m+k} \pi^{m-k})\)

= \(q^{-(r-1)/2 - 1/2} = q^{-1/2} q^{-(m+k)/2}\)

\(\sum \Sigma[\pi^{m}, \pi^{m}] = q^{(m+k)/2} \sum_{k=0}^{m-n} \lambda_{m,k} \lambda_{m-k} q^{-(m+k)/2} (x^{m+k} y^{m+k} + x^{m} y^{m})\)

From this it follows that \(\text{Im} \Sigma = C \left[ x, y, x^{-1} y^{-1} \right] \Omega_{2}\) and that \(\Sigma(z) = q^{4}(x^{m+k} y^{m+k} + x^{m} y^{m})\)

\(\Sigma(\pi) = xy\)
Remark 4. \( \mathcal{H}(G,K) \) is called the unramified or spherical Hecke algebra.
("Spherical" is in analogy with the real case when \( K = \text{SO}(2) \) & \( F^2 \) which are bi-invertible.) Quite important for infinite tensor products.

If \( (\pi,V) \) is an unramified rep, then any non-zero \( \phi \in V^K \) is called a spherical vector. The sign class of such \( \pi \) is hence determined by \( \pi(T) = \chi_\pi(x\psi) \) and \( \pi(S) = x\beta \), where \( x, \beta \) are \( \eta(x), \eta(y) \), or in other words, by the conjugacy class of the semi-simple matrix \( (\beta \hat{x}) \in \text{GL}_2(F) \). \( x, \beta \) are sometimes called the unramified parameter for \( \pi \).

Ex. \( \mathcal{H}(G_K) \cong \mathcal{C}[\text{GL}_2(F)] \)

In fact, in general, \( \mathcal{H}(G,K) \cong \mathcal{C}[G] \) for a certain gp \( G / K \).

Example. \( \pi(g) = X(\det g) \), \( \pi: F^x \to \mathcal{C}^x \) is an unramified character.

Then (watch out for 2 kinds of \( \pi \).) \( \pi(T) = \pi(\hat{x}^0) = \chi_\pi(x^2) \)

\[ \pi(S) = \pi(\hat{x}^0) + \sum_{\text{bad } \hat{S}} \pi(\hat{S}) \]

\[ = \eta(x) \chi_\pi(x) \]

\( \therefore \) the parameter \( \alpha = \begin{pmatrix} \eta(x) & 0 \\ 0 & \eta_\pi(x) \end{pmatrix} \)

So every unramified rep. with parameter \( x, \beta \) s.t. \( x\beta^4 + y^2 \) must be 2-dim.

Most of this lecture will deal with 1 kind of rep of \( G \), the principal series.

S4. The principal series of \( G \)

If \( a = (\beta, \alpha) \in A \), then \( i(a) \cdot 10 \).

The idea: \( B = (\alpha) \) - a 1-dim rep of \( B \) defined by 2 chars (along the diagonal) can be induced up to a rep of \( G \).

A def. of a rep is coming up.
Let $\mu_1, \mu_2: \mathbb{F}^* \to \mathbb{C}^*$ be 2 characters (ch. homs).

Let $G$ act by right multiplication translation $g$.

Let $\mathcal{B}(\mu_1, \mu_2) = \{ \text{loc. cl. } \phi: \mathbb{F}^* \to \mathbb{C} | \phi(\text{ang}) = \mu_1(\alpha_x) \mu_2(\alpha_y) \delta(x) \delta(y) \}$

$(\mathcal{B}(\mu_1, \mu_2), \alpha \in \mathbb{F}, \nu \in \mathbb{R}^+, \phi \in \mathcal{B})$

Letting $G$ act by right translation gives a representation $\rho(\mu_1, \mu_2)$ of $G$ on this space. This is admissible, since every $\mathcal{B}$ is determined by its restriction $\mathcal{B}|_K$ to $K = \text{GL}_2(\mathbb{F})$, which is $G$-locally at $f^*$ on $K$.

Note that $\rho(\mu_1, \mu_2)(f^*) = \mu_1(\mu_2(a))$ is a scalar.

However, $\rho(\mu_1, \mu_2)$ may not be irreducible.

**Determine when $\rho(\mu_1, \mu_2)$ is irreducible.**

Recall the Bruhat decomposition $G = BN \cup WN$, $W = \{ g0 | g \in G \}$

So $\mathcal{B}(\mu_1, \mu_2)$ is determined by $\varphi(x) = \mathbb{E}(w(\mu_1, \mu_2))(\varphi(w)) = \mathbb{E}(\mu_1, \mu_2)(\varphi(w))$.

So by the idea $\mathcal{B} \rightarrow \varphi$ we can replace $\mathcal{B}(\mu_1, \mu_2)$ by a space $V = V(\mu_1, \mu_2)$ of locally at $f^*$'s on $F$.

Write $\nabla$ for the rep of $G$ on $V$.

Then $(\nabla(\mu_1, \mu_2)) \varphi(x) = \mu_1(\alpha_x) \mu_2(\alpha_y) \delta(x) \delta(y) \varphi(\alpha_x \alpha_y) \quad (1)$

since $(\alpha_x, \alpha_y) = (\alpha_x \alpha_y, \alpha_y)$.

Also, $(\nabla(\mu_1, \mu_2)) \varphi(x) = \mu_1(-x) \mu_2(-x) \delta(-x) \delta(x) \varphi(x) \quad (2)$

since $(\alpha_x, \alpha_x) = (\alpha_x, \alpha_x, \alpha_x)$.

We get $\varphi(x) = c \mu_1^2 \mu_2(x) \delta(x)$ for $|x| > 0$.

So we can actually work out which functions we have here.
It's easy to see that $V = C_c(E) = \mathcal{S}(E)$, the Schwartz space.

Hence $V = \mathcal{S}(E) \otimes C \mu_\infty$, with $\varphi_\mu(x) = \begin{cases} \mu_x^\mu(x) \cdot |x|^{-d} & \text{for } |x| > 1 \\ 0 & \text{for } |x| \leq 1 \end{cases}$.

If $n = (\delta_i) \in \mathbb{N}$ then $\varpi(n) \varphi(x) = \varphi(x \cdot a_1) \cdot b_1(x)$.

Note $\mathcal{S}(F) \subseteq V$ is invariant under $B$ by (1).

Let's find the $B$-invariant subspaces of $\mathcal{S}(F)$.

We need

The Fourier transform

Pick a non-trivial additive character $\varphi: E \to \mathbb{C}^*$

e.g. $\varphi(x) = \exp(2\pi i \cdot \text{Tr}_{F_{\mathbb{R}^p}}(x))$

Then $\varphi \in \mathcal{S}(F)$ given by

$\hat{\varphi}(y) = \int \varphi(x) \cdot \overline{\varphi(x)} \cdot |x|^{-d} \cdot \varphi(x) \cdot dx$

and $\hat{\varphi}(x) = c \cdot \varphi(-x)$, and $c = 1$ for a suitable choice of $(\varphi, dx)$.

Hence

$\varphi(x) = \int \hat{\varphi}(y) \varphi(-xy) \cdot dy$

and $\varphi(x \cdot a_1) = \int \varphi(-ay) \cdot \hat{\varphi}(y) \cdot \varphi(-xy) \cdot dy$

$\text{Span} \{ \varphi(-ay) \cdot \hat{\varphi}(y) \mid a \in F \} = \left\{ \Theta \in \mathcal{S}(F) \mid \text{supp } \Theta \subseteq \text{supp } \varphi \right\}$

and so the $N$-invariant subspaces of $\mathcal{S}(F)$ are in 1:1 correspondence with the open subsets $\Sigma \subseteq F$.

$\Sigma \leftrightarrow \left\{ \Theta \in \mathcal{S}(F) \mid \text{supp } \Theta \subseteq \Sigma \right\}$

$\pi(\sigma_0) \varphi(x) = \mu_x(a) \cdot |a|^{-d} \cdot \varphi(ax) = \mu_x(a) \cdot |a|^{-d} \cdot \int \hat{\varphi}(y) \cdot \varphi(-ayx) \cdot dy$

$= \mu_x(a) \cdot |a|^{-d} \cdot \int \hat{\varphi}(a^2y) \cdot \varphi(-xy) \cdot dy$

So $U_\Sigma$ is invariant under $\pi(\sigma_0) \leftrightarrow 0 \Sigma = \Sigma$.

So the $B$-invariant subspaces of $\mathcal{S}(F)$ are:

$U_{\psi} = \{ 0 \}$, $U_\Sigma = \mathcal{S}(F)$, & $U_{F \psi} = \{ \varphi \mid \int F \varphi(x) \cdot dx = 0 \}$

$\mathcal{S}(F)^\circ$. 

So any \( G \)-invariant subspace \( U \subseteq V \) either contains \( S(F)^0 \) or is finite-dimensional & hence 1-dim.

If \( \dim U > 1 \): then \( \tau(g) |_U = \pi(\text{det } g) \), so if \( U \subseteq C \phi \) say, we get

\[
\tau(N) \psi \phi \psi = \psi \cdot N \psi \cdot j^* \psi \quad \text{& the transformation formula (1) implies that } \mu_j^* \mu_j = 1.
\]

Conversely, if \( \mu_j^* \mu_j = 1 \) then \( V \simeq \{ \text{constants} \} \) as a \( G \)-invariant subspace.

**Other case: \( U \supseteq S(F)^0 \)**

Reduce to first case by a duality argument: \( U \) has finite codimension

\[
\langle \cdot , \cdot \rangle : V(\mu_j, \mu_j) \times V(\mu_j^* , \mu_j^* ) \to \mathbb{C} \quad \text{defined by}
\]

\[
\langle \psi , \psi' \rangle = \int_F \psi(x) \overline{\psi}(x) \, dx
\]

(NB from (3) \( \langle \psi(p(x)) \rangle \ll 1 \); \( \mu_j^* \mu_j \) pairing by (4) & (2))

(NB if we hadn't put in that \( \delta(x^2) \) factor then we'd get some silly factors

on \( \mu \) instead of \( \mu_j^* \mu_j \).

So \( U \) is a f.d. unit subspace of \( V(\mu_j^* , \mu_j^* ) \) & so \( \mu_j^* \mu_j = 1 \) if \( U \not= V \)

& we get \( U = \{ \phi \in V \mid \langle \phi , \phi \rangle = 0 \} \).

**Def./Thm. 5.** Let \( \mu_j, \mu_j : F^* \to \mathbb{C}^* \) be characters. Then \( \rho(\mu_j, \mu_j) \) is

indecomposable, &

(i) \( \mu_j^* \mu_j = 1 \) \( \Rightarrow \tau(\mu_j, \mu_j) = \det \rho(\mu_j, \mu_j) \) is irreducible

(ii) \( \mu_j \mu_j^* = 1 \) \( \Rightarrow \rho(\mu_j, \mu_j) \) has a unique 1-dim. subrepresentation,

\( \tau(\mu_j, \mu_j) \), denoted by \( \tau(\mu_j, \mu_j) \), \( \text{ie} \), to \( (\mu_j, 1, 1)^* \).

& the quotient \( \sigma(\mu_j, \mu_j) \) is irreducible

(iii) \( \mu_j^* \mu_j = 1 \) \( \Rightarrow \rho(\mu_j, \mu_j) \) has 0 \( 1 \)-dim. irreducible quotient \( \tau(\mu_j, \mu_j) \)

\( \text{& the kernel } \sigma(\mu_j, \mu_j) \) is irreducible.

We have essentially done all the details. (just need to check those irreducibility claims or details)

**Def.** The reps \( \sigma(\mu_j, \mu_j) \) are called special reps.

NB calling the \( \tau(\mu_j, \mu_j) \) \( \tau(\mu_j, \mu_j) \) even in the special case is a little

perverse, but it will become clear once the Jacquet-Langlands stuff is done. why

this is the natural thing to do.
Theorem G (i) \( \pi(\mu_1, \mu_2) \cong \pi(\mu'_1, \mu'_2) \iff \{ \mu_1, \mu_2 \} = \{ \mu'_1, \mu'_2 \} \)

(ii) \( \sigma(\mu_1, \mu_2) \cong \sigma(\mu'_1, \mu'_2) \iff \{ \mu_1, \mu_2 \} = \{ \mu'_1, \mu'_2 \} \)

(iii) \( \pi_1 = \pi_2 \) for any \( \sigma \).

He may prove this later, once we've got some machinery.

NB these reps are quite easy but they're not all the collections of irreducible reps of \( G \), only a small subset. The rest are the supercuspidal reps.

Sometimes principal & special \( \hookrightarrow \) reducible reps of Galois reps.

Supercuspidal \( \hookrightarrow \) irreducible reps.

Finally note that all 1-dim reps have shown up in the \( \pi(\mu_1, \mu_2) \).

Unramified principal series: let \( K = GL_2(\mathbb{O}) \). Suppose \( \mathbb{B}(\mu_1, \mu_2)^K = \{0\} \).

Because \( G = BK \), every \( \xi_f \) in \( \mathbb{B}(\mu_1, \mu_2)^K \) must be a multiple of

\[
\mathbb{E}^{\text{spher}}(g) = \mu_1(\xi_f) \mu_2(\xi_f) |q_{11}^{a_1}, q_{12}^{a_2}, q_{21}^{a_3}, q_{22}^{a_4}, k \in K,
\]

& because \( (\xi_f, \xi_f) \leq K \), \( \mathbb{E}^{\text{spher}}(1) \in K \) \( \Rightarrow \mu_1(1^+) = \mu_2(1^+) \)

ie \( \mu_1 \) must be unramified character.

(i) The case \( \mu_1 \mu_2 = 1 \): then \( \pi(\mu_1, \mu_2) \) is an irreducible unramified rep

(ii) \( \mu_1 \mu_2 = 1 \). Then \( \pi(\mu_1, \mu_2) = (\mu_1, 1^+) \). But is unramified so

\[
\mathbb{E}^{\text{spher}}(1) = \mathbb{E}^{\text{spher}}(1^+) \Rightarrow \sigma(\mu_1, \mu_2) \in \mathcal{O}_f \& \mathbb{E}^{\text{spher}}(1) \subset \mathcal{O}_f \]

(iii) similarly.

Let's work out \( T \& S \):

\[
T_{\pi} \mathbb{E}^{\text{spher}}(1) = \mathbb{E}^{\text{spher}}(1^+) + \sum_{b \in \mathcal{O}_f} \mathbb{E}^{\text{spher}}(1^+) \]

\[
= \left( \mu_1(\pi) q^{\langle \alpha \rangle} + \mu_2(\pi) q^{\langle \beta \rangle} \right) \mathbb{E}^{\text{spher}}(1^+)
\]

\[
= q^{\langle \beta \rangle} \left( \mu_1(\pi) + \mu_2(\pi) \right) \mathbb{E}^{\text{spher}}(1^+)
\]

\[
S_{\pi} \mathbb{E}^{\text{spher}}(1) = \mathbb{E}^{\text{spher}}(1^+) = \mu_1 \mu_2(\pi)
\]

So \( \pi(\mu_1, \mu_2) \rightarrow \) conjugacy class of \( (\mu_1(\pi), \mu_2(\pi)) \) in \( GL_2(\mathbb{O}) \)

Hence \( \{ \pi(\mu_1, \mu_2) \} \), \( \mu_1 \) unramified, \( \mu_2 \) exhaust the set of (equiv. classes of)

unramified characters.
Today, he's going to try & tell us about \( GL(1,R), GL(2,C) \). His knowledge of the situation here is much less than the p-adic case! He won't be speaking with as much authority. The analysis looks much more unfriendly to the average number theorist.

The first remark to be made is that a \( GL(1,R) \) module \( V \) is a rep of \( GL(2,R), \) \( \mathfrak{g} \) is a rep of \( GL(2,R), \) is not a rep of any group at all, in fact.

## II. Representations of \( GL(1,R), GL(2,C) \)

### \( \mathfrak{g}, K \)-modules

\[ \mathfrak{g} = GL(1,R) \otimes K = O(2) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad K \text{ the maximal split.} \]

Note that if \( V \in \mathfrak{g} \)-finite for some action of \( G \) on \( V \), then

\[ \mathfrak{g}^K \text{ is a span } \{ \tau_{(k)}v \mid k \in K \} \text{ s.t.} \]

\( \tau(g) v \) is \( \mathfrak{g} \)-finite, \( K \) this in general has rather small intersection with \( K \), as \( K \) and \( \mathfrak{g} \) are more or less orthogonal.

So the main difference between \( R \) & p-adic case is that in general, \( G \)
will not act on any space of \( K \)-finite vectors. This leads us to the idea of a \( (\mathfrak{g}, K) \)-module.

Here \( \mathfrak{g} \) is the Lie algebra of \( G \); \( \mathfrak{g} = \mathfrak{gl}(2,R) \); \([X,Y] = XY - YX\)

A representation of \( \mathfrak{g} \) is a linear map \( \rho: \mathfrak{g} \rightarrow \text{End}(V) \)

\[ \text{st. } \rho([X,Y]) = \rho(X) \rho(Y) - \rho(Y) \rho(X). \]

Recall \( \text{exp}: \mathfrak{g} \rightarrow G, \quad x \mapsto e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

Now let \( \pi: G \rightarrow GL(V), \) \( V \) a complex Hilbert space. \( \text{st.} \)

\[ \pi(g) \text{ are bounded } V_g \]

\( \text{st. in the sense that } V \text{ the map } g \mapsto \pi(g) v \text{ s.t.} \)

Assume also that \( \pi|_K \) is a rep by unitary transformations of \( V. \)

Then \( V = \bigoplus V(p), \) each \( V(p) \) a sum of copies of some irreducible reps of \( K. \)

Then \( V = V^* = \bigoplus V(p), \) an algebraic direct sum. \( V^* \) is no longer a Hilbert space.

\( V^* \text{ is the set of } K \)-finite vectors in \( V. \)
Let $(\pi, V)$ be as above, & say $V$ is irreducible.

(i) If $v \in V^0$, then $v \mapsto \pi(g)v \in C^\infty$, even real analytic, so if $X \in \frak{g}$ we can define
\[
\left. \left( \frac{d\pi}{dt} \right)(x)v \right|_{t=0} = \frac{1}{i} \left( \pi(e^{ix})v \right)
\]

(ii) $(d\pi)(x)$ takes $V^0$ onto itself, & gives a rep of $\frak{g}$ on $V^0$.

(If there is not an action of $G$ on $V^0$ but there is an action of $\frak{g}$,

[iii] Now assume $(\pi, V)$ is unitary & irreducible. Then $\dim V(p) < \infty$.

i.e. topologically irreducible non-trivial closed irreducible subspace.

Moreover, $V^0$ is jointly irreducible as a $g \mathfrak{g}$ & $K$-module. Moreover, if $V'$ is another
irreducible unitary rep, then $V \cong V'$ as unitary $G$-modules $\iff$ $V \cong V'$ which
commutes with the action of $g \mathfrak{g}$ & $K$.

(Nota bene: this is a theorem of Harish-Chandra. It is true
for any reductive real Lie $G$ with maximal compact $K,$ although it is possible to extract a
self-contained pf for $SL_2(\mathbb{R})$ from Lang's book.

So given a rep of $G$ we extract a $(g \mathfrak{g}, K)$-module. The $(g \mathfrak{g}, K)$-module is
much more algebraic, e.g. $V(p)$ is f.d. $V_p.$ There is some sort of way of
going back, but you need conditions on $(g \mathfrak{g}, K)$ $V^0$. He'll go into this later.

**Def:** An **admissible** $(g \mathfrak{g}, K)$-module is a complex vector space $V$, together with

\[\varphi_g: \frak{g} \to \text{End}(V), \text{ a Lie algebra HM} \]

\[\varphi_K: K \to \text{GL(V)}, \text{ making } V \text{ the direct sum of } \text{f.d.} \text{ copies of } K\]

such that

(1) $\varphi_g|_K: K \to \text{End}(V)$ & $\varphi_g|_\frak{g}$ are equal. (i.e. $\frak{g}$ is $\text{Lie alg of } K$)

(2) $\varphi_g k = \varphi_g(k x k^{-1}) = \varphi_g(k) \varphi_g(x) \varphi_g(k^{-1})$

**Notation:** Write $V = \bigoplus V(p)$ where $p$ runs over inequivalent f.d. irreps of $K$,

\[\text{each } V(p) = \text{sum of } \mathbb{C} \text{ copies of } p\]

$V(p)$ are called $K$-types. $V$ **admissible** $\iff$ $\dim V(p) < \infty.$

All this goes through for general $G, \mathfrak{g}.$
Now we specialize to $GL_2(\mathbb{R})$. Note then we understand the $\mathfrak{sl}_2(\mathbb{C})$ reps of $K$.
(I think he said they're all 1+2-dim, or something). We want to classify the irreducible reps of $G$ in this case.

$G = GL_2(\mathbb{R})$. We complexify to start off with.

Let $g_0 \in \mathfrak{so}_2(\mathbb{C})$, which acts on any $(q, K)$-module by linearity.

A basis (of $C$-basis, that is) for $g_0$ is $\{ e_i, f_i, h_i \}$ (not the identity as $h_i$). $H := (0, 0)$

& \quad X_\pm = \frac{1}{2} (e_i \pm i f_i)

Then $e^{i\theta} g_0 e^{-i\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

$[J, g_0] = 0$; $[H, X_+] = 2 X_+$; $[H, X_-] = -2 X_-$; $[X_+, X_-] = H$

Let $z = (z_0, z_1) \in \mathbb{C}$. Now $g_0 \cong g_z$, the complexified Lie algebra of $SL_2(\mathbb{R})$

\[ \{ X \in g_0 \mid \text{tr} X = 0 \plusn \mathbb{C} \} \text{ spanned by } X \equiv H, X_+ \equiv \plusn \mathbb{C} \]

(NB: we'll do a lot for $SL_2(\mathbb{R})$ and then show that $GL_2(\mathbb{R})$ follows.)

An admissible $(q, K)$-module, in the $GL_2(\mathbb{R})$ case, can be described as

- A graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$, where $V_0$ is finite.
- Operators $\pi(X_+), \pi(H), \pi(J)$ on $V$ satisfying the commutation laws, and so:
  \[ V_n = \ker(\pi(H) - n) \cdot V \]
- Operator $\pi(E)$, with square 1, satisfying certain relations.

Now, by Schur's lemma, if $(V, \pi)$ is an irreducible $(q, K)$-module, then

$\pi(J) = c \in \mathbb{C}$ (since it has an eigenvector on some $V_n$)

So it is sufficient to classify $(q, K)$-modules.

If we restrict further to $(q, K')$, where $K' = SO(2)$, then either

- $V$ remains irreducible, or
- $V = W \oplus W^\perp$, where $W$ is a $(q, K')$-module, irreducible.

& \quad \& W^\perp = \text{conjugate of } W \text{ by the automorphism } \begin{pmatrix} e \quad & \quad 0 \\ 0 & e^{-1} \end{pmatrix} \in G \times SL_2(\mathbb{R})$
So now we will classify the irreducible \((q^2, K')\)-modules \((\tau, V)\)

\[ \tau(X_+) : V_n \to V_{n+2} \quad \text{since if } v \in V_n, \text{ then} \]

\[ \tau(H) \tau(X_+) v = \tau(EH, X_+) v + \tau(X_+) \tau(H) v \]

\[ = (2 + \eta) \tau(X_+) v \]

where \(\eta \in \{0, 1\} \text{ s.t. } V = \mathbb{C} \oplus V_n \text{ for } n \equiv \eta \mod 2\)  

Now we'll define (some multiple of) the Casimir operator, namely

\[ D = X_+ X_- + X_- X_+ + \frac{1}{2} H^2. \]

Formally \(D \in U(q^2)\), the universal enveloping algebra of \(q^2\). Alternatively, think of it as composition of operators. It is nothing to do with multiplication of matrices.

\(D\) acts on any rep space \(V\) of \(q^2\), & \(D\) commutes with \(q^2\) (in fact the center \(Z(q^2)\) of \(U(q^2)\) is \(\mathbb{C} \Gamma(D)\)).

So by Schur's lemma, \(\tau(D) \cdot v \in \mathbb{C} \cdot v\),

\[ D = 2X_+X_- + H + \frac{1}{2} H^2 = 2X_-X_+ + H + \frac{1}{2} H^2. \]

\[ \tau(X_+) \tau(X_+) \big|_{V_n} = \frac{1}{2} (d_3 n^2 / 2) \]

& \[ \tau(X_-) \tau(X_-) \big|_{V_n} = \frac{1}{2} (d_3 n^2 / 2) \]

Now let \(v \in V_k \), some \(k\). Then the span of \(\{ \tau(X_+) v, \tau(X_-) v \mid r \geq 0 \}\)

is stable under \(q^2\), and so equals \(V\) by irreducibility.

Hence \(\dim V_n = 1 \cdot V_n\).

Moreover, since \(V\) is irreducible, if \(V_k \neq 0\) then both \(\tau(X_+) : V_k \to V_{k+2}\)

are surjective (else \( \otimes \), \( V_n \) or \( \otimes \), \( V_n \) are invariant submodules).

So there are 3 cases left to consider, the same number as there are minutes

left in the lecture.
Case (ii) \( \exists k \in \mathbb{N} : V_k \neq 0 \), \( \pi(X_0) V_k = 0 \). Then \( V_n = 0 \) \( \forall n \neq k \), so \( \pi = 0 \), and

\[ \pi_X \pi_X \big|_{V_k} = 0 \], so \( d = \frac{k^2-k}{2} \).

If \( k \leq 0 \) then \( \pi(X_0) \pi_X \big|_{V_k} = \frac{1}{2} (d - (k-1)^2) \cdot 0 \),

so \( V = \bigoplus V_n \) in finite-dimensional.

If \( k > 0 \) then \( \pi(X_0) \pi_X \big|_{V_n} = 0 \) for all \( n \neq k \), so

\[ V = \bigoplus V_n \] in co-dim \( n \neq k \).

Let \( v \in V_k \), \( v \neq 0 \). If we write

\[ \pi_X v = \begin{cases} \frac{2^k}{(k-1)!} \cdot \pi(X_0) v & \text{if } n = k+2r \text{ for } r \geq 1 \\ 0 & \text{otherwise} \end{cases} \]

then \( V = \bigoplus V_n \) such that

\[ \pi(X_0) \pi_X v = \frac{1}{2} (k+1) \cdot \pi v, \]

\[ \pi(X_0) v = \pi v. \]

\( \text{discrete series } \pi \), \( d = \frac{1}{2} k(k-2) \).

Lecture 6

Recall we were looking at \((\mathfrak{g}, K) = (\mathfrak{s} \mathfrak{l}_2(\mathbb{R}), \mathbb{SO}(2))\) -modules \((\pi, V)\),

\[ V = \bigoplus V_n \] under \( K \).

Recall \( \pi_X \pi_X \big|_{V_n} = \frac{1}{2} (n+1) - \frac{1}{2} n, d \) - eigenvalue of \( D \), Casimir operator.

\[ \dim V_n = f \cdot \forall n \in \mathbb{N}, \text{ s.t. } V = \bigoplus V_n \]

(i) \( \exists k \in \mathbb{N} : V_k \neq 0 \), \( \pi(X_0) V_k = 0 \).

\[ \frac{d}{d \pi_X} \bigg|_{V_k} = 0 \text{ then } \pi(X_0) V_k = 0 \] \( \& \) \( V \) is co-dim \( k \).

Now on to case (ii).
Case(ii) \( \exists k \text{ s.t. } V_k \neq 0 \text{ but } \pi(x_+(V_k)) = 0 \)

2 cases again:
1. \( k < 0 \) (\( -k > 0 \)) & \( V = \oplus V_n \) so \( f.d. \) (same as (i))
2. \( k > 0 \), in which case

\[
V = \bigoplus_{n \equiv k \mod 2} V_n
\]

\& we get an equiv. class of reps \( D_k \)

\( D_k \) is the conjugate of \( D_k^+ \) by \( \varepsilon \).

Case(iii) \( V_n = 0 \) for all \( n \equiv m \mod 2 \)

So \( V = \bigoplus_{n \equiv m \mod 2} V_n \), \( \dim V_n = 1 \), \( V_n \equiv m(2) \).

Then \( \pi(x_+), \pi(x-) \) injective \( \Rightarrow d \equiv n \mod 2 \) for any \( n \equiv m \mod 2 \).

\( \therefore \) we can write \( d = \frac{n \pm t}{2} \), \( s \equiv t \mod (\text{odd}) \), \( s \equiv t + m \mod (\text{mod 2}) \)

\( V \) is then determined uniquely by \( m \) \& the action of \( \pi(x_+), \pi(x-) \)

\( \pi(x_+), \pi(x-) \) \( \pi(x_+) \)

An explicit basis: let \( v \in V_m(10) \); define

\[
\varphi_n = 2^{(n-m)/2} \frac{\Gamma(n+1)}{\Gamma(n+1)} \pi(x_+(n-m/2)) v
\]

Then \( \pi(x_+), \pi(x_-) \)

This rep is denoted \( \mathbb{B}_3^{(10)} \), the principal series.

So we have proved

**Theorem 3.** Every \( \text{uced ad} \) \((q, k')\)-modules of \( v \) dimension is isomorphic to 

\( \mathbb{B}_k \) or a \( \mathbb{D}_k \); the only equivalence are 

\( \mathbb{B}_k \equiv \mathbb{D}_k \)

\( \mathbb{D}_k \) is \( \varepsilon \)-conjugate of \( \mathbb{D}_k^+ \)

\( \mathbb{B}_3 \) is \( \varepsilon \)-conjugate to its \( \varepsilon \)-conjugate. \( \varphi_n \leftrightarrow \varphi_m. \)

\( \square \)

NB we seem to have admissibility to show that the Carimir operator acts

\( \varphi_n \leftrightarrow \varphi_m. \)

\( \square \)

It is a thm of Harish-Chandra that used \( \varepsilon \)-admits for real reducible gps.
We'll now go back to our original task.

Classification of (only admissible, presumably) \((g, K)\)-modules, \(G = GL(n, \mathbb{R})\).

\(\mu_i, \mu_j : \mathbb{R}^x \rightarrow \mathbb{C}^x\)

\(\mu_i(t) = 1 + \sum_{m \in \mathbb{Z}} (\text{sgn} t)^m, \ s_i \in \mathbb{C}, \ m_i = 0 \text{ or } 1\)

Set \( s = s_i - s_j, \ m = |m_i - m_j|\)

\[\mathcal{B}(\mu_i, \mu_j) = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} \varphi_{m} \]

with action of \((g, K)\):

\[
\begin{align*}
\tau(H) \varphi_n &= n \varphi_n \\
\tau(R_g) \varphi_n &= \exp(2\pi i s) \varphi_{m} \\
\tau(\varepsilon) \varphi_n &= (-1)^{m} \varphi_{n} \\
\tau(J) \varphi_n &= (s_i + s_j) \varphi_{n}
\end{align*}
\]

\(\mathcal{B}(\mu_i, \mu_j)\) can be identified with the space of right \(K\)-finite functions

\[\mathcal{F} \rightarrow \mathbb{C} \quad \text{s.t.} \quad \mathcal{F}(x) g = \mu_i(a_1) \cdots \mu_i(a_r) \frac{\partial^s}{\partial x} \mathcal{F}(g)\]

by \(\varphi_n \rightarrow \mathcal{F}_n \quad \text{s.t.} \quad \mathcal{F}_n(r_0) = e^{2\pi i n}.\)

Thm 9 (i) \(\mathcal{B}(\mu_i, \mu_j)\) is an admissible \((g, K)\)-module, \& every irreducible, admissible \((g, K)\)-module is a submodule of some \(\mathcal{B}(\mu_i, \mu_j)\).

(ii) \(\mathcal{B}(\mu_i, \mu_j)\) is irreducible if \(s \not\equiv m \mod 2\), \(s \not\equiv 0 \mod 2\). Call this \(\mathcal{B}(\mu_i, \mu_j)\), \(\mathcal{F}(\mu_i, \mu_j)\), irreducible, is the principal series.

(iii) If \(\mu_i \mu_j = \text{sgn}(1)\), and \(s < 0\) is an integer then \(\mathcal{B}(\mu_i, \mu_j)\) has a unique, finite-dimensional submodule, denoted \(\mathcal{F}(\mu_i, \mu_j)\). It is equal to \(\bigoplus_{m \geq 0} \mathcal{F}_m\). Then let \(\mathcal{F}(\mu_i, \mu_j)\) be the

finite-dimensional quotient.

(iv) \(\mu_i \mu_j = \text{sgn}(1)\), \(s > 0\) integer. Then \(\mathcal{B}(\mu_i, \mu_j)\) has a unique, finite-dimensional submodule \(\mathcal{F}(\mu_i, \mu_j) = \bigoplus_{m \leq 0} \mathcal{F}_m\), and the quotient \(\mathcal{F}(\mu_i, \mu_j)\) is irreducible.

The \(s\)'s are discrete series.
(v) Finally, a silly case. If \( \mu'_s, \mu'_s = \text{sgn} \), then \( s = 0 \) so the f.d. submodule disappears, and we get
\[
\mathcal{B}(\mu_s, \mu_s) = \mathcal{B}(\mu, \mu) \text{ is irreducible.}
\]
This one is called limit of discrete series.

**Thm 10.** Let \((\mathfrak{g}, V)\) be an admissible \((\mathfrak{g}, K)\)-module and \((\mathfrak{h}, V)\) a \(K\)-inst inner product, s.t.
\[
\mathfrak{h}(x)V = \{ x \mathfrak{m}_s, \mathfrak{m}_s \} \Rightarrow \{ \mathfrak{m}_s, \mathfrak{m}_s \} = \{ \mathfrak{m}'_s, \mathfrak{m}'_s \}
\]
\[\sigma(\mathfrak{m}_s, \mathfrak{m}_s) \Rightarrow \{ \mathfrak{m}'_s, \mathfrak{m}'_s \} = \{ \mathfrak{m}_s, \mathfrak{m}_s \} \text{ or } \{ \mathfrak{m}_s, \text{sgn}, \mathfrak{m}_s, \text{sgn} \}
\]
No \( \mathfrak{m} \) is equivalent to a \( \mathfrak{m}' \).

**Thm 11.** Let \((\mathfrak{g}, V)\) be an admissible \((\mathfrak{g}, K)\)-module, and \((\mathfrak{h}, V)\) a \(K\)-inst inner product s.t.
\[
\mathfrak{h}(x)V = \{ x \mathfrak{m}_s, \mathfrak{m}_s \} \text{ if } x \in \mathfrak{g}_{\mathfrak{r}'}, x, v \in V.
\]
Then there's a unique unitary rep of \(G\) on the completion \( \hat{V} \) s.t. \((\mathfrak{g}, \hat{V})\) is the \((\mathfrak{g}, K)\)-module of \(K\)-finite vectors in \( \hat{V} \). 

This is true for any reductive (real?) group \(G\) due to Harish-Chandra.

So \((\text{irred unitary rep of } G, \text{up}) \mapsto (\text{irred admissible } (\mathfrak{g}, K)\text{-modules with an inst inner product})\).

**Thm 11.** \((\mathfrak{g}, K) = (\mathfrak{g} \mathfrak{p}^E(R), O(2))\), the irreducible \((\mathfrak{g}, K)\)-modules associated to unitary reps of \(G\) are

\[
\rho(\mathfrak{m}_s, \mathfrak{m}_s), \mathfrak{m} \text{ unitary}
\]
\[
\rho(\mathfrak{m}_s, \mathfrak{m}_s) \text{ where } m = 0, s = 1, \rho = -s, 0 < s < 1
\]
\[
\sigma(\mathfrak{m}_s, \mathfrak{m}_s) \text{ with } |\mathfrak{m}_s, \mathfrak{m}_s(i)| = 1
\]

Pl by using thm 10 & trying to attach inner products. Not too bad. Hill and it.
Anyway, we've talked about $(g, K)$-modules. There's an action of $g$ & one of $K$. It would be nice to find a $g$ object & $K$ action instead. The Hecke algebra, of course. Analysis sort of disappears - it goes into construction.

### 86. The Hecke algebra at infinity

We have $g$, & a Lie algebra structure $[.,.]$.

There is an associative algebra $\mathcal{U}(g)$, the universal enveloping algebra, with a unit $1$ & a linear map $\delta : g \to \mathcal{U}(g)$, $\delta (x) = x x_1 - x_1 x$, & st. any rep $\rho$ of $g$ extends to $\rho^g$ to a rep of $\mathcal{U}(g)$.

In fact, if $g$ has a basis $\{ x_i \}$, then $\mathcal{U}(g) = \langle \mathcal{G} \rangle / \langle x_i x_j - x_j x_i - [x_i, x_j] \rangle$

**Thm 12. (Poincaré-Birkhoff-Witt)**

A basis for $\mathcal{U}(g)$ is $\{ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \}$. $\Box$

$\mathcal{U}(g)$ has a center $Z \subset C$. For $g = g^0_2 (C)$, $Z = C [x, y]$. [12]

In any indep. f.d. rep of $\mathcal{U}(g)$, or any indep. admn $(g, K)$-module, the elt of $Z$ act via a char $\chi : Z \to C$.

$\chi$ is called the infinitesimal char of the module.

**Corollary.** If $G = GL_n (C)$, then there is only a finite # of equiv classes of $(g, K)$-modules with given infinitesimal character.

- $B(\mathfrak{gl}_n (C))$, $\pi (f) = s_i f$, $\pi (x) = \frac{s_i + s_j}{2}$, $s = s_i s_j$, so

  there's only finitely many choices for $s_i, s_j$, $4(8 \times \mathfrak{gl}_8)$.

**Thm 13.** $\exists$ associative algebra $H = H(g, K)$, without a unit, & a directed family of commuting idempotents $E \subset H$ s.t.

- $H = \bigoplus H e$, & there is an equivalence of categories

  $(g, K)$-mod $\leftrightarrow$ non-degenerate reps of $\rho$ of $H$

  s.t. $\exists$ a non-degenerate rep.

  adm. $\pi (v)$ of finite rank $\mathfrak{V}_S$. 

- $W \times \mathfrak{gl}_n (C)$, $\pi (w)$, $\pi (f)$.
We have $(\tau, V)$ a $(\mathfrak{g}, K)$-module.

(No we need the thm to understand the decomposition of a global rep into local reps - see John's lecture next Monday)

Let $A_K$ = algebra of left+right $K$-finite functions on $K$ under convolution. $\rho : K \rightarrow GL_n(\mathbb{C})$ irred rep. \ [$\alpha_{ij}(k) \in A_K$ ]

\[ \rho(k) = (\alpha_{ij}(k)) \]

and $A_K$ is spanned by such matrix coeff. (all $\rho$)

\[ e_{ij} = \frac{1}{d_{ij}} \text{ is an idempotent. } A_K = \bigoplus_{\rho} M_{d_{ij}}(\mathbb{C}), e_{ij} \text{ a projector.} \]

$V$ as above, $\phi \in A_K$

\[ \pi(e_{ij}) \mathcal{V} \rightarrow \mathcal{V}_{e_{ij}}. \text{ So } \forall \mathcal{V} \exists \forall \mathcal{E} = \text{ finite sum of } e_{ij}s, \text{ s.t. } \pi(e_{ij}) \mathcal{V} = \mathcal{V} \]

$V$ is admissible $\Rightarrow \pi(\mathfrak{g})$ has finite rank $\forall \mathfrak{g} \in \mathfrak{h}$

$U(\mathfrak{g})$ also acts on $V$ (since $\mathfrak{g}$ does)

The algebra $\mathfrak{h}$ will be composed of products $X \ast \mathfrak{g}$, $\mathfrak{g} \in U(\mathfrak{g})$

Note $\mathfrak{g} \in A_K$, $\phi \in A_K$ $\Rightarrow \pi(X) \pi(\phi) = \pi(L_X \phi)$

\[ \text{where } (L_X \phi)(k) = \frac{1}{d} \phi(ke^{-t}k) \mid_{t=0} \]

$\pi(\mathfrak{g}) \pi(X) = \sum \pi(x_j) \pi(m_{ij}\phi)$ if $ad(k): X_i \rightarrow \sum m_{ij} X_j$

\[ X \in \mathfrak{g} \]

Define $\mathfrak{h} = U(\mathfrak{g}) \otimes A_K$ (vector space). The $\otimes \mathfrak{g}$ over $U(\mathfrak{g})$ which acts on $U(\mathfrak{g})$ by right mult & on $A_K$ by $L_X$ (to ensure\( \square \) holds)

The product on $\mathfrak{h}$ is $\phi \ast X_i = \sum X_j \ast (m_{ij}\phi)$.

Then $(\mathfrak{h}, \mathcal{E})$ has the required properties. \( \square \)
To conclude the archimedean theory, he'll spend 2 minutes on

3.7 $GL_n(C)$

Let $\mu_1, \mu_2 \in C^* \to C^*$ be the HMs

$$B(\mu_1, \mu_2) = \{ \text{right } K\text{-finite } \phi \in GL_n(C) : GL_n(C) \to C \text{ st. } \phi(\mu \cdot \phi(g)) = \mu_1(\alpha_1) \mu_2(\alpha_2) \frac{\alpha_1}{\alpha_2} \phi(g) \}$$

Here $|z|_C = z\bar{z}$, the local norm.

Here $K = \text{max } \text{split subgp } U(1) \subseteq G$

$B(\mu_1, \mu_2)$ is an admissible $(\mu_1, \mu_2)$-module

Theorem 4.14
(i) If $\mu_1 \mu_2^* \neq Z^p Z^{*p}$ for $p \in \mathbb{Z}$, $\mu_1 \mu_2^* \neq Z^p Z^{*p}$, then $B(\mu_1, \mu_2)$ is irreducible; call it $\pi(\mu_1, \mu_2)$.

(ii) If $\mu_1 \mu_2^* = Z^p Z^{*p}$, $\mu_2 \neq 0$, then $B(\mu_1, \mu_2)$ has a subquotient $\pi(\mu_1, \mu_2)$, which is isomorphic to

$$\text{Sym}^{*p} \otimes \text{Sym}^{*p} \otimes (\mu_1 \otimes \mu_2 \otimes \text{det})$$

& the $\pi(\mu_1, \mu_2)$s exhaust all irreducible $(\mu_1, \mu_2)$-modules.

(iii) $\pi(\mu_1, \mu_2) \neq \pi(\mu_2, \mu_1)$ & there are no other equivalences. □

Hence reps are classified by conjugacy class of semisimple HMs

$$C^* \to GL_n(C), \quad \pi(\mu_1, \mu_2) \leftrightarrow (\mu_1^{(1)} \otimes \mu_2^{(1)})$$

That is all & more than he wants to say about the archimedean case.
III. The Kudla model ( & Atkin-Lehner theory )

(although it's not really what Atkin-Lehner had in mind)

Notation on I : \( E/G_p \), \((\tau, \nu)\) irreducible rep of \( G = GL_2(F) \).

Let \( \gamma : F \to \mathbb{C}^\ast \) be a non-trivial additive character.

**Theorem 15** Assume \( \tau \) is co-dim 1. Then \( \exists! \) space \( K(\tau) \) of functions on \( F^\ast \), 

\( \langle \tau \rangle \) \( \mathbb{C}^\ast \) \( L(\alpha x) \) \( \forall \alpha \in F^\ast \), \( \forall \gamma \in F, \forall \epsilon \in K(\tau) \).

The support of any function in \( K(\tau) \) is contained in a split subset of \( F \), and the \( \epsilon \)'s are locally cusp. Moreover, \( K(\tau) \) contains \( \mathcal{S}(F^\ast) = \mathcal{C}_0(F^\ast) \), as a subspace of finite codimension.

Idea of pf

(i) Construct a linear form \( \lambda : V \to \mathbb{C} \) s.t. \( \lambda(\tau(\nu)) = \gamma(x) \lambda(\nu) \)

(ii) Deduce existence of a space \( W(\tau) \) of \( \epsilon's on G \), the \( \text{Whittaker model} \), \( \epsilon \) \( \in \) \( V \), by

\( v \to \phi, \phi(g) = \lambda(\tau(g)v) \)

(\( \Rightarrow \phi(\tau(1)g) = \gamma(x) \phi(g) \))

(iii) \( K(\tau) \) is obtained by restricting \( \epsilon's on W(\tau) \) to \( (\alpha \epsilon) \)

**Example** \( \tau \circ \pi \), \( \pi \) irreducible series \( \sigma(\mu_1, \mu_2) \) with \( \mu_1 \mu_2 = 1/2 \)

(\( \Rightarrow \tau \in \text{subrep of } B(\mu_1, \mu_2) \))

Then \( B(\mu_1, \mu_2) \Rightarrow V(\mu_1, \mu_2) \), space of \( \epsilon's on F \), satisfying

\( \langle \sigma_{a \nu} \rangle \phi(x) = \mu_1(a) \left\lfloor \frac{1}{2} \right\rfloor \phi(\frac{x}{a}) \)

(I think he said \( \phi(x) \) \( \epsilon \) cusp treated at \( \mathfrak{p} \))

The Fourier transform, formally, \( \Rightarrow \mathcal{F}(\phi)(y) = \mu_1(a) \left\lfloor \frac{1}{2} \right\rfloor \phi(\frac{a}{\nu} y) \phi(\frac{a}{\nu} y) \)

The theory is, the Fourier's in \( \mathcal{B} \) with support not cusp have \( \phi \) diverging.
You have to be careful with details for \( \mathfrak{F} \) then.

So, to get \( K(\mathfrak{F}) \), define \( \mathfrak{F} = \mu_{1}(x) |x|^{-k} + \phi(x) \) for a mapping \( V(\mu_{1}, \mu_{2}) \rightarrow \mathfrak{F} \) (form of \( \mathfrak{F} \)).

We need to define \( \phi \) carefully when \( \mathfrak{F} \in S(\mathfrak{F}) \).

Anyway, in this way we can explicitly work out what \( K(\mathfrak{F}) \) is:

Proposition: \( K(\mathfrak{F}) = \{ \mathfrak{F} \in C \text{ s.t. } \mathfrak{F}(x) = 0 \text{ for } |x| \rightarrow 0 \} \), with behaviour as \( |x| \rightarrow 0 \) open below. \( \phi \) given.

(i) \( \odot(\mu_{1}, \mu_{2}) \) irreducible polynomials \( (\mu_{1} \mu_{2}^{2} + 1)^{1/2} \) & \( \mu_{2} \pm \mu_{1} \).

Then \( \mathfrak{F}(x) = c_{1} \mu_{1}(x) + c_{2} \mu_{2}(x) \) for \( |x| \ll 1 \)

(\( \mathfrak{F}(\mathfrak{F}) \) has cod 2)

(ii) \( \odot(\mu_{1}, \mu_{2}) \) irreducible polynomials \( (\mu_{1} \mu_{2}^{2} + 1)^{1/2} \) & \( \mu_{2} \pm \mu_{1} \).

Then \( \mathfrak{F}(x) = c_{1} \mu_{1}(x) + c_{2} \mu_{2}(x) \) for \( |x| \ll 1 \)

(\( \mathfrak{F}(\mathfrak{F}) \) has cod 2 again)

(iii) \( \sigma(\mu_{1}, \mu_{2}) \), \( \mu_{2} \mu_{1}^{2} = 1 \).

Then \( \mathfrak{F}(x) = c_{1} \mu_{1}(x) \) for \( |x| \ll 1 \)

(\( \mathfrak{F}(\mathfrak{F}) \) has cod 2)

Because of uniqueness of \( K(\mathfrak{F}) \) we can use this to deduce the 6.

(eg. \( \mathfrak{F} \) or on cod. \( S(\mathfrak{F}) \) is different)

Remark: \( \exists \mathfrak{F}(\mathfrak{F}) \Rightarrow \mathfrak{F}(ax) = \mathfrak{F}(x) \) for a.e. open subset of \( \mathfrak{F} \). Use this & the fact that \( K(\mathfrak{F}) \geq \mathfrak{F}(\mathfrak{F}) \) to prove.

Then 16. See next page.
Theorem 16 (Local Atkin-Lehner-Thm) Define for $k \geq 0$

$$G_k^{(k)} = \{ \varphi \in \text{GL}_1(\mathbb{O}) \mid \varphi \equiv (\alpha, \beta) \mod m_k \}$$

$(\pi, V)$ irreducible admissible $\overline{\rho}$-rep of $G$. Then there exists a unique $f \geq 0$ (the conductor of $\pi$) s.t.

$$V_{G_k^{(k)}} = \{(0) \mid f \leq f' \}, \quad V_{G_k^{(k)}} \text{ is } 1\text{-dim}.$$  

We have $\text{cond}(\omega_\pi) \leq f$.

Assume $\text{cond}(\varphi) = 0$ ($\Leftrightarrow \varphi|_{F^\times} = 1$ \& $\varphi|_{\text{Frob}_\pi} = 1$), \& let $\pi \in K(\pi)^{G_k^{(k)}}, \pi \neq 0$. Then $\text{supp} \leq 0$, and $\text{I}(1) = 0$.

The fact that $\text{I}(1) = 0$ is the local analogue of the fact that $a_\pi = 0$ for a newform $\Sigma_{0,q}$.

Proof: Identify $V$ with $K(\pi)$, \& write $\pi$ for $\pi'$. Assume $\text{cond} \varphi = 0$.

Let $V_\infty = \{ \varphi \in V \mid \varphi(0,1) \equiv 0 \mod \omega_\pi \}$  

$$\text{ie } V_\infty \text{ is } 1\text{-dim} \& \text{supp } \leq 0$$

$$\Rightarrow k \equiv \text{cond}(\omega_\pi) \leq 0.$$  

$V_\infty \neq \{0\}$ because, e.g., $\text{char} \varphi \in S(F^\times) = K(\pi) = \text{null}$.

If $\varphi \in V_\infty$, then $\varphi$ is induced by $(\frac{0}{1}, 0, 0)$ for some $k \geq 1$ [Note $V$ is admissible!]

$$\Rightarrow k \equiv \text{cond}(\omega_\pi) \leq 0.$$  

$$\Rightarrow \text{I}(1) = 0.$$

Let $\varphi \in V^{G_k^{(k)}}$ \& assume $\text{I}(1) = 0$, so $\text{supp } \leq 0$.

Then $\varphi = \pi^{(\frac{0}{1}, 0, 0)} \leq 1$ \& supp $\leq 0$.

$$\Rightarrow \varphi \leq \pi^{(\frac{0}{1}, 0, 0)} \leq 1.$$

In particular, $\leq 1$ \& supp $\leq 0$.

$$\Rightarrow k \geq \text{cond}(\omega_\pi) \leq 0.$$  

So let $f = \min \{ k \mid V_{G_k^{(k)}} = \{0\} \}< \infty$. We have to prove dim $V_{G_k^{(k)}} = 1$.

Let $\varphi \in V^{G_k^{(k)}}$ \& assume $\text{I}(1) = 0$, so $\text{supp } \leq 0$.

Then $\varphi = \pi^{(\frac{0}{1}, 0, 0)} \leq 1$ \& supp $\leq 0$.

$$\Rightarrow \varphi \leq \pi^{(\frac{0}{1}, 0, 0)} \leq 1.$$
So $s'$ is unit by $\langle \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \rangle.$

Now $f \neq 1 \Rightarrow s'$ unit by $G(s-1) \Rightarrow s' \neq 0 \Rightarrow s = 0 \Rightarrow f \neq 1.$

$s = 0 \Rightarrow s'$ unit under $\langle \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \rangle \Rightarrow s' \in SL_2(F)$

So $s' = 0$ or $V$ is $\infty$-dim; it does not act via c character!

$s = 0$ again $\neq 0.$

Hence $s(1) = 0.$

Finally, if $s, s' \in V^{G(s)}(t)$ then some linear combination of $s, s'$ vanishes at $1 \Rightarrow \text{(by preceding bit)}$ this linear combination is zero and $s, s'$ are lin. dependent.

Hence dim $V^{G(s)}(t) = 1.$

---

Remark. The unique non-zero vector (up to scalar multiple) in $V^{G(s)}(t)$ is called the new vector or new vector. In the Kuriw. model, it can be normalized by $s(1) = 1.$

Example. Unramified $\pi_1(\mu_r, \mu_r).$ Then $f = 0$ and the new vector $\tilde{s}$ is just the spherical vector $\left( \langle \mu_r(1) = \mu_r(0) \rangle \right)$

Then $(T_{s_1} \tilde{s})(\pi) = \sum_{b \mod \pi} q(\pi x) \tilde{s}(\pi x) + \tilde{s}(\pi/\pi) \omega(\pi) \left( \left( \begin{smallmatrix} \pi/\pi \\ 0 \end{smallmatrix} \right) & \left( \begin{smallmatrix} 0 \end{smallmatrix} \right) \right) = q \tilde{s}(\pi x) + \tilde{s}(\pi/\pi) \omega(\pi) \quad \text{if } x \in 0.$

$= \lambda \tilde{s}(\pi) \text{ for some } \lambda = q^{r/2} \left( \mu_r(\pi), \mu_r(\pi) \right)$

$\lambda$ is the eigenvalue of $T_{s_1}.$

So if $A_m = q^m \tilde{s}(\pi^m),$ $\tilde{s}(1) = 1$

then $A_{m+1} = \lambda A_m - q \omega(\pi) A_{m-1}$

$\Rightarrow \sum_{m=0}^{\infty} A_m q^{-m} = \frac{1}{1 - \lambda q^{-1} \omega(\pi) q^{-1}}$

$= \text{Am eigenvalue of } \mathbb{F}[x, x^{-1}] (z = T_{s_1}).$
Recall this morning for $\varpi \in \text{dim}^2 (\mathfrak{p}, V)$ fixed. Let $K = \mathbb{Q}(\zeta_{\varpi})$, $\varpi \in \text{cond}(\mathfrak{p})$, $S(\varpi) = 1$.

If $S$ is not under $O^*$ and support $\subseteq \mathbb{C}$ is determined by $\{ \mathcal{S}(\pi^*) | n \neq 0 \}$

then $q^N$ in Fourier expansion.

e.g. $\pi_1, \pi_2$ irreducible (i.e. $\pi_1$ unram. & $\pi_2 = 1$)

$$\mathcal{S}(\pi^*) = \begin{cases} 0 & n < 0 \\ q^n \alpha_n & n \geq 0 \end{cases}$$

Then $\sum \alpha_n q^{-n} = \frac{1}{1 - \lambda q^{-1} \eta^{(\omega)}(\varpi)} = \frac{1}{1 - \mu_1(\varpi) q^{-\frac{1}{2}}(1 - \mu_2(\varpi) q^{\frac{1}{2}})}$

$\alpha_n$ eigenvalue of $T_{\pi, n}$

Case when $\pi$ has unramified central character $\varpi = \pi^*$.

$$(\pi^*(p))$$

Then $S$ is not by the Iwahori subgroup $H = \{ g \in GL_2(\mathcal{O}) | g \equiv (c \ c) \text{ mod } \mathfrak{p} \}$ of $G$.

($H$ is usually denoted $B$ but for us $B$ in Borch)

$\eta = (\varpi \ c)$ normalises $H$, so $\pi(\eta) S = c S$ for $S$ a new vector.

$\eta = (\varpi \ c) \mapsto c \omega(\varpi)$

$w = (\varpi \ c)^{-1} \mapsto (\varpi^{-1} \ c) \varpi \ c$.

Then $S(w) S(x) = \omega(\varpi^{-1}) c S(\varpi x) = c S(\varpi x)$

$GL_2(\mathcal{O}) = \bigcup_{a \mod \varpi} H \cup H(\varpi, c)$

If $e = \text{idem} = e_{ginv}$, then $\pi(e) S = 0$ as its $e \in GL_2(\mathcal{O})$.

$$S = \sum_{a \mod \varpi} \gamma(\alpha x) c^{-1} S(\varpi x) + S(x)$$

This implies that the only $(\pi, V)$ with $V^H = 0$ are subreps of unramified principal series.
Theorem 17. If \((\pi, V)\) irreducible admits \(c\)-dim., \(TFAE\):

1. \(\pi\) is not \(\cong\) to a subrep of a \(B(\mu_1, \mu_2)\)
2. \(K(\pi) \cong S(F^*)\)
3. \(\forall v \in V, \forall n > 0, \int \pi(\phi^n)v dy = 0\)
4. matrix coefficients \(\langle \pi(g)v, w \rangle\), \(v, w \in V\), are cpt supported and the centre of \(G\).

**Definition.** \(\pi\) is supercuspidal if (iv) holds. This is picked because it generalizes.

**Idea of pt. (1).** From here we dropped \(K(\pi)\) for \(\pi\) in \(G\).

Now, \(K(\pi) \cong S(F^*)\)

(\(T\) not under \(B\)) \(\Rightarrow\) quotient \(K(\pi)/S(F^*)\) f.d. \(\Rightarrow\) we get a 1-dim. \(B\)-int setup as \(B\) is solvable.

So \(\pi \cong \text{Ind}_B^G(\chi)\), \(\chi\) char of \(B\).

\(\Rightarrow\) \(\pi\) is supercuspidal.

So (iv) \(\Rightarrow\) (iv).

(iii) \(\Rightarrow\) (iv) isn't so bad either. \(\forall x \in K(\pi) : \int \pi(\phi^n)x dy = \int x(y) dy\)

\(\int \text{vanishes} \Rightarrow \int \text{vanishes in odd of } 0. \Box\)

He won't say anything more about supercuspidal reps, which is a bit sad because they're the key.

**Note:** if a global object contains a supercuspidal local object then the global object is cuspidal. That's why they're called supercuspidal.

Finally, something on \(L\) \& \(\varepsilon\) factors.
59 Local factors & local Langlands

None has had the time to talk about Tate's thesis, & he'd have to assume it (GL, local L-funs or sthg).

\((r, S)\) so dim 0 admit -red. \(\mathcal{S} \in \mathcal{K}(r)\).

Define \(L(r, S) = \int_{\mathcal{S}(r)} |x|^S d\nu x \in \mathcal{C}(q) \) & uniform \(C_{[q]}\) if \(\mathcal{S} \in \mathcal{S}(r)\).

Now say \(S = S_{\text{new}}\), new vector with \(\mathcal{S}(d) = 1\).

\(L(r, S) = M(S_{\text{new}}, S) = \sum_{n \geq 0} S_{\text{new}}(n) q^{n(1-d)} \quad \text{NB here we're assuming cond} \ q = 0. \text{ Then it indep of} \ q.\)

e.g. unramified \(\mathcal{S}(\mu, \mu)\) is -dim 1:

\[
L(r, S) = \frac{1}{(1 - \mu(r) q^{-1})(1 - \mu(r) q^{-1})}
\]

Our local functional eqn:

\(\mathcal{S} \in \mathcal{K}(r) \to \mathcal{S}(x) = \omega(x)^{-\frac{1}{2}} \pi(x) \mathcal{S}(x) \in \mathcal{K}(r), \pi = w^{-d} \tau.
\)

Then \(M(S, S) \in \mathcal{S}(r, r, S) = \mathcal{M}(S, r, 1-d)\) for certain \(\mathcal{S}(r, r, S)\), indep of \(S\).

\(\mathcal{R} \to L(r, S), \in \mathcal{S}(r, r, S)\)

NB 1) Can do this at all primes & multiply to get a global thing. Or something.

2) It all works if \(\infty\) too. Need an understanding also to do local \(\to\) global. However, local Langlands at \(\infty\) is easy & he wants to talk about local Langlands.

Now a rep \(\rho\) of \(\mathcal{W}_F\) on \(U\) s.t. \(U\).

We get \(\rho|_{\mathcal{W}_F} : \mathcal{W}_F \to \mathcal{G}(U)\) s.t. \(\rho(N) \in \text{End} \mathcal{U}\) finite

s.t. \(\rho(\pi) \rho(N) \rho(w^{-1}) = \|\mathcal{W}_{\rho} \rho(N)\|\).

Now we get \(L(\rho, S) = \det(1 - \rho(\pi) q^{-1} | U_{\mathcal{W}_F})^{-1}\)

If \(\dim \rho = 1, \rho : F^\circ \to \mathcal{C}, L(\rho, S) = \left\{ \begin{array}{ll}
\frac{1}{(1 - \rho(\pi) q^{-1} \text{ ramified})}\end{array}\right.\)
Take these gives us a def of $\varepsilon(p,\chi,\lambda)$ for $p: F^* \to C^*$

A deep truth of Deligne & Langlands (Langlands proved it first & no one has read his proof, Deligne subsequently proved it & lots of people have read Deligne's proof because it has finite length) implies that we can define $\varepsilon$ for $\dim(p) > 1$ as well, agreeing with Tate's & compatible with induction in degree 0.

**Theorem (Local Langlands conjecture for $GL_2$)**

There exists a 1:1 correspondence between isom. classes

\[
\begin{array}{c c c}
\text{(2-dim reps of)} & \leftrightarrow & \text{(irred admin)} \\
\text{WD}_F, F/F & \text{rep of } G & \text{(in cluding 1-dim ones)}
\end{array}
\]

$p \mapsto \pi(p)$

St (i) If $X: F^* \to C^*$ is a character, then $\pi(p \circ X) \cong \pi(p) \otimes X_{\text{det}}$

& also $w_{\pi(p)} = \det p$

(ii) $L(\pi(p), s) = L(p, s)$

$\zeta(-\sigma) = \zeta(-\sigma)$.

Example:

1. $p = X_{\text{det}}, 1\text{-dim} \mapsto p = X_{1/1} \times X_{1/1}$
   
   (Note ugly $1, 1$)

   These serious problems,

   with the normalizations)

2. Reducible $p = \mu_1 \otimes \mu_2$ of $W_f = F^*$

   Then $\pi(p) = \pi(\mu_1, \mu_2)$.

   $\mu_1, \mu_2$ unramified, $L(\mu_1 \otimes \mu_2, s) = L(\mu_1, s)L(\mu_2, s)$

(ii) $p$ indecomposable but reducible

\[ p = \text{sp}(2) \otimes X, \quad p |_{W_F} = X_{1/0}, \quad p(M) = (x^2) \]

special rep $\sigma$ of $\mu_1, \mu_2$.

Note that this why we have WD group need to find reps which correspond to the special case. Not to solve the problem.

NB if we actually defined $L$-ps & $\varepsilon$-factors in this case then he might have said something about them. Is that $\varepsilon$ are equal is harder than $L$s, (as usual?).
Finally,

3) Irreducible $\rho \leftrightarrow$ supercuspidal $\tau$ (unsurprising as they're the only ones left!!)

Say $p=2$. Then $\rho = \text{Ind}_{E/F}(\theta)$, $\theta$ a char $E$, $E/F$ quadratic

The well rep attached to $\theta$ a rep of $\mathbb{G}$.

Hoddest case: $p=2$. There are other irreductible $\rho$ & other supercuspidals.
After partial results, the pf was completed by Knapp, about 15 years ago.

He has ~ 5 minutes left, so he'll just mention the infinite case.

$R, C : W_E = C^\times$, $W_{E^F} = \langle C^\times, F, F^2-1, FzF^{-z} \rangle$

$W_F = W_{E^F} = C^\times$.

We define $L$ & $e$-factors (they involve $\Gamma$).

Irreducible admissible $(\rho, K)$-modules are parametrized by semisimple $2$-dual reps of $W_F$, $F = C, R$

$F = C : \rho = (\mu, \mu)$; correspond $(\rho, K)$-module is $\tau(\mu, \mu)$

$F = R : \rho$ factors through $W_{E^F} = R^\times$ so $\mu \oplus \mu \sim \tau(\rho) = \tau(\mu, \mu)$

or 2) $p$-adic. In this case, restriction to $C = \mathbb{Z} \to \left( \frac{\mathbb{Z}}{\mathbb{Z}} \right)^3 |z|^k$

with sc $\mathbb{Z}$, $k \in \mathbb{C}$

Then $\tau(p) = \alpha(\mu, \mu) \times \mu_1 \mu_2(\chi) = x^2 \text{sgn}(x)$, $x = s_1 s_2$

Much easier!
There will be a success party 8.30pm a week Saturday @ 8.30pm @ John's house.
There is a sale of Berkshire books outside afterwards.

He wants today to talk about the classical theory of modular forms. He will be sticking to $\mathbb{Q}$ in this case, but a lot of the adelic approach goes through for a general number field.

$H$ = upper half plane = $\{z: \text{Im}(z) > 0 \}$, $\text{SL}_2(\mathbb{Z})$ is a handy group, $q = e^{2\pi i z}$ is a handy notation.

**Modular forms & stuff**

**Ex 1** $\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$. Note that if $\sigma = (c_2, 0) \in \text{SL}_2(\mathbb{Z})$ then $\Delta((c_2, 0)z) = (cz + d)^{12} \Delta$.

**Ex 2** $\varphi(z) = q \prod_{n=1}^{\infty} (1-q^n)^{\frac{1}{n}} (1-q^{rn})$

If $c = 0 \pmod{11}$ then $\varphi(z) = (cz + d)^{12} \varphi(z)$

If we write $\Delta(z) = \sum_{n=1}^{\infty} c(n)q^n$, $\varphi(z) = \sum_{n=1}^{\infty} c(n)q^n$ then $c(n)$ & $c(n)$ have great arithmetical importance.

Deligne attached reps of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\Delta$, reflecting properties of $c(n)$.

In this it showed $|\varphi(p)| < 2p^{1/2}$

Also, $p - c(p) = \# \text{solns of } \begin{cases} y^2 \equiv x - x^2 \pmod{p} \end{cases}$

If $E: y^2 = x^3 - x^2$ & $\text{Ep} : \text{Ker}(E(\overline{\mathbb{Q}}) \to E(\overline{\mathbb{Q}}))$

then we get $\text{Gal}(E)$

If $p + 5$ this vanish non-ab ext.

No one even knows which primes split etc. John hopes that the Langlands circle of ideas will solve this problem at some stage.

**Notation.** $\text{GL}_2^+(\mathbb{R}) = \{ \sigma = (c_2, 0) \in \text{GL}_2(\mathbb{R}) \text{ with } \text{det}(\sigma) > 0 \}$

It operates on $H$ via $\sigma(z) = \frac{az + b}{cz + d}$. We will also define $J(0, z) = cz + d$.

**Notation.** Same as the books e.g. Shomura has a $(det)^{1/2}$ factor in $h_i$. Note that for John's $J$ we have $J(0, 0, z, 2) = J(0, 0, z, 2) M(0, z)$. 
New say \( k \in \mathbb{Z} \) is an integer, & \( f: \mathbb{H} \to \mathbb{C} \).

**Def.** \( f \mid \sigma_k: \mathbb{H} \to \mathbb{C} \) is defined by

\[
(f \mid \sigma_k)(z) = f(\sigma(z)) j(\sigma(z)^{-1})(\det \sigma)^{-k/2}.
\]

John thinks this is the most usual def.

It works nice adelically; the centre of \( \text{GL}_2(\mathbb{R}) \) acts trivially.

Note that \( f \mid \sigma_k (\sigma_0) = (f \mid \sigma_k) \mid_{\sigma_0}. \)

Reference: Shimura, Miyaoka books.

Now say \( \Gamma = SL_2(\mathbb{Z}) \) of finite index.

Define \( V_k(\Gamma) = \{ f: \mathbb{H} \to \mathbb{C} \text{ such that } (i) f \mid \sigma_k = f \text{ \forall } \sigma \in \Gamma \text{ \ and } \sigma \text{ \ is } \text{holo} \text{ on } \mathbb{H} \}. \)

Cups are \( \mathbb{P}^1(\mathbb{Q}) \) in this case.

Say \( k \) is even. Say \( \text{SL}_2(\mathbb{Z}) \).

Then \((f \mid \sigma_k) \text{ is invariant under } \left( \begin{array}{cc} 1 & N \alpha \\ 0 & 1 \end{array} \right) \) some \( N > 0, N \in \mathbb{N}(\alpha). \)

\[
(f \mid \sigma_k)(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z/N} \text{ This is the classical way of treating things.}
\]

If \( f \) is meromorphic at the cups if there are only a finite number of \( a_n \) with \( n > 0 \) which are non-zero.

If \( f \) is holomorphic if \( a_n = 0 \) for \( n > 0 \).

The cup forms are the holomorphic \( f \) with \( a_n = 0 \ \forall n \neq 0 \).

**Notation** \( M_k(\Gamma) = \{ f \in V_k(\Gamma); f \text{ is holo \& cups} \} \)

\( S_k(\Gamma) = \{ f \in M_k(\Gamma); f \text{ vanishes \& cups} \} \)

\( \Delta \in S_1^0(\text{SL}_2(\mathbb{Z})) \), \( \psi \in S_2(\Gamma_0(1)) \)

He'll really only be talking about cup forms because they lie at the heart of the theory.
Lemma: Assume \( f \in \mathcal{V}_k \). Then \( f(z) \) is a cusp form \( \iff |f(z)| (\text{Im } z)^k \) is bounded on \( \mathcal{H} \).

(Proof in all the books.)

Now say \( N \geq 1 \)

\[
\Gamma_0(N) = \left\{ \sigma = (a,b) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}
\]

\[
\Gamma_1(N) = \left\{ \sigma \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N, \quad d \equiv 1 \mod N \right\}
\]

By using the Riemann-Roch theorem & stuff, & Riemann surfaces, we can deduce:

Fact: \( M_k(\Gamma_0(N)) \) & \( S_k(\Gamma_1(N)) \) are free \( / \mathbb{C} \). In fact, if \( k \geq 2 \) there's a nice formula for their dimension.

Petersson inner product over \( S_k(\Gamma_0(N)) \)

\[
(f, g) = \frac{1}{\text{vol}(\mathcal{H} \setminus \mathbb{H})} \int \frac{f(z) \overline{g(z)}}{y^k} \, \text{d}x \, \text{d}y
\]

He wants to talk about Hecke operators & diamond operators on \( S_k(\Gamma_0(N)) \)

Note \( 0 \to \Gamma_2(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \to 0 \)

\[
(a, b) \mapsto a \mod N
\]

If \( \omega \in (\mathbb{Z}/N\mathbb{Z})^* \) then \( \tilde{\omega} \sigma \omega (\text{this is a def.}) \in \Gamma_0(N) \) s.t. \( \sigma \omega \to \omega \).

If \( f \in M_k(\Gamma_0(N)) \) then define \( \langle \omega f \rangle = f|_k \sigma_\omega \). Note - this is well-defined.

Hecke operators are difficult to explain classically. It's all tied up with double cosets, but it's a bit contrived. Adeleically it's much easier.

He'll just give the formula for Hecke operators.

Note that if \( f \in M_k(\Gamma_0(N)) \) then \( f|_k e \in M_k(\Gamma_0(N), \chi) \) & similarly for \( S_k \).

Hence \( M_k(\Gamma_0(N)) = \bigoplus \chi \in (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C} \) \( M_k(\Gamma_0(N), \chi) \) & similarly \( S_k \).

Anyway, back to Hecke operators. Here's a new description, but it's not clear using this description why it fixes \( M_k \) or \( S_k \).
Say $n \geq 1$

**Def.**

$$ T_n = n^{d-1} \sum_{\Delta \in \mathbb{P}} \sum_{\Delta > 0} \sigma_0(\Delta) \frac{d-1}{d} \mathcal{O}_k(\Delta) $$

Here $\Gamma = \Gamma_c(N)$

**Fact.**

$M_k(\Gamma) \& S_k(\Gamma)$ are f.d. / C

$S_k(\Gamma) \& M_k(\Gamma)$ are stable under $T_n$

The $T_n$ commute.

**Def.**

$H_k(\Gamma) =$ Z-algebra in $\text{End}(M_k(\Gamma))$ generated by all $T_n$, $n = 1, 2, ...$

$h_k(\Gamma) =$ Z-algebra in $\text{End}(S_k(\Gamma))$ generated by all $T_n$, $n = 1, 2, ...$

**Def.**

If $(m, N) = 1$, put $S_m = m^{k-2} < m>$

**Fact.**

$S_m \in h_k(\Gamma)$ for all $(m, N) = 1$

**Fact.**

$T_n$ with $(m, N) = 1$ are self-adjoint w.r.t. $(\cdot, \cdot)$, (so we can simultaneously diagonalize them)

**Theorem.** Assume $f(x)$ is a non-zero elt of $S_k(\Gamma_c(N))$. Then TFAB:

1. $f(x)$ is an eigenform for all the Hecke operators
2. $f(x)$ is a Dirichlet character $\chi$ mod $N$, s.t. $f \in S_k(\Gamma_c(N), \chi)$

Moreover, if $f(x) = \sum c_i(x)q^i$ then $c_i(x) \neq 0$, & the following formal identity holds:

$$ \text{Mellin Transform of } f = \sum_{n \geq 1} \frac{c_i(x)}{n^s} = \frac{c_i(x)}{s} \prod (1 - \frac{\epsilon(p)}{p^s} \chi(p)p^{k-1}x) $$

where, $\epsilon(p) = c_i(x)/c_i(1)$

Moreover, when (1) & (2) hold, $T_n f = c_i(x)f$

$$ \frac{c_i(x)}{c_i(1)} $$
in the lecture
He finally wants to talk about

**Primitive forms in $S_k(\Gamma_0'(N))$**

Atkin-Lehner

NB: This would have been classical. It still has a nice adelic interpretation, though.

The crime that Atkin-Lehner committed was to call them newforms. It's a crime against the English language, especially to make it I want. John is plumping for primitive forms, which is what the French call them.

Define $S_k(\Gamma_0'(N))^{\text{old}} \equiv \text{old forms}$ if $M | N$ & $d$ is a divisor of $N/M$

then define $\varphi_d : S_k(\Gamma_0'(M)) \rightarrow S_k(\Gamma_0'(N))$

$\varphi \mapsto \varphi|d(z)$

& set $S_k(\Gamma_0'(N))^{\text{old}} = \bigoplus_{M \text{ with } M+K \text{ for all } M/M} S_k(\Gamma_0'(M))^{\varphi_d}$

Now set $S_k(\Gamma_0'(N))^{\text{new}}$ is the adjoint's newforms: the orthogonal complement of $S_k(\Gamma_0'(N))^{\text{old}}$ under $(,)$.

Fact: $S_k(\Gamma_0'(N))^{\text{new}}$ is stable under the whole of $k_k(\Gamma_0'(N))$

**Def:** A primitive form of level $N$ is an elt of $S_k(\Gamma_0'(N))^{\text{new}}$ which is an eigenform of $k_k(\Gamma_0'(N))$ & s.t. $c_k(y) = 1$.

If $g \in S_k(\Gamma_0'(N))$ is an eigenform for $k_k(\Gamma_0'(N))$ & $g \in S_k(\Gamma_0'(M))$ is an eigenform for $k_k(\Gamma_0'(M))$ & $g = \sum c_\gamma(y(q))q^\gamma$, $g' = \sum c_\gamma'(q)q^\gamma$, then say $g \sim g'$ if $c_\gamma'(q) = c_\gamma(q)$ for all but a finite no. of $p$.

**Theorem:** If $\phi$ is primitive form in each equivalence class, say $\phi$. If $g \sim \phi$ then the level of $\phi$ divides the level of $g$.

(1) If $M$ is any integer divisible by the level of $\phi$, then $\exists g$ of level $M$

**st.** $g \sim \phi$.
One of the big problems of the automorphic side of things is that there's no decent reference - either there's no proof, or it's too difficult for the beginner. John himself has been guided by a set of lecture notes of Richards, although he's normalised things differently.

Say \( \mathbb{A} \) are the adeles of \( \mathbb{Q} \). We'll stick with \( \mathbb{Q} \) although one of the advantages of the adelic approach is that it goes through for any field, number field.

Say \( \mathbb{A}^o \) is the finite adeles (this crummy notation is due to Richard Taylor, so John takes no blame)

We have \( \mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}^o) \times \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{A}^\infty) \), \( \mathrm{GL}_n(\mathbb{Q}) \), \( \mathrm{GL}_n(\mathbb{Q}) \), embedded diagonally.

If \( F \) is a field, sett \( \mathbb{A}(F) = \{ (a_F) \in \mathrm{GL}_n(F) \} \) \& \( \mathbb{B}(F) = \mathbb{B}(F) \cap \mathrm{SL}_n(F) \).

Define \( \hat{\mathbb{Z}} = \prod \mathbb{Z}_p \). Then \( \mathrm{GL}_n(\hat{\mathbb{Z}}) \cong \mathrm{GL}_n(\mathbb{A}^o) \).

We need various little results & we'll put them together in a big lemma.

Lemma

(1) \( \mathrm{GL}_n(\mathbb{Q}) = \mathbb{B}(\mathbb{Q}) \mathrm{GL}_n(\hat{\mathbb{Z}}) \)
\( \mathrm{SL}_n(\mathbb{Q}) = \mathbb{B}(\mathbb{Q}) \mathrm{SL}_n(\hat{\mathbb{Z}}) \)

(2) \( \mathbb{A}^o = \mathbb{A}^\infty \mathbb{A}^o \mathbb{A}^\infty \)

(3) \( \mathrm{GL}_n(\mathbb{A}^o) = \mathbb{B}(\mathbb{Q}) \mathrm{GL}_n(\hat{\mathbb{Z}}) \)
\( \mathrm{SL}_n(\mathbb{A}^o) = \mathbb{B}(\mathbb{Q}) \mathrm{SL}_n(\hat{\mathbb{Z}}) \)

(4) If \( N \neq 1 \), then \( \mathrm{SL}_n(\mathbb{Z}) \rightarrow \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z}) \), surjective.

(5) (Strong approx. for \( \mathrm{SL}_n \)) \( \mathrm{SL}_n(\mathbb{Q}) \mathrm{SL}_n(\hat{\mathbb{Z}}) \) in dense in \( \mathrm{SL}_n(\mathbb{A}) \)

Proof of (5) Every open subgroup contains, for some \( N \), the subgroup \( V_N = \{ g \in \mathrm{GL}_n(\hat{\mathbb{Z}}) \mid g \equiv I \mod N \} \).
So it suffices to show that \( \mathrm{SL}_n(\mathbb{Q}) V_N = \mathrm{SL}_n(\hat{\mathbb{Z}}) V_N \forall N \).

But \( \mathrm{SL}_n(\mathbb{Z}) V_N = \mathrm{SL}_n(\hat{\mathbb{Z}}) \), so \( \mathrm{SL}_n(\mathbb{Z}) V_N \rightarrow \mathrm{SL}_n(\hat{\mathbb{Z}}) V_N = \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z}) \)
so \( \mathrm{SL}_n(\mathbb{Q}) V_N \supseteq \mathbb{B}(\mathbb{Q}) \mathrm{SL}_n(\hat{\mathbb{Z}}) = \mathrm{SL}_n(\hat{\mathbb{Z}}) \).

(5) If \( U \) is any subgroup of \( \mathrm{GL}_n(\mathbb{A}^o) \) s.t. \( \text{det } U = \hat{\mathbb{Z}}^\ast \), then we have \( \mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{Q}) U \mathrm{GL}_n(\mathbb{R}) \)

Proof of (5) follows quickly from (i). If \( g \in \mathrm{GL}_n(\mathbb{A}) \), then \( \text{det } g = \omega(\text{det } w) \), \( \omega \in \mathbb{Q}^\ast \), \( w \in \mathbb{U} \), \( \beta \in \mathbb{R}_{>0} \)
by part (w).
So \((\delta_1^o)^{-1} g u_1^o(\delta_1^o) \in SL_3(A)\)
\(\delta_1^o\) & it's an elt of \((\delta_1^o)^{-1} g u_1^o(\delta_1^o) \cap SL_3(A)\) which is open & non-empty.

\((\delta_1^o)^{-1} g u_1^o(\delta_1^o) = \bigvee_{\delta} \gamma_1, \gamma_1 \in SL_3(A), \forall \gamma \in SL_3(A^o), \delta \in SL_3(A)\)

\(g = \left(\frac{\delta_1^o}{\gamma_1}\right) v u_1^o x \left(\frac{\delta_1^o}{\gamma_1}\right)\)

\(u, u_1 \in \text{GL}_3(A)\)

\((vii)\) If \(U\) is any open subset of \(\text{GL}_3(A^o)\), and suppose \(g = \bigvee_{\gamma} \gamma_1^o u_1^o \text{det}(U) R^o\). Then

\(\gamma_1^o \text{ det}(\text{det}(U)) = I, \text{ for all } \gamma \in \text{det}(U)\).

If we choose \(g, \gamma \in \text{GL}_3(A^o)\) s.t. \(\text{det}(g) = 1\). Then, we have

\(\text{GL}_3(A) = \bigvee_{\gamma} \text{GL}_3(A) g \gamma \text{GL}_3(A)\)

This is just a generalization of \((vii)\).

**Def.** \(U_2(N) = \{ g \in \text{GL}_2(A) \mid g = (\delta_2^o) \text{ mod } N \} \subseteq \text{GL}_2(A^o)\)

**Remark.**

(i) \(U_2(N) \cap \text{GL}_2(A) = \Gamma_2(N)\)

(ii) \(\text{det}(U_2(N)) = 2^N\)

**Conclusion.** If \(g \in \text{GL}_3(A) = \text{GL}_3(A) U_2(N) \text{GL}_2(A)\) then \(J = \bigvee_{g \in \text{GL}_3(A)} \bigvee_{u \in U_2(N)} \bigvee_{w \in \text{GL}_3(A)}\)

This decomposition is not unique.

Now say \(J \in \text{GL}_3(A) \subseteq \Gamma_2(N)\), \(k \in I\). We will begin the translation.

**Def.** \(\phi : \text{GL}_3(A) \rightarrow C\)

\(\phi^* : \text{GL}_3(A) \times C^k \rightarrow C\)

Here \(u \in \text{GL}_3(A)\) is acting on \(C^k\), \(w \in C^k\).

NB (i) not clear that it is well-defined yet.

(ii) It sort of doesn't matter what power of \((\text{det}(w))\) you put. It all boils down to personal taste. John likes \(k/2\) best.

Let's deal with well-definedness. Say \(s \in \Sigma U_2 U_1 \wedge 2, 2_1 \wedge 2_2 W_1, W_2, T_1 \in \text{GL}_2(A)\)

\(u_1 \in U_2(N)\)

\(w \in \text{GL}_3(A)\)

Then \(\delta = \delta_2^o \delta_1^o \in U_2 U_1\wedge 2 W_1 W_2, W_2, T_2 \in \text{GL}_3(A^o)\)

\(\text{det} \delta > 0\).

\(\delta = u_1^o \wedge 2 W_1 W_2, W_2, T_2 \in \text{GL}_3(A)\)

\(\text{det} \delta > 0\).

\(\delta = u_1^o \wedge 2 W_1 W_2, W_2, T_2 \in \text{GL}_3(A^o)\)

\(\text{det} \delta > 0\).
So \( \delta \in \text{GL}_1(\mathbb{R}) \cap U_1(N) = \Gamma_1(N) \) (slight abuse of notation - we're identifying \( \delta \) with the finite part or the infinite part)

Hence \( \hat{g}_{|_F}^F \delta = \delta \) as \( \delta \in S_F(\Gamma_1(N)) \).

In \( \hat{g}(\delta(z)) \hat{j}(\delta(z))^k = \hat{g}(z) \)

So if \( z = w_i \cdot \delta = w_i \cdot w_i^* \)

\[ \hat{g}(w_i) \hat{j}(w_i) \hat{j}(w_i)^* \frac{(\det w_i)^{k/2}}{2} \]

\[ = \hat{g}(w_i) \hat{j}(w_i) \hat{j}(w_i)^* \frac{(\det w_i)^{k/2}}{2} \]

\[ = \hat{g}(w_i) \hat{j}(w_i) \hat{j}(w_i)^* \frac{(\det w_i)^{k/2}}{2} \]

Hence \( \phi_\beta \) is indeed well-defined.

(Recall \( \text{GL}_1(\mathbb{R}) \) acts on \( \mathbb{H} \) & also that \( j(a, \omega_i, z) = j(a, \omega_i, z) = j(a, z) \))

The stability subgroup of \( \iota \) in \( \text{GL}_1(\mathbb{R}) \) is \( \mathbb{R}^* \cdot \text{SO}_2(\mathbb{R}) \).

The stability subgroup of \( \iota \) in \( \text{SL}_2(\mathbb{R}) \) is \( \text{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \mid u^2 + v^2 = 1 \right\} \)

Set \( U_\infty = \mathbb{R}^* \cdot \text{SO}_2(\mathbb{R}) \).

Then we get an identification \( \text{GL}_1(\mathbb{R})/U_\infty \rightarrow \mathbb{H} \)

\[ \beta : U_\infty \longrightarrow \beta i \]

Properties of \( \phi_\beta \)

\[ \hat{g}(w) \hat{j}(w_i) \frac{(\det w_i)^{k/2}}{2} \]

1) \( \phi_\beta \) is left unit by \( \text{GL}_1(\mathbb{R}) \) (as \( \hat{g} = \hat{j} w \) & \( \phi_\beta(\zeta) \) doesn't depend on \( \zeta \).

2) \( \phi_\beta \) is right unit by \( U_1(N) \) (as \( \phi_\beta(\zeta) \) doesn't depend on \( \zeta \) either)

3) For all \( \zeta \in U_\infty \), we have \( \phi_\beta(\zeta \cdot \iota) = \phi_\beta(\iota) \hat{j}(\zeta, \iota^*) \frac{(\det \zeta)^{k/2}}{2} \) (easy by def.)

(Recall \( \hat{g} = \hat{j} w \), \( \iota \in \text{GL}_1(\mathbb{R}), \omega_1 \equiv U_1(N), \omega \equiv \text{GL}_1(\mathbb{R}) \))

4) Fix \( g \in \text{GL}_1(\mathbb{R}) \). Take any \( \zeta \in \mathbb{H} \), & pick any \( \omega \in \text{GL}_1(\mathbb{R}) \) s.t. \( \omega(w) = \zeta \).

Then the function

\[ \zeta \mapsto \phi_\beta(g \omega) \hat{j}(w_i) \frac{(\det w_i)^{k/2}}{2} \] is well defined.

\[ \text{and in fact it's homomorphic as a function of } \zeta \text{, for any } g \in \text{GL}_1(\mathbb{R}) \]

This is because if \( g \in \text{GL}_1(\mathbb{R}) \), then \( g = \hat{j} w \cdot \frac{1}{w_i} \), so \( \omega g w = \hat{j} w \cdot \frac{1}{w_i} \cdot \omega \cdot w_i = \hat{j} \).
Hence \( \varphi_g (gw) j(w, z)^k (\det w)^{-k/2} = f(\gamma^t w, z) j(\gamma^t w, z)^{-k} \det(\gamma^t w)^{1/2} = f(\gamma^t w, z) j(\gamma^t w, z)^{-k} \det(\gamma^t w)^{1/2} = \int_{\mathbb{H}} \gamma^t(z) \) which is holomorphic on \( \mathbb{H} \).

5) Fix \( g \in \text{GL}_2(\mathbb{R}) \). Then the function on \( \text{GL}_2(\mathbb{R}) \) given by \( w \mapsto (\varphi_g gw) \) is bounded on \( \text{GL}_2(\mathbb{R}) \).

\( \Box \) We just showed \( \varphi_g (gw) j(w, z)^k (\det w)^{-k/2} = \int_{\mathbb{H}} \gamma^t(z) \), \( z = w \).

Hence \( |\varphi_g (gw)| = \left| \int_{\mathbb{H}} \gamma^t(z) \right| \frac{|\det w|^{1/2}}{|j(w)|} \).

If \( z = w \) then \( \text{Im} \ z = \text{det} w \).

Hence \( \varphi_g (gw) = \left| \int_{\mathbb{H}} \gamma^t(z) \right| |\text{Im} \ z|^{-k/2} \).

Now \( \int_{\mathbb{H}} \gamma^t(z) \) is a cusp form for \( \Gamma_1(N) \Gamma^t \cap \text{SL}_2(\mathbb{R}) \).

\& so by a famous property of cusp forms, \( \varphi_g (gw) \) is indeed bounded.

Next time we will show why properties 1) - 5) are true sense character \( \varphi_g \), in that any \( \varphi \) with these properties is \( \varphi_g \) for some \( g \).

Recall \( N \in \mathbb{Z} \), \( g \in \text{SL}_2(\mathbb{Z}) \), \( \text{GL}_2(\mathbb{R}) = \text{GL}_2(\mathbb{R}) \cap \text{GL}_2(\mathbb{Q}) \)

\( U_2(N) = \{ g \in \text{GL}_2(\mathbb{Z}) \mid g \equiv (z \ z) \mod N \} \).

\( \varphi_g : \text{GL}_2(\mathbb{R}) \rightarrow \mathbb{C} \)

1) \( \varphi_g \) is an integral lift by \( \text{GL}_2(\mathbb{R}) \).
2) \( \varphi_g \) is unit on right by \( U_2(N) \).
3) \( \varphi_g (\gamma z) = (\varphi_g (z)) j(\gamma, z)^k (\det \gamma)^{1/2} \) \( \forall \gamma \in U_2(N) = \mathbb{R}^+ \text{SO}_2(\mathbb{R}) \).

4) Fix \( g \in \text{GL}_2(\mathbb{R}) \). Then the function \( \mathbb{H} \rightarrow \mathbb{C} \) given by

\( z = w \mapsto (\varphi_g gw) j(w, z)^k (\det w)^{1/2} \), \( w \in \text{GL}_2(\mathbb{R}) \)

is holomorphic.

5) Fix \( g \in \text{GL}_2(\mathbb{R}) \). Then the function on \( \text{GL}_2(\mathbb{R}) \) given by

\( w \mapsto (\varphi_g gw) \) is bounded.
Lemma. Given $\phi : \text{GL}_n(\mathbb{R}) \to \mathbb{C}$ satisfying (1)-(5), \( \exists f \in \mathcal{S}_k(N) \) s.t. \( \phi \cdot f \).

Proof. Given \( \phi \), define \( f(z) = \phi(w) j(w, i)^k (\det w)^{ik} \), \( z \to w \), we have \( \text{GL}_n(\mathbb{R}) \)

- Homomorphism \( H \to \mathbb{C} \) by \( 4 \)
- Well-defined by \( 3 \)

\( \forall w \in \Gamma_k^2(N) \) we have \( \int_f \alpha = \phi \).

But note \( \alpha \in \Gamma_k^2(N) \Rightarrow (\alpha, 1) \in \mathcal{U}_1(N) \)

Note: \( \phi((\alpha, 1), w) = \phi(w(\alpha, 1)) \cdot \phi(w) \) by \( 2 \)

\[ \phi(\alpha^{-1}(1, \alpha w)) = \phi(1, \alpha w) \) by \( 1 \)

\[ = f(\alpha w, i) j(\alpha w, i)^k (\det w)^{ik} \]

\[ = \int f(wz) j(w, i)^k (\det w)^{ik} \]

Hence \( \int f(z) j(w, i)^{-k} (\det w)^{-ik} = \phi(w) \cdot \int f(w) j(w, i)^k (\det w)^{ik} \)

\[ = \int f(\alpha z) j(\alpha, z)^{-k} (\det w)^{ik} \]

\[ = (\int_f \alpha)(z) \]

Finally, we want to show that \( f \) is cuspidal.

Note that \( |f(z)| = |\phi(w)| \left( \frac{|j(w, i)|^k}{\det w} \right)^{ik} \)

\[ z = w \in \mathbb{R} \Rightarrow \text{Im} z = \frac{\det w}{|j(w, i)|^k} \]

\[ |f(z)| = |\phi(w)| (\text{Im} z)^{ik} \]

\[ \Rightarrow f(z) \text{ is a cusp form by } 5 \]

Definition. Let \( \mathcal{S}_k \) be the vector space of all functions \( \phi : \text{GL}_n(\mathbb{R}) \to \mathbb{C} \) satisfying (1)-(5), except that we weaken 2). \( \Leftrightarrow \) (2):

- \( \phi \) is right-invariant under some open subgroup \( U \subset U(N) \)

So we've encapsulated \( \mathcal{S}_k(N) \) for all \( N \), & other stuff, too!
Note. There is also an action of $GL(V)$ on $S_n$ by right multiplication, e.g.,

$$\phi \in GL(V), \text{ try defining } \phi_\phi(g) = \phi(\phi g)$$

Then $\phi_\phi(f g) = \phi(f) \phi_\phi(g)$ & to check property 3) we have to commute $f \& g$.

In general we can't commute them, so 3) does not hold.

If $g \in GL(V)$, we can commute them.

Hence $S_n$ is a $GL(V)$-module under right translation.

Lemma (it's really a remark—he should avoid name inflation!)

$$S_n \left( \Gamma_k(N) \right) \cong S_k^{U_k}$$

$$f \mapsto \phi_f$$

So there's some sort of admissibility condition here, like in Topo lectures.

Say $M$ is a v.s. over $C$ with an action of $GL(V)$

**Def.** We say that $M$ is admissible if

1. $M_U$ is a.d. over $C$ for every open subgroup $U$ of $GL(V)$
2. The stabilizer of any $z \in M$ is open in $GL(V)$

Lemma. $S_n$ is an admissible $GL(V)$-module.

**Prop.** Condition (2) is obvious by def.: stabilizer $\cong U(\phi)$

1. Say $U$ is any open subgroup of $GL(V)$

Then $\exists \phi_0, \ldots, \phi_{r-1} \in GL(V), \text{ s.t. } GL(V) = \bigsqcup_{i=0}^{r-1} GL(V) \phi_i UGL(1)$

(by (viii) on page III.7)

For $1 \leq j \leq r$, define $\Gamma_j = \phi_i U \phi_j^{-1} \in GL(C)$

Define $\theta: S_n^U \rightarrow \prod_{j \leq r} S_n(\Gamma_j)$

$$\theta(f) = (f_{\phi_0}, f_{\phi_1}) \quad \text{where, for } 1 \leq j \leq r,$$

$$f_j(z) = \phi_i g_j(w) j(w, i)^k (det w)^{k/2}, \quad z \in \phi_i, \phi_j \in GL(C), \phi: H \rightarrow C$$

Check $f_j \in S_k(\Gamma_j)$. 
We need to check \( B \) is injective, then we'll be done.

Say \( B(\varphi) = 0 \). We need to show \( \varphi = 0 \).

Say \( f \in \text{GL}_2(A) \). We need \( B(f) = 0 \).

But \( f = \gamma g_j u w \) for some \( j \), \( \gamma \in \text{GL}_2(k) \), \( u \in U \), \( w \in \text{GL}_2(k) \).

Then \( \varphi(f) = \varphi(g_j u w) = \varphi(g_j u) = 0 \) as \( \gamma \neq 0 \). \( \Box \)

In fact, we can also show that \( B \) is injective. It's just a generalization of the fact that \( \text{Sh}(\Gamma \backslash A) \cong \text{Sh}(k) \).

John now wants to talk about the interpretation of the classical diamond \& Hecke actions in this new setting.

**Action of \((\mathbb{Z}/N\mathbb{Z})^*\) on \( S_k U_i(k) \)**

Define \( U_0(N) = \{ g \in \text{GL}_2(\mathbb{Z}) \mid g \equiv \begin{pmatrix} \alpha & \eta \\ \gamma & \delta \end{pmatrix} \pmod{N} \} \) for \( \alpha, \beta, \gamma, \delta \in \mathbb{Z} \).

Then \( 0 \to U_1(N) \to U_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \to 0 \)

\[ (a, b) \mapsto \bar{a} \pmod{N} \]

**Lemma** If \( f \in S_k(\Gamma \backslash A) \), then \( \varphi_{\text{Sh}} f \sigma = \sigma^{-1} \sigma^k f \). \( \Box \) Easy lemma.

Hence, as in the classical case, \( S_k U_i(k) = \bigoplus_{\chi \in (\mathbb{Z}/N\mathbb{Z})^*} S_k U_i(k)^x \)

Now let's look at the action of the center, \( \{ (a, 0) : a \in (\mathbb{Z}/N\mathbb{Z})^* \} \)

**Lemma** \( \chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^* \) then \( \overline{\chi} : A^* \to A^*/\mathcal{O}_{A}^{\times} \mathcal{O}_{A}^{\times} \to (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^* \)

is the associated character.

**Lemma** If \( \varphi \in S_k U_i(k)^x \), then \( (a, 0) \varphi = \overline{\chi}(a) \varphi \) for all \( a \in (\mathbb{Z}/N\mathbb{Z})^* \).

Note that this is the advantage of the \((\det)^k\) factor that John has gone for - there's no extra \((a)\)'s floating around.
Let \( Z = \omega \mathbb{H} \), \( \omega \in \mathbb{O} \), \( \omega \neq 0 \) unqg, \( \eta \in \hat{\mathbb{Z}} \).

Then \( \eta \in \mathbb{Z}^{\omega} \subset \hat{\mathbb{Z}} \) (\( \exists ! \lambda \in \mathbb{Z} \) s.t. \( \lambda \equiv \omega \mod N \)).

\[
\begin{pmatrix}
\omega & 0 \\
0 & \zeta
\end{pmatrix} =
\begin{pmatrix}
\mathbb{O} & 0 \\
0 & \mathbb{O}
\end{pmatrix}
\otimes
\begin{pmatrix}
\alpha \\
\zeta
\end{pmatrix}, \quad \forall 
\in U_1(N)
\]

Then \( \varphi(\begin{pmatrix}
\omega & 0 \\
0 & \zeta
\end{pmatrix}) = \varphi(\begin{pmatrix}
\mathbb{O} & 0 \\
0 & \mathbb{O}
\end{pmatrix}) \otimes \begin{pmatrix}
\alpha \\
\zeta
\end{pmatrix} \).

\[
\varphi(\begin{pmatrix}
\omega & 0 \\
0 & \zeta
\end{pmatrix}) = \begin{pmatrix}
\varphi & 0 \\
0 & \varphi
\end{pmatrix}
\begin{pmatrix}
\mathbb{O} & 0 \\
0 & \mathbb{O}
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\zeta
\end{pmatrix}.
\]

Hecke operators on \( \mathbb{S}_k \)

Say \( U_1, U_2 \) are open subgroups of \( \text{Gl}_2(\mathbb{A}^\infty) \).

Then \( [U_1, U_2] : \mathbb{S}_k U_1 \rightarrow \mathbb{S}_k U_2 \)

defined thus: \( U_1 g U_2 = \bigcap_{j \in J} g_j U_2 \)

Then \( [U_1, U_2](\varphi) = \sum_{j \in J} g_j \varphi \).
"Is David Reid here?" 11 people want sandwiches tomorrow. He'll ask again at the end.

I think he said he'll point off 2-things today.

Recall Hecke operators. Recall $S_k = \{ \varphi: GL_2(\mathbb{A}) \to \mathbb{C} \mid \varphi(1,1), 3, 4, 5, 7) \}$

Hecke operators: $U_1, U_2 \in GL_2(\mathbb{A}^\text{ab}) \cdot \mathfrak{g} \cdot GL_2(\mathbb{A}^\text{ab})$

Then $[U_2 g U_2] q = \sum_{j \in \mathbb{Z}} g_{j} q^j$, where $U_1 g U_2 = \sum_{j \in \mathbb{Z}} g_{j} q^j U_2$

$[U_2 g U_1]: S_k \to S_k$

Now say $U_1, U_2 \in \mathbb{P} U_1(N)$, $N \geq 1$

If $p$ is a prime, define $\pi_p \in \mathbb{A}^{\text{ab}}$ by $(\pi_p)_q = 1$ for $q \neq p$

$\pi_p^0 = \mathbb{F}_p = \left[ U_1(N) \mathbb{C}_{\left( \pi_p \right)} \right] U_1(N)$

Of course $(\pi_p^0)_{\pi_p^0} \in \text{centre of } GL_2(\mathbb{A}^\text{ab})$

$U_1(N) \mathbb{C}_{\left( \pi_p \right)} U_1(N) = (\pi_p^0 \pi_p^0) U_1(N)$

Lemma. If $\varphi \in S_k^{U_1(N)}$ then $S_k^p \varphi = \delta_{\pi_p^0} \varphi$ for $(p, N) = 1$.

Proof. Recall $U_1(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 0$ & $\delta_p \mapsto p \text{ mod } N$

So we have $\pi_p = p/(1/p, 1/p,\ldots, 1/p,\ldots)$

Pick $d \in \mathbb{Z}$ s.t. $dp \equiv 1 \text{ mod } N$. Then $S_d^p \varphi = S_d \delta_p \varphi$ (easy check).

Now say $p$ is any prime again.

Define $T_p = [U_1(N) \mathbb{C}_{\left( \pi_p \right)}] U_1(N)$. Recall $S_k^p (U_1(N)) \to S_k^{U_1(N)}$

We can compose the action of $T_p$ with that of $T_p$.

Here's where John's action normalisation looks strange. Of course, it's a non-trivial situation if this looked right then something else would look wrong.
Prop. \( \forall \sigma \in S_k(\Gamma\delta(N)) \), we have \( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) = \varphi^{(n; \varphi)}_\sigma \chi \).

Example
\[
f = \Delta = q \prod_{n=1}^{\infty} (1-q^n)^{\omega_n} = \sum_{n=1}^{\infty} \tau(n)q^n, \quad k+12.
\]
Then \( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) = \frac{-\tau(n)}{n^k} \varphi^{(n; \varphi)}_\sigma \chi \).

Proof of prop. Firstly, for \( \lambda \in \lambda \in \Lambda \), by \( (\omega_j)_E = 0 \), let \( p \) and \( (\omega_j)_P = j \).

Write \( B = U_2(N)U_2(N) \).

Fact

1. If \( (p, N) \neq 1 \), then \( B = \prod_{j=0}^{p-1} \left( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) \right) U_2(N) \).

2. If \( (p, N) \neq 1 \), then \( B = \prod_{j=0}^{p-1} \left( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) \right) U_2(N) \).

Convince yourselves. Away from \( p \) things look easy. At \( p \) the classical decomposition of double cosets for the usual \( T_p \), essentially. John claims that Tony said something about this (?).

Let's do \( (p, N) = 1 \). Write \( \eta_\sigma \), \( \eta_\rho \), \( \eta_\sigma \otimes \eta_\rho \) for \( \left( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) \right) \).

Then \( \left( \varphi^{(n; \varphi)}_\sigma \chi(\varphi) \right) = \sum_{j=0}^{p-1} \eta_\sigma(\varphi) \chi(\varphi) = \sum_{j=0}^{p-1} \varphi^{(n; \varphi)}_\sigma(\varphi) \chi(\varphi) \).

Now \( \psi \in GL_2(\mathbb{M}) = GL_2(\mathbb{G})U_2(N)GL_2(\mathbb{G}) \).

Say \( \psi = \gamma \psi_1. \) Then \( \eta_\sigma = \gamma \eta_1 \).

We want to understand \( \psi_1 \), but \( \eta_1 \in B \).

\[
\psi_1 = \eta_\sigma \psi_1, \quad \psi_1 = \varphi(\psi) \psi_1 \in U_2(N) \quad \text{or a permutation of } [0, p-1]
\]

\[
\delta \eta_1 = \eta_\sigma \psi_1, \quad \psi_1 = \varphi(\psi) \psi_1 \in U_2(N) \quad \text{understand } \eta_1.
\]

Case. \( \eta_1 = \left( \varphi^{(n; \varphi)}_\sigma \right), \quad h \neq 0, 1, \ldots, p-1. \)

Then \( \psi = \varphi^{(n; \varphi)}_\sigma \chi \).

\[
\eta_1 = \varphi^{(n; \varphi)}_\sigma \chi \psi_1 = \gamma \eta_1 \psi_1 = \psi^{(n; \varphi)}_\sigma \chi \psi_1.
\]
\[ S\eta = \delta \eta'w = \gamma(\delta^k)(zw, (w^* - h^2)w) \]

where \((z\eta)^* = (w^* - h^2), q \neq p \) & \((z\eta)^* = 1\). Note the fact that \(z \in U_2(n)\)

\[ \varphi_d(S\eta) = \bar{f} \left( \omega \right) \left( j(w, w^*) \right)^k (det w)^{k/4} \]

Con 2. \(\eta' = (0, 0)^T\), \((p, n) \neq 1\)

Then \(S\eta = \delta \eta'w = \gamma(0, 0)^T \left( zw, \sigma_p(0, 0)^T w \right) \)

where \((z\eta)^* = \sigma_p(0, 0)^T \) of \(q \neq p\) & \((z\eta)^* = 0_p\)

Note \(z \in U_2(n)\) again.

\[ \varphi_d(S\eta) = \bar{f} \left( w, w^* \right) \left( j(w, w^*) \right)^k (det w)^{k/4} \]

\[ = \varphi_{d|k}(w, 0)^T \] (3)

Hence \(\mathcal{T}_p(\varphi_d)(S) = \sum_{k=0}^{p-1} \varphi_d(S\eta) = \sum_{k=0}^{p-1} \varphi_{d|k}(w, 0)^T + \varphi_{d|k}(w, 0)^T\)

Next note \((\delta^T)(\delta^T) = (\delta^T)^2\)

so this is \(\sum_{k=0}^{p-1} \varphi_{d|k}(w, 0)^T + \varphi_{d|k}(w, 0)^T\)

\[ = \sum_{k=0}^{p-1} \varphi_{d|k}Sf \] (plN slightly easier)

Now well understand \(S_k\) - a lot more as a rep of \(GL(n, \mathbb{R})\) (it'll turn out to be a direct sum of irreducible Adams \(\varphi_p\))

There is an inner product on \(S_k\). It doesn't make \(S_k\) into a Hilbert space or anything.

Say \(\varphi_p, \varphi_q \in S_k\). They behave well under right translation by \(U = n^*SO(n)\).

Hence \(\varphi_p \varphi_q\) is invariant on the right by \(U_n\). (Easy check).
\[
\text{Def: } (\varphi_1, \varphi_2) = \int_{\mathbb{R}^n \setminus \det(\mathbb{R}^n)} \overline{\varphi_1(s)} \varphi_2(s) \, ds
\]

There is a fair chance that this could converge - cf. Richards Quaternion Algebra.

It does indeed converge.

Now say \( f_1, f_2 \in S_k \Gamma_1(n) \). Define \( \varphi_i = \varphi f_i \).

Then if \( I = \mathcal{F}w \), usual notation:

\[
\mathcal{F}F_k \Gamma_1(n) = \varphi_1(I) \varphi_2 \Gamma_1(n) \text{ etc. Set } z = w_i.
\]

\[
\begin{align*}
\varphi_k(z) \overline{\varphi_k(z)} = \sum_{j} f_j(z) \overline{f_j(z)} \left( \frac{\det w}{|\det w|} \right)^k \\
= f_k(z) \overline{f_k(z)} (\text{Im } z)^k
\end{align*}
\]

It's not too difficult to check that

\[
(\varphi_k, \varphi_\ell) = c_k \times \int \frac{\prod_{i \neq j} f_i(z) \overline{f_j(z)} \, dy^1 dz_1 \ldots dy^n dz_n}{f_k(z) \overline{f_k(z)}}, \quad z = x + iy
\]

Properties of adelic \((,)\) (NB \( \varphi_k, \varphi_\ell \) on \( \text{reg general elt of } S_k \) now)

1) \((,)\) is \( \text{GL}(\mathbb{A}^n) \)-mult, i.e. \( g(\varphi_k, \varphi_\ell) = (\varphi_k, \varphi_\ell) \) \( \forall g \in \text{GL}(\mathbb{A}^n) \)

2) \((,)\) restricted to \( \mathcal{F}U \times \mathcal{F}U \), for \( U \) any compact open subgroup, is non-degenerate.

He's running out of time. He wanted to give a little algebraic argument, which would have yielded

Then \( S_k = \mathcal{F}W \), \( W \) admissible irreducible \( \text{GL}(\mathbb{A}^n) \)-subspaces, which are orthogonal under \((,)\).

He'll talk more about this next time.
Recall we're talking about $S_k = \{ (p, GL_d(A)) : d \to s \) hold $, $k \geq 1$

$GL_d(A)$ acts via right translation.

$$(p, (a, b)) \mapsto \int_{GL_d(A)} \rho \left( \begin{array}{c} a \end{array} \right) \left( \begin{array}{c} b \end{array} \right) ds.$$

Not $GL_d(A)$-inv. It doesn't make $S_k$ complete.

However, $(, )$ restricts to a non-degenerate inner product on $S_k^U \times S_k^U$.

We will use $(, )$ to prove a theorem, coming up, which will convince us that $S_k$ is the $GL_d(A)$-module to be looking at.

Then $S_k \cong W$, where $W$ is an irreducible $GL_d(A)$-module.

Notation If $W \in S_k$, set $W^=(V) = \{ v \in V : (v, w) = 0 \forall w \in W \}$.

Lemma If $W \in S_k$ and $GL_d(A)$-invariant subspace of $S_k$, we have $V = W \oplus W^=(V)$.

Proof of lemma. Take $v \in V$. We want $v = w + \rho \in W$, $(\rho, W) = 0$.

Now $v \in V^U$ for some open subset $U$ of $GL_d(A)$, & $V^U$ is f.d.

Then $(, )$ is non-degenerate on $V^U : V^U \cong W^U \oplus W^1(V^U)$

Hence $\rho = v + \alpha$, $\alpha \in W^U \in W$, $\rho \in W^1(V^U)$ is $W^U \cong W^U \alpha = 0, (\rho, W^1) = 0$.

He claims $(\rho, W) = 0$. Take $w \in W$; we must show $(\rho, W) = 0$.

Now $w \in W^U \aleph W^U$ & wlog $U \subseteq U$, $U$ normal in $U$, $U$ open

wlog $U / U$ is a finite group. Hence $V^U = V^U \oplus W$. While $w \aleph W^U \oplus W$.

Now $w \aleph W^U : (\rho, w_2) = 0, (\rho, w_0) = (\rho, w) = 0 \forall w \in W^U \Box$ of lemma.

Now hopefully a Zorns lemma-type argument will finish it off.
Proof

Now pick a max family \( \{ V_i \} \) s.t. (i) \( V_i \) is an irreducible \( \text{GL}_n(A^\mathbb{R}) \)-submodule of \( S_k \) & (ii) \( \Sigma V_i = \Theta V \) (Zorn)

Define \( V = \Theta V_i \), a subspace of \( S_k \)

By our lemma, \( S_k = V \oplus V^\perp \). Set \( X = V^\perp \). Must show \( X = 0 \)

Suppose for a contradiction that \( X \neq 0 \)

Then there's a compact open \( U \) s.t. \( X^U \neq 0 \) & \( U^U \) is minimal w.r.t. not being \( 0 \)

With this \( U \) fixed, pick \( A \in X \) s.t. \( A^U = 0 \) \( \subset \text{ker} \left( A \right) \) (2) \& \( U \)

Consider all \( \text{GL}_n(A^\mathbb{R}) \)-invariant subspaces \( B \) of \( X \) s.t. \( B^U = A^U \)

Pick minimal such \( B \) (Zorn) Claim: \( B \) is irreducible.

For if \( B < B \) is a \( \text{GL}_n(A^\mathbb{R}) \)-invariant subspace, then \( A^U = B^U = B_+^U \Theta (B^{+}_x(B))^{U} \)

\[ \text{Minimal of } A^U \Rightarrow B_+^U = A^U \text{ or } (B^{+}_x(B))^U = A^U \]

Minimal of \( B \) vs \( B_+^U \)

If \( \exists B_+^U \neq A^U \) then by minimality of \( B \) we see \( B_+ = B \)

If \( (B^{+}_x(B))^U = A^U \Rightarrow B^{+}_x(B) = B \) by minimality of \( B \)

\[ \Rightarrow B_+ \in B^{+}_x(B) \Rightarrow B_+ = 0. \]

Factorization

Say \( \phi : \text{GL}_n(A) \rightarrow \mathbb{C}^n \). Then \( \phi = \text{Tr} \phi \), \( \phi : \text{GL}(G) \rightarrow \mathbb{C}^n \)

We want to do the same for \( \text{GL}_n(A^\mathbb{R}) \)-modules - e.g. \( W_i \oplus \otimes W_i \).

We will study \( \text{GL}_n(A^\mathbb{R}) \) & \( \text{GL}_n(E^\mathbb{R}) \) (cf Tony)

Both of these are locally profinite groups.

Say \( G \) is a locally profinite gp. We can define \( \mathbb{H}(G) \) to be the locally \( \text{cmpt} \) supported \( * \)-algebra on \( G \). \( \mathbb{H}(G) \) becomes an algebra under \( * \) and we've fixed a Haar measure.

\[ \mathbb{H}(G) = \bigoplus_k \mathbb{H}(G, K), \text{ & } \mathbb{H}(G, K) \text{ has unit } ex = (\text{charf of } K) \sqrt{\text{vol}(K)} \]
Fact 1. If $V$ is a smooth $G$-module, we can endow $V$ with a structure of $\mathcal{H}(G)$-module, i.e., $V$: $\mathcal{H}(G) V$ (check?):

$$\eta: G \to \text{Aut}(V) \cong \eta: \mathcal{H}(G) \to \text{End}(V)$$

(in fact, isomorphism between smooth homomorphisms $\mathcal{H}(G)$ and non-degenerate $G(K)$-modules $\mathcal{H}(G,K)$ or something)

$$\eta(f) V = \int g \in G \eta(g) v dg$$

Fact 2. Irreducibility criterion: A smooth $G$-module $V$ is irreducible $\iff V^K$ is an irreducible $\mathcal{H}(G,K)$-module for all $K$.

They mentioned this.

We will also need:

If $G = G_1 \times G_2$, $G_1$ & $G_2$ locally profinite,

$W_1$ is an irreducible $G_1$-module, $W_2$ an irreducible $G_2$-module,

$W = W_1 \otimes W_2$ is an irreducible $G$-module.

Theorem. Let $W_1$ be an irreducible $G_1 \times G_2$-module, then $\exists$ a unique $G$-module $W_2$ st. $W_1 \cong W_1 \otimes W_2$.

Moreover $W_1$ & $W_2$ are ! up to iso$m$.

(they said something about isotypy etc.)

Classical result (Bourbaki, Algebra, Chap IV, p.94)

If $\mathbb{F}$ is any algebraically closed field, $A,B$ algebras, $A/B$, $M$ a simple $A \otimes_B M$-module, of f.d. $/k$.

Then $M = M_1 \otimes_B M_2$, $M_1$ a simple $A$-module. $M_2$ a simple $B$-module.

Proof (of claimed result)

We have $M$ as an $A$-module or an $B$-module. So if we pick an irreducible $A$-submodule $P$ of $M$ (f.d.), we get

$\text{Hom}_A(P,M)$, endowed with the structure of a $B$-module.

Pick $R \subseteq \text{Hom}_A(P,M)$ a simple $B$-module.

Then $P \otimes R \hookrightarrow P \otimes \text{Hom}_A(P,M) \to M$ an $A \otimes_B$-HM. Thus $R \subseteq M$. 

Proof then:

Now say $G = G_1 \times G_2$

$K = K_1 \times K_2$, $K_1 \subset G_1$, $K_2 \subset G_2$. Then $\mathcal{H}(G,K) \cong \mathcal{H}(G_1,K_1) \otimes \mathcal{H}(G_2,K_2)$.

$W$ a $G$-module $\Rightarrow W$ gets an $\mathcal{H}(G)$-module structure.

Pick $K$ s.t. $W^K = 0$, $W^K$ a $\mathcal{H}(G_1,K_1) \otimes \mathcal{H}(G_2,K_2)$.
Then there exists (by the lemma) an $\mathcal{H}(G_2, K_2)$-module $W_2(K_2)$ and an $\mathcal{H}(G_1, K_1)$-module $W_1(K_1)$, irreducible.

Let $\varphi: W^K \to W_1(K_1) \otimes W_2(K_2)$

If $K' = K_1 \times K_2$, then we get $W_1(K_1)$.

Set $W_1 = \lim K \to W_1(K_1)$, $W_2 = \lim K \to W_1(K_2)$.

Tensor products commute with direct limits.

$$W = W_1 \otimes W_2 = \lim \left( W_1(K_1) \otimes W_2(K_2) \right)$$

$W_i$ are also irreducible.

Finally, we want to understand the case $G = \text{GL}_2(F^2) = \text{TT} \text{GL}_2(Q)$.

Note that if we have a $\text{TT} \text{GL}_2(Q)$-module $W$, irreducible, then $W = \otimes_{\text{res}} W_i$.

with $W_i$ an irreducible $\text{GL}_2(Q_i)$-module.

Tensor products of infinite families of $\times C$

Say we are given \( \{ W_\omega \}_{\omega \in \Lambda} \) and

(i) a finite subset $\Lambda_0 \subseteq \Lambda$,

(ii) for each $\omega \in \Lambda \setminus \Lambda_0$, an $x_\omega \in W_\omega$, $x_\omega \neq 0$.

Say $S$ is a finite subset of $\Lambda$ containing $\Lambda_0$.

Set $W_S = \bigotimes_{\omega \in S} W_\omega$.

If $S \subseteq S'$, define $f_{SS'}: W_S \to W_{S'}$ by $f_{SS'}(\otimes_{\omega \in S} W_\omega) = (\otimes_{\omega \in S} W_\omega) \otimes_{\omega \in S'} x_\omega$.

Define $\bigotimes_{\omega \in S} W_\omega = \lim \frac{W_S}{S}$

We can change $x_\omega$ to $a_\omega x_\omega$, $a_\omega \in C^*$.

$\bigotimes_{\omega \in S} W_\omega$ only depends on the $C$-vector spaces generated by the $x_\omega$.

It makes sense to talk about $\otimes_{\omega \in S} W_\omega$ so long as $x_\omega = x_\omega$ for all but a finite number of $\omega$. 
The same idea: work for...

**Algebra.** Given \( \{ A_\lambda \}_{\lambda \in \Lambda} \) with an idempotent \( e_\lambda \in A_\lambda \) for all but a finite number of \( \lambda \).

We can give \( \bigotimes_{\lambda \in \Lambda} A_\lambda \) the structure of an algebra by defining
\[
(\otimes a_\lambda)(\otimes b_\lambda) = \otimes a_\lambda b_\lambda
\]
for all but finitely many \( \lambda \).

Take now \( \Lambda \) the set of finite places of \( \mathbb{Q} \).

For \( \nu \in \Lambda \) set \( K_\nu = \text{GL}_1(\mathbb{Z}_\nu) \) & define \( e_\nu = e_{K_\nu} \in \mathcal{H}(\text{GL}_1(\mathbb{Q}_\nu)) \)
\[
e_{K_\nu} = (\text{charf of } K_\nu)/\text{vol}(K_\nu).
\]

We get \( \bigotimes_{\nu \in \Lambda} \mathcal{H}(\text{GL}_1(\mathbb{Q}_\nu)) \)

**Remark.** \( \bigotimes_{\nu \in \Lambda} \mathcal{H}(\text{GL}_1(\mathbb{Q}_\nu)) \cong \mathcal{H}(\text{GL}_1(\mathbb{A}^\infty)) \) (he said canonical isomorphism)

This seems to be because \( \bigotimes_{\nu \in \Lambda} \mathcal{H}(\text{GL}_1(\mathbb{Q}_\nu)) \cong \mathcal{H}(\bigotimes_{\nu \in \Lambda} \text{GL}_1(\mathbb{Q}_\nu)) \)

Say \( J_\nu \subset \bigotimes_{\nu \in \Lambda} \text{GL}_1(\mathbb{Q}_\nu) \) is open in \( \text{GL}_1(\mathbb{A}^\infty) \)

Then \( J_\nu = K_\nu \) for all but a finite no. of \( \nu \).

\[
\mathcal{H}(\text{GL}_1(\mathbb{A}^\infty), J) \cong \bigotimes_{\nu \in \Lambda} \mathcal{H}(\text{GL}_1(\mathbb{Q}_\nu), J_\nu)
\]

He might have fixed some compatible system of measures \( \nu \times J \) works...

Next note that if we are given \( \mathcal{V} \) an admissible \( \text{GL}_1(\mathbb{Q}) \)-module \( W \) s.t.
\[
\dim_\mathbb{Q} W_\nu = 1 \quad \text{for all but a finite no. of } \nu.
\]

Pick \( x \in W_\nu \setminus 0 \) for these \( \nu \) s.t. \( \dim_\mathbb{Q} W_\nu = 1 \)

Then \( W = \bigotimes_{\nu \in \Lambda} W_\nu \) is an irreducible \( \text{GL}_1(\mathbb{A}^\infty) \)-module.

However, the converse is also true...
Theorem. Let \( W \) be an irreducible \( GL_n(\mathbb{A}^\infty) \)-module. For each finite \( v \), there exists an irreducible \( GL_n(\mathbb{A}^v) \)-module \( W_v \) st.

(i) \( \dim_{\mathbb{Q}} W_v = 1 \) for all but a finite no. of \( v \)

(ii) \( W_v \otimes W \) relative to \( x_v \), where for all but finitely many \( v \), \( 0 \neq x_v \in \mathbb{A}^v \).

Moreover, the factors \( W_v \) are uniquely determined by \( W \).

This is rather a miraculous result, but we essentially proved it already. Something about lifting up to dense uniqueness. We'll see that it's easier to pass to the Hecke algebras to prove this later.

\( W \) is a module over \( \mathbb{H}(GL_n(\mathbb{A}^\infty)) \). Choose a split open \( J = \bigoplus J_v \) of \( GL_n(\mathbb{A}^\infty) \) st. \( \mathbb{A}^\infty \setminus \{0\} \). Of course, \( J_v \) for all but finitely many \( v \).

\( W^J \) is \( J \)-d./\( C \) and an irreducible \( \mathbb{H}(GL_n(\mathbb{A}^\infty), J) \)-module (modulo \( \mathbb{H}(GL_n(\mathbb{A}^v), J_v) \)).

Hence, \( \otimes_{v} \mathbb{H}(GL_n(\mathbb{A}^v), J_v) \) acts on \( W^J \).

If \( S \) is sufficiently large, i.e., \( S \) contains \( v \) st. \( J_v \) is \( \mathfrak{m} \)

then \( W^J \) will be irreducible. \( \otimes_{v} \mathbb{H}(GL_n(\mathbb{A}^v), J_v) \) as for \( v \neq S \) everything acts as scalars.

Hence by our result for finitely many things, \( W^J \cong \otimes_{v} W_v(J_v) \), \( W_v(J_v) \) an irreducible \( \mathbb{H}(GL_n(\mathbb{A}^v), J_v) \)-module.

Now pass to the inductive limit.

Also note that \( \dim_{\mathbb{Q}} W_v(J_v) = 1 \) when \( J_v \neq \mathfrak{m} \).

Tony saved his bacon on this next one.

Tony Scholl defined Hecke algebras at infinity:

\[ H_v = \mathbb{H}(G_v, K_v), \quad K_v = \mathbb{G}_a(L) \]

Tony explained all this. It's a nasty tensor product of lots of measures with a universal enveloping algebra. We get \( \{\mathfrak{m}\} \) \( \mathbb{H}_0 \) & \( \mathbb{H} \leq \mathbb{H}(\mathfrak{m}) \).

John's conscience is clear (he won't give the details). He hopes Tony's is also.
\[ \text{Def.} \quad \mathcal{H} = \mathcal{E}_v \quad \mathcal{H}_v, \quad \mathcal{H}_w \text{ if } v > 0 \]
\[ \mathcal{H}(\text{GL}_v(\mathbb{A})) \text{ if } v < 0 \]

The modules that Richard has been talking about are tailor-made for this setting. If we have an admissible irreducible \( \text{GL}_v(\mathbb{A}) \times (\mathcal{E}, \mathcal{K}_v) \)-module \( (\mathcal{E}_v, \mathcal{K}_v)\), then we will be able to factorize it. On the other hand, a module \( \text{GL}_v(\mathbb{A}) \times (\mathcal{E}_v, \mathcal{K}_v) \)-module is admissible if

(i) \( \mathcal{V} \) is a smooth \( \text{GL}_v(\mathbb{A}) \)-module
(ii) \( \mathcal{V} \) is a \( (\mathcal{E}_v, \mathcal{K}_v) \)-module
(iii) The actions above commute

If \( \rho \) is any irreducible rep of \( K = \mathbb{F}_v \), then \( V(\rho) \) is \( \text{d}/\text{c} \).

Via the \( \epsilon \) Hecke algebras, we get a modification of the last thm:

If \( \mathcal{W} \) is an irreducible \( \text{GL}_v(\mathbb{A}) \times (\mathcal{E}_v, \mathcal{K}_v) \)-module, then for each finite \( \mathcal{W}_v \) there exists \( W_v \) st. \( W_v \otimes W_v \) is just a \( \text{d}/\text{c} \). This just rubbishes it all off but I'm sure it's clear what John is saying.

The point of all this is that the interesting \( \text{GL}_v(\mathbb{A}) \)-modules are the ones that appear in the decomposition \( S = \bigoplus \mathcal{W}_v \), \( W_v \otimes W_v \).

\textbf{The space} \( A^0 \)

The problem with \( S \) is that it was sort-of invented to model modular forms. There are funny non-holomorphic things worked out by Messi in the '30s that aren't accounted for. Our action at \( 1 \) is too simple. Also we're not a \( (\mathcal{E}_v, \mathcal{K}_v) \)-module yet.

There's problems with \( A^0 \). Normalizations vary. Just as in \( S \) case. He hopes what he's written down is correct. He's sure one of the experts will correct him otherwise.

\( \text{Define } A^0 = \{ \rho: \text{GL}(\mathbb{A}) \to \mathbb{C} | \text{norm... well... we'll come to this } \} \)

We've been on having \( S \in A^0 \). This won't be immediately obvious.

Write \( \text{GL}(\mathbb{A}) = \text{GL}_v(\mathbb{A}) \times \text{GL}_2(\mathbb{R}) \)

\[ \mathcal{S} = \left( S^v, S_w \right) \]

Here are our conditions:
For fixed $S$, $p(S)$ is locally c.d. in $S$, & for a fixed $S'$ we have $p(S') \in S'$ in $S$. (He calls this 'if small', but the slide next to me thinks he only meant that 'it is smooth in both directions'.)

$\varphi$ is left-invariant by $GL_2(\mathbb{A})$.

$\varphi$ is inst. on the right by an open subgp $U(p)$ in $GL_2(\mathbb{A})$.

For fixed $S$, the function $S \mapsto p(S)$ is bounded on $GL_2(\mathbb{R})$.

NB. He just put 'slightly increasing': it would give us $A$, not $A'$. A has non-cuspidal things in.

In fact cont. 1-6 still don't quite force cuspidality - e.g. act for $\varphi$ on still in.

So impose.

$\varphi$ is cuspidal, i.e. $\int_{GL_2(\mathbb{A})} \varphi((t_s)S) \, d\mu = 0 \quad \forall \, \varphi \in GL_2(\mathbb{A})$.

(behaviour under right translation by $K_0 \subset K_0 \times \mathbb{R} \rightarrow GL_2(\mathbb{R}$)

And $\varphi$ is $K_0\bar{\mathbb{R}}$-finite. $K_0\bar{\mathbb{R}}$: finite

He must continue on his quest to define $A_0$, despite Brian's cry of 'come back, John!'

If $g_{m, c}$ Lie algebra of $GL_2(\mathbb{A})$ ie $g_{m,c} = M_2(\mathbb{R})$, then

$$(x, \varphi)(S) = \frac{d}{dt} \varphi(\exp(tX))\big|_{t=0}$$

& extend by C-linearly to a rep of $g_{m,c}$

$U_{m,c} = \text{Universal enveloping algebra of } g_{m,c}$

Then $g_{m,c} \subset U_{m,c}$

$X \mapsto X'$

and $\varphi$-centre of $U_{m,c}$. $\varphi$ is $\bar{\mathbb{R}}$-finite.

He now tells us why this is a reasonable def. Actually, he firstly wants to tell us why it's a def of anything at all, i.e. he wants to check the def makes sense. There's just one point: $(X, \varphi)$ is bounded for $X_{eq}$. 
Lemma. Assume \( \eta : \text{GL}_n(\mathbb{R}) \to C^0 \) satisfies:
1. \( K_0 \)-finite
   \[ \int_{K_0} \eta_w \, dw \] is finite.
2. \( \eta \) has compact support.
3. \( \eta \) has no \( \text{K}_0 \)-invariant on the right.
4. \( \eta * \alpha = \eta \).

If Richard talked about it (in a more general setting) \( \square \)

Now define \( \eta' : (S^0, S_0) \to C^\infty \).

Then \( (X, \eta)(S_0) = \frac{d}{dt} \left( \eta \left( S_0 \exp(tx) \right) \right)_{t=0} \)

\[ = \left( \int_{K_0} \eta(w) \frac{d}{dt} (w^{-1} S_0 \exp(tx))dw \right)_{t=0} \]

This (evidently) justifies the fact that \((X, \eta)\) satisfies 4). Evidently.

So we have some draft space \( N. \) We also had \( S_k \), John asserts that \( S_k \subseteq N. \) The thing is, there was no Lie algebra action on \( S_k \). We'll have to show that \( S_0 \) of \( S_k \) are \( K_0 \)-finite.

Define \( A^0_k = \{ \eta : A^0 \to A^0 \mid \eta(S_0) \subset S_0 \} \) \( V \in U = \text{SU}_3 (\mathbb{R}) \).

Recall for \( S \in \text{GL}_n(\mathbb{R}) \) we're writing \( S^0(S_0, S_0) \), \( S_0 \in \text{GL}_n(\mathbb{R}) \).

Define \( H = \mathbb{C} \smallsetminus \mathbb{R}^2 \); \( z = x + iy \in H \)

\[ S_0 = \begin{pmatrix} x + iy & 1 \cdot i y & 1 \cdot i y \\ 0 & 1 \cdot y & 1 \cdot x \\ 1 \cdot y & 1 \cdot x & 1 \end{pmatrix} \]

where \( r \in \mathbb{R}^+ \) and \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \).

Write \( \tilde{\eta} (S^0, z) = \phi(z^0, j(S_0, \alpha)(\text{det} \, S_0)^{-1/2}) \).

\[ (z, \omega) \quad \text{Here} \quad j(S_0, \alpha) = (1 \cdot y, \text{re}^{i\theta}, \text{det} \, (S_0) = r(sgn \, y)^{1/2}) \]

Formula: \( \tilde{\eta}(z) = \phi(z^0, z)^{1/2} \) \( e^{i\theta} \), \( \phi \in A^0_k \).

\( j(w) \) has a basis \( (e^0), (e^i), \frac{1}{2}(e^i), \frac{1}{2}(e^i) \).

Well put dashes on things if they're in \( \mathbb{R} \), e.g., \( j(x) \).

\( j(x) \)
\[ (\exp tJ) = (e^t \ 0) \quad , \quad \frac{d}{dt} \left( \psi(s) \exp(tJ) \right) = 0 \quad \Rightarrow (J, \psi) = 0 \]

Action of \( H = -iA \), \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

\[ [H, \psi(x)] = (-i) (A, \psi(x)) = (-i) \left. \frac{d}{dt} \psi \left( \exp(tA) \right) \right|_{t=0} \]

Now \( A^\dagger = (-i, 0) \) \( \exp(tA) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \)

\[ (H, \psi)(s) = -i \left. \frac{d}{dt} \left( \psi \left( s \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right) \right) \right|_{t=0} = -i \left. \frac{d}{dt} \left( s \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right) \psi \right|_{t=0} \]

\[ \psi(s) = \begin{pmatrix} k \psi(s) \\ \bar{\psi}(s, z) \end{pmatrix} \]

Action of \( X_r, X_l \)

\[ U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_r = \frac{i}{2} (U + V), \quad X_l = \frac{i}{2} (U - V) \]

\[ (U, \psi)(s) = \frac{d}{dt} \left( \psi(s \exp(tU)) \right) |_{t=0} \]

\[ U^2, I, \quad \exp(U) = \sum_{k=0}^{\infty} \frac{U^k}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{U^{2k}}{(2k)!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \]

\[ \sum_0 (U, \psi)(s) = \frac{d}{dt} \left. \psi(s \exp(tU)) \right|_{t=0} \]

\[ \& \quad \psi(s \exp(tU)) = \bar{\psi}(s, z) \exp(tU)(i) \langle j(s \exp(tU), i) \rangle^k \]

\[ x \chi(\psi) \]

\[ \frac{d}{dt} \left. j(s \exp(tU), i) \right|_{t=0} = k j(s, i) e^{2z} \]

\[ \exp(tU)(i) = e^{\theta} \frac{(1-e^{2\theta}) \cos \theta + i e^{2\theta} \sin \theta}{1 - (1-e^{2\theta}) \sin \theta} \quad \text{(this is all)} \]

\[ (U, \psi)(s) = k (\cos \theta + i \sin \theta) \psi(s) + \int \left( 2 y \cos \theta \frac{\partial}{\partial y} - 2 y \sin \theta \frac{\partial}{\partial x} \right) \bar{\psi}(s, z) \]

\[ j(s, i) k \psi(s) \]

\[ (s, i)^{k^2} \]
A totally similar calculation gives us $V\phi \cdot V^2=I$ so can do $\exp(\theta V)$ explicitly, & get

$$(V\phi)(\mathbf{z}) = k(\sin 2\theta - i\cos 2\theta) \phi(\mathbf{z}) + j(\mathbf{z}, i)^k (\det \mathbf{z}_w, \mathbf{z}_w + i\mathbf{z}_w \frac{\partial}{\partial \mathbf{z}} y)$$

$$\Rightarrow (\mathbf{z}_w, \mathbf{z})$$

$$X_+ \phi = \frac{1}{2}(U\phi + iV\phi)$$

$$= j(\mathbf{z}_w, i)^k (\det \mathbf{z}_w, \mathbf{z}_w + i\mathbf{z}_w \frac{\partial}{\partial \mathbf{z}} y) e^{z_0} (y(\partial^2 y + i\partial^2 x) + k) \Rightarrow (\mathbf{z}_w, \mathbf{z})$$

& $(X_+ \phi)(\mathbf{z}) = j(\mathbf{z}_w, i)^k (\det \mathbf{z}_w, \mathbf{z}_w + i\mathbf{z}_w \frac{\partial}{\partial \mathbf{z}} y) e^{z_0} (y(\partial^2 y + i\partial^2 x)) \Rightarrow (\mathbf{z}_w, \mathbf{z})$

Now the Cauchy–Riemann eqns give

Consequence: $\Rightarrow (\mathbf{z}_w, \mathbf{z})$ is holos of $\mathbf{z}_w \Rightarrow (X_+ \phi) = 0$

Recall that there was some holomorphicity cond. on $\mathbf{z}_w$ & so this is good-looking stuff.

Then $\mathbf{z}_w = \mathbf{A}_w \Rightarrow \mathbf{A}_w$. In fact, $\mathbf{z}_w = \{ \phi \in \mathbf{A}_w \mid (X_+ \phi) = 0 \}$

RHS = LHS by def. of $\mathbf{z}_w$. The problem for $\Rightarrow \mathbf{z}_w$-finiteness.

The question: what is $\mathbf{z}_w$? Well $J$ is in the centre of $\mathbf{z}_w$ so clearly $\Rightarrow \mathbf{z}_w$. Also the $J$-Carneval operator $J = H^2 - 2H^1 + 1 + 4x^2 x^1$.

Note $[H^1, x^1] = 2x^1$, $[H^1, x^2] = 2x^2$, $[x^1, x^2] = H^1$

& we see that $[J^2, \text{other}\text{ stuff}] = 0$ so indeed $J^2 \in \mathbf{z}_w$

In fact $\mathbf{z}_w = \{ J^2 \} &$ he'll talk about this in his next lecture & then finish the $\Rightarrow$ of the thm.

Lecture 8
Tues 24th Feb 1943
2:30pm

Recall the survivor's party, 8:30pm, 104 Mason Road. Near railway station.

To finish off his just gonna quote these etc.

His trying to show $\mathbf{z}_w = \{ \phi \in \mathbf{A}_w \mid (X_+ \phi) = 0 \} \Rightarrow \mathbf{z}_w$ done $\Rightarrow$.

We need to check out $\mathbf{z}_w$ action on $\mathbf{z}_w$.

John's defined $J^2 = H^2 - 2H^1 + 1 + 4x^2 x^1$. Tony's Carneval operator, as

it happens, was $J^2 = \frac{\partial^2 - 1}{\partial x^1}$.

$J^2 \in \mathbf{z}_w$. 

\[ J^2 \in \mathbf{z}_w \]
Note $H^\psi \cdot k \psi & x_\psi \cdot \psi = 0$, so $H^\psi \cdot (k \cdot \psi) & J_\psi \cdot \psi = 0$

So clearly, $S_\psi \subset C[R_\psi, J_\psi] \Rightarrow S_k \subset A^o_k$ & we're home.

So it remains to prove that $S_\psi \subset C[R_\psi, J_\psi]$.

We will use a theorem of Harish-Chandra which is probably true in much greater generality.

Write $T = \mathfrak{g}_G \subset \mathfrak{e}$: vector space of diagonal matrices.

$C_2$ acts on $T$ by interchanging the diagonal elt.

Set $T^\times = \text{Hom}(T, C)$.

Theorem (H-C) $S_\psi \rightarrow (\text{Polynomial } f's \text{ on } T^\times)^{C_2}$.

\[
C[x_1, x_2]^\times \cong C[x_1 + x_2, x_1 x_2]
\]

\[
\downarrow \\
J' \equiv \#(J'^2 \cdot \mathfrak{r})
\]

If we believe all that then we're clearly done.

\[
S_k \subset A^o_k \subset A^o
\]

He wants to give us some facts about $A$ before talking a bit about the more arithmetic $S_k$.

$A^o$ is a $\text{Gl}_n(A^o) \times (\mathfrak{g}_G, K^o)$-module.

$H = \otimes H_v$

Def: An automorphic rep of $H$ is an irreducible subquotient of the rep of $H_v$ on $A$.

Hennirot wrote something recently & John is nailing off this. This is where his facts are from.

Fact: An automorphic rep of $H$ is admissible.

He seems to be writing $A$ for $A^o$ now.
\[ \mathcal{A}(X) = \left\{ \phi \in \mathbb{A} \mid (\exists z) \phi = X(z) \phi \right\} \]

Then (a) Every auto. rep. of \( \mathfrak{H} \) occurs in \( \mathcal{A}(X) \) for some \( X \).

(b) For fixed \( X \), \( \mathcal{A}(X) \) is a direct sum of irreducible \( \mathfrak{H} \)-modules, each occurring with multiplicity \( 1 \). Each \( W \) which occurs has infinite dimension.

\[ W = \bigoplus W_i \]

**Strong multiplicity 1 then.**

Let \( \pi_1, \pi_2 \) be irreducible reps of \( \mathfrak{H} \).

Say \( \pi_1 = \bigotimes \pi_{i_1}, \pi_2 = \bigotimes \pi_{i_2} \).

Then \( \pi_1 \cong \pi_2 \iff \pi_{i_1} \cong \pi_{i_2} \) for all but a finite no. of \( i \).

He doesn't really want to talk about non-holomorphic forms & stuff. He does want to look more at \( \mathfrak{S}_k \) & get some arithmetic facts out.

5. \( \mathfrak{S}_k \subseteq A^0 \)

\[ \bigoplus W_i, \text{ W. irreducible } \text{GL}_2(\mathbb{A})\text{-modules} \]

Take \( W \) to be one of these \( W_i \).

\[ W = \bigotimes W_i, \text{ W. an irreducible } \text{GL}_2(\mathbb{A})\text{-module, } \nu \text{ to } \nu = (\mathfrak{g}_0, K_0)\text{-module, } \nu \text{ on } \]

\[ W_0 ? \text{ Well, } W \in \mathfrak{S}_k \text{. We have that an action of } \text{SO}_0(\mathbb{R}) \& \mathbb{R}_+ \text{ which we understand. } \]

\[ \text{We act on } \text{ by } \chi \text{. } \phi = 0 \forall \phi \in \mathfrak{S}_k \text{. } W_0 = \mathfrak{D}_k^+, \text{ a } (\mathfrak{g}_0, K_0)\text{-module.} \]

Thinking about a bit \( \Rightarrow \)

**Fact.** \( W_0 = \sigma_{(\mu, \mu)} \), \( \mu = 1/2 \), \( sgn \).
Say \( W \) is an irreducible submodule of \( S_k \).

Then \( W = \bigoplus W_v \).

\( \therefore \) \( \exists \) \( f \), a primitive form of wt \( k \) for \( \Gamma_1(N) \), a suitable \( N \).

We appeal to Tony's theorem to construct \( f \).

We know \( \dim_{Q_p} W_v^{GL_1(Z_p)} \leq 1 \) for all but a finite \( v \) of \( \mathbb{N} \).

For \( h \equiv 0 \pmod{\mathbb{Z}_p} \):

\[ K_{p,h} = \{ \gamma \in GL_1(Z_p) \mid \gamma \equiv \left( \begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) \pmod{\mathbb{Z}_p} \} \]

**Theorem.** For every \( p \), \( \exists \) \( j \), \( f \) s.t. \( W_f^p \mid_{K_{p,h}} = \left\{ 0 \right\} \) if \( h \neq j \).

\& \( W_f^p \mid_{K_{p,h}} \) has dimension \( 1 \).

Tony proved that this using Klingen models.

\( V_p \) \( \ni \) \( \exists \eta_p \) to be a non-zero elt of \( W_f^p \).

It's obvious that \( \bigoplus \eta_p \) makes sense in \( W = \bigoplus W_v \subset S_k \).

Then \( U_1(N) \leq GL_1(Z) \) \& \( N = \prod_{p \mid N} \mathfrak{p}_p \). Then \( U_1(N) = \prod_p T_\mathfrak{p}_p \).

Say \( \eta = \bigoplus \eta_p \).

The conclusion is that \( \eta \in S_k \) \( U_1(N) \equiv S_k(\Gamma_1(N)) \).

\( \eta = (p) \) for some \( \xi \in S_k(\Gamma_1(N)) \).

It's pretty easy to check that \( f \) is a primitive form.

He wants to spend the last few minutes talking about another fact: the Taniyama-Weil conjecture or a recipe that Tony told John.

\( E/Q \) an elliptic curve. For every \( p \), \( \nu_p \neq 0 \), we can attach \( \gamma_p \), an irreducible \( GL_1(Z_p) \) module, \& \( \nu_p \neq 0 \) gives us \( m_q \), an irreducible \( (\mathbb{Q}_p, \mathbb{Z}_p) \) module \( \therefore \) \( H_p = \sigma(\mu_q, \mathbb{Q}_p, \mathbb{Z}_p) \).

They are almost all unramified.

\( \nu_p, p \neq \infty \), \( \gamma_p \) \( \ni \) \( V_1(E) = T_1(E) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \), an \( \ell \)-adic rep.

If \( \eta \) \( \in \) there's an action of \( Gal(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) \) on \( V_1(E) \).
The usual Grothendieck trick as explained by Martin Taylor gives us a 2 dim rep of $\text{WD}_p$

$$\sigma(\pi_p): \text{WD}_p \rightarrow \text{GL}_1(E).$$

Then Loyal Langlands attach $\pi_p$, an irreducible rep of $\text{GL}_1(\mathbb{Q}_p)$

$$\text{So } E \rightarrow \{ V_\ell(E) \} \rightarrow \{ \sigma(\pi_p) \} \rightarrow \{ \pi_p \} \rightarrow \pi \otimes \pi_p.$$ 

$\pi$ is an admissible irred rep of $\text{GL}_1$. 

Taniyama-Wiles $\Rightarrow \pi$ is automorphic.

Tony has this recipe & John will explain it.

$\pi_p \text{ unramified } \iff E \text{ has good reduction at } p$

$\pi_p = \sigma(\mu_\ell, \mu_p) \iff E \text{ has potential multiplicative red. at } p$ \text{(Image of I)}

$\sigma(\mu_\ell \text{ unramified } \iff \text{mult red.})$

$\pi_p \text{ ramified only } \iff E \text{ has potential good}$

$\sigma(\text{unramified p-adic eisen} / \text{super-cusp.}) \rightarrow \text{ p-adic red.}$

$\sigma(\text{ramified only}) \rightarrow \text{ good red. over a non-abelian ext. / } \mathbb{Q}_p$

$\sigma(\text{unramified only}) \rightarrow \text{ good red. over a non-abelian ext. / } \mathbb{Q}_p$

Hopefully $\varepsilon$-factors would match up too.

It would be nice if this were true as the Galois $\varepsilon$-factors are nasty to work out.
IV. Quaternion Algebras

The analysis is much easier for quaternion algebras, although they're less familiar objects. There are 3 parts to this course:

1) Quaternion algebras, generalities (3 lectures, last one on Sat; just 2 are easy)
2) Functional analysis (3 lectures; what he wished the analysis had taught him as an undergraduate)
3) Automorphism forms (2 lectures; trace formula etc).

4.1. Generalities

D is a quaternion algebra over a field F if

1) D is an F-algebra, associative with 1 but not necessarily commutative
2) D is central in F, i.e., F is the centre of D
3) D is simple, i.e., D has no non-trivial 2-sided ideals
4) dim_F D = 4

Example (exercise) \( M_2(F) \) is a 2x2 matrices over F.

This simple example is exceptional in many ways. We say \( M_2(F) \) is split.

\( M_2(F) \) is to \( GL_2(F) \) as D is to weird gys we'll look at later.

Lemma 4.1: If A and B are simple F-algebras & if A has centre F then \( A \otimes_F B \) is simple.

6-line proof but he won't waste time. Check out references.

References:
- Artin Algebra
- Non-commutative rings - Herstein
- Associative Algebras - Pierce
- Algebras de Quaternion - Vigneras
- Bures no they - Unit

They get more arithmetic as you go down.

Lemma 1.1 is 12.14.8 of Pierce & Thm 4.11 of Herstein.

Cor: If \( E/F \) is a field extension then \( D \otimes_F E = D_E \) is a quaternion algebra over \( E \).

Note: \( D_E \) may be split, even if \( D \) isn't.

We say \( E \) split \( D_E \) if \( D_E \) is split.
Lemma 2. If $D$ is not split, $\delta \in D \cdot \{0\}$, then $\delta$ has a 2-sided inverse.

**Proof.** The map $D \to D$ is linear. Say its kernel is $I_\delta$ & its image is $D\delta$.

Then $D \to \text{End}_F(D\delta)$ & $D \to \text{End}_F(I_\delta)$ by left multiplication.

If $D$ is simple, so either these maps are 0 or they're injections.

Hence $\dim D\delta = 0$ or $\geq 2$

$\dim I_\delta = 0$ or $\geq 2$

However, if one of them has dimension 2 then the map is an iso.

And $D$ splits. They both can't have dimension $\geq 2$; one has dimension 0.

Hence $D\delta = 0$ ($\neq 0$ in $D\delta$)

or $I_\delta = 0$ ($\subseteq \delta$) & $D\delta = D$ so $\delta$ has an inverse.

It must be 2-sided by the same argument or something. $\square$

Cor. If $D$ is not split & $\delta \in D \cdot \{0\}$, then $F(\delta)$ is a field.

Cor. If $F$ is algebraically closed, then $D$ must be split.

Somehow the non-split quot. alg. are related to the fact that $F$ isn't algebraically closed.

We want to talk about $\prod_{i=1}^n E_i$ where $E_i/F$ is a finite field ext.

We will call $\prod_{i=1}^n E_i$ a POF, i.e. a product of fields.

He doesn't know of a better notation. They're the semisimple commutative $F$-alg. but this is worse!

Lemma 3. If $E \subseteq D$ is a POF & $E+F$ then $\dim E \cdot 2$. $E$ is its own centralizer.

in $D$, $E$ splits $D$, & if $E$ is not a field, then $D$ is split.

NB he'll leave it to our imagination as to what "$E$ splits $D$" means.

I guess $D \otimes_F E \cong M_2(E)$.?

**Proof.** Suppose $E$ is a field. Then $E \cong \dim E \cdot [E:F] = \dim D \cdot [E:F] = 2$.

Thus if $\delta \in D \cdot E$, then $D \cdot E + E \delta$, & $D$ is not commutative: $E$ does not commute with $E$. $E$ is its own centralizer.

Next note $D \otimes E \to \text{End}_E(D)$.

$\delta \otimes x : y \mapsto \delta xy \quad (E$ acts on $D$ on the right$)$

Thus $\delta \otimes 0$ so $\delta$ is injective as $D \otimes E$ is simple.

By dimension counting its iso. $D_\delta \cong \text{End}_E(D)$.
If \( E \) is not a field then it is all an exercise: \( E \otimes D \simeq F \otimes E \), \( \Delta = 1 \) 
& hence \( D = H(E), \Delta = \frac{1}{\Delta} \).

So any subfield \( F \) of dimension \( 2 \langle F \rangle \) splits \( D \). There is a converse.

Lemma 4: If \( E/F \) is quadratic & splits \( D \) then \( E \otimes D \) (best have \( \text{char } F = 0 \)).

\( \text{If } E \text{ isn't a field then } E \otimes F \) & \( D \) must be split already. Embed diagonally.

Say \( E \) is a field then. Set \( E = F(\sqrt{\delta}), \delta \in F \).

\( E \) splits \( D \Rightarrow \exists \delta_0, \delta_1, \delta_2 \in \mathbb{D}_E \), \( \delta_0, \delta_1, \delta_2 \neq 0,1 \), & \( (\delta_0, \delta_1, \delta_2) = \delta_0 \otimes \delta_1 \otimes \delta_2 \Rightarrow \sqrt{\delta} \in \mathbb{D} \). Time is short so he won't go in to details.

Defn. There's a canonical involution \(*: D \rightarrow D\) defined thus:

\( \delta \in E, \delta^* = \delta \) 

If \( D \) is split, \( (a, b)^* = (a, b) \) => adjugate of \( (a, b) \).

If \( D \) not split \& \( \delta \in D \) then \( \star \) or (the field) \( F(\delta) \) is the non-trivial alt of the Galois gop of \( F(\delta)/F \).

Rem: \( \delta^* = \delta \Rightarrow \delta \in F \).

Define \( \text{Tr} \delta = \delta + \delta^* \in F \) - the reduced trace.
& \( \text{the reduced norm } \text{nr} \delta = \delta \otimes \delta^* \in F \) - the reduced norm.

They fulfill for non-split \( D \) what we have & do do for split \( D \).

Lemma 5: If \( E \) splits \( D \) then \( D \otimes D \).

\( \text{tr}, \sqrt{\text{det}} \)

\( F \otimes E \) commutes

\( \delta \) is an exercise. So easy - if \( D \) is non-split then \( \delta \) = trace of \( \delta \otimes \text{End}_F(F(\delta)) \).

So it easy to reduce facts about \( \delta \) & \( \sqrt{\text{det}} \) to facts about trace \& det.

Cor: \( \text{tr} (\delta_0 + \delta_1) = \text{tr} \delta_0 + \text{tr} \delta_1, \chi (\delta_0 \delta_1) = \chi (\delta_0) \chi (\delta_1), (\delta_0, \delta_1)^* = \delta_0^* \delta_1^* \)
Write $D^*$ = units in $D$ (= $D^* \setminus \{0\}$ in non-split case
  
  $\sigma_2$ in split case

$D^t = \ker \psi : D^* \to E^*$

Another elementary but useful fact is

**Lemma (Noether--Skolem)** (NB he has lemmas & props, everyone else has theorems & props!!)

If $M/F$ a quadratic POF (NB he sometimes uses $M$ & sometimes $E$, $E$ is usually $C_D$)

& if $\sigma_1, \sigma_2 : M \to M$, then $\exists \delta \in E^*$ s.t. $\delta \sigma_2(x) \delta^{-1} = \sigma_1(x)$ \forall x \in M

If $M$ is a field then $D$ is split & do an exercise: any 2 bases are $GL_2$-equivalent.

If $M$ is a field then $D$ is a $D \otimes_F M$-module in 2 ways:

$\delta \cdot m : x \mapsto \delta \sigma_2(x) m$ or $\delta \cdot \sigma_1(m)$

But $D \otimes_F M = M_2(M)$ has a unique module of dimension $2/M$. (exercise)

\[ \exists \psi : D \to M \text{ between the 2 actions} \]

\[ \psi(\delta \cdot \sigma_2(m)) = \delta \psi(\sigma_1(m)) \forall \delta \in E^* \forall m \in M. \]

So $\psi(\sigma_2(m)) = \sigma_1(m) \psi(\delta) = \psi(\sigma_1(m)) \sigma_2(m)$

$\exists \in E^*$ with $\psi(\delta) : 1 \to x \psi(\sigma_1(m)) \delta^{-1}$ is invertible (only important if $D$ split). 

So $\sigma_2(m) = \psi(\delta) \sigma_1(m) \psi(\delta)$ \forall $m \in M$. \[ \square \]

**Exercise** If $\sigma_1, \sigma_2 : D \to E^*$, $E/F$ finite field ext., then $\sigma_1$ & $\sigma_2$ are conjugate by an elt of $E^*$.

**Rem.** $D \otimes_F D \cong \text{End}_F(D)$

$\delta \cdot \sigma_2 = (x \mapsto \delta x \delta^{-1})$. This is a hint too.

**Examples** of Noether--Skolem then in action (NB don't need char $0$, evidently)

1) If $#F < \infty$ & $D/F$ is a quaternion algebra then $D$ is split

Pf (exercise) Let $M/F$ be the quadratic ext. If $D$ is not split then

$D^* = \bigcup_{\delta \in \delta^*} \delta M \delta^{-1}$. Now count. Etc. on both sides $\Rightarrow \#

2) $F = \mathbb{C}$ (or $F = F$) any quart alg is split
3) $F = \mathbb{R}$, $D$ is either split or $D \cong H = \{ (a, b) \mid a, b \in \mathbb{C}, j^2 (a, b) = C \otimes C \}$ where $j^2 = -1, j \in \mathbb{C}$. 

Proof: If $D$ is not split, then $E \cong D \otimes \mathbb{C}$ and complex conjugation gives 2 embeddings, so Neither - Speiser $\Rightarrow$ 2 split $\Rightarrow$ $D \cong \mathbb{R}^2$.

Then $D = C \otimes C \otimes \mathbb{C}$, where $\delta_1 = \pm 1$.

If $\delta_1 = 1$, then $D \cong M_2(\mathbb{R})$.

If $\delta_1 = -1$, then $D \cong H$.

One final lemma for today. Real alg can't be split by odd degree field ext's.

Lemma 7: Suppose $E/F$ has odd degree, $D_E \cong D$, then $D \cong D$.

Proof: Omitted. It's interesting to note that although the proof appears to be elementary, it is embedded in many books on the subject that it's difficult to extract. Eg:

- Sublemma in then 445 Herstein
- Lemma 34 of Pierce

Cor: Suppose $E/F$ is odd degree ext, $D, D'$ real alg $/F$ with $D \cong D'$. Then $D \cong D'$.

Exercise:

Two things he forgot to say last time:

$$H^2 = \ker (\chi : H^n \to \mathbb{R}_+^* ) \cong SU(2)$$

Now if we have $D/F$ & $E/F$ a quadratic ext. with $E/F$ separable, $E = F(\sqrt{d})$ & $E$ split $D$, then $E \cong D$.

The proof is $(\delta_{1, \delta_{2, \delta_{3, \delta_{4}}}})^2 = 0$,

Then $(\delta_{1, \delta_{2, \delta_{3, \delta_{4}}}}) = d$.

1.2 Quaternion algebras over local fields (sketch proofs coming up)

$F / \mathbb{Q}_p$ a finite ext. $\&$ $v : F^* \to \mathbb{Z}$ a valuation.

Say $D/F$ is a non-split quaternion algebra.

Define $v_0 : D^* \to \mathbb{Z}$ by $v_0 = v_0 v$, i.e. $D^* \to F^* \to \mathbb{Z}$.

We'll note some easy properties of $v_0$. 


\begin{align*}
\nu_D(xy) &= \nu_3(x) + \nu_3(y) \\
\nu_3(1+x) &\geq \min(0, \nu_3(x)) \quad \text{(work in } F(s); \nu_3 N_{E/F} \text{ a valuation)} \\
\nu_3(xy) &\geq \min(\nu_3(x), \nu_3(y)) \\
\nu_D(x^n) &= \nu_3(x) \\
\text{Set } \mathcal{O}_D &= \{ x \in \mathcal{D} \mid \nu(x) \geq 0 \} \\
\mathcal{M}_D &= \{ x \in \mathcal{D} \mid \nu(x) > 0 \} \\
\mathcal{O}_D^* &= \{ x \in \mathcal{D} \mid \nu(x) = 0 \} \\
\mathcal{O}_D \text{ is free of rank } 4 \text{ over } \mathcal{O}_F
\end{align*}

(\mathcal{O}_D \text{ spans } \mathcal{D} \cdot \cdot \cdot \mathcal{O}_3 \supset \Lambda, \text{ a rank } 4 \text{ free } \mathcal{O}_F. \text{ So either } \mathcal{O}_3 \not\supset \mathbb{P}^{1} \Lambda \text{ for some } \mathbb{P} (\text{so we're OK)} \text{ or } \mathcal{O}_D \text{ has elements of arbitrarily small valuation (not possible))}

Next note \(\mathcal{O}_D / \mathcal{M}_D\). \text{ It's a division algebra ring over the residue field } \mathcal{O}_F.

\(\mathcal{O}_D / \mathcal{M}_D\) \text{ is a field extension of } \mathcal{O}_F / \mathcal{M}_F. \text{ Say its got degree } f.

Now all two-sided ideals of \(\mathcal{O}_D\) are powers of \(\mathcal{M}_D\).

We have \(\mathcal{M}_F \mathcal{O}_D = \mathcal{M}_D^e\) for some \(e\).

Cf. 4. (exercise) \text{ (just as in local field extension case)}

Certainly \(e \leq 2\). \text{ It's slightly trickier to prove } f \leq 2.

Hence \(e = f = 2\).

If \(E_0/F\) is the unramified quadratic extension, then \(E_0 \rightarrow \mathcal{D}\) (as above)

Then \(\exists j \in \mathcal{D}^* \text{ st. } jxy^2 = x^2 \forall x \in E_0\)

Then \(D^* E_0 \oplus E_0 j^* \text{, } j^2 \in F. \text{ Scaling } j \text{ on the right by an element of } E_0^* \text{ changes } j^2 \text{ by an elt of } N_{E_0/F} \text{.}

Hence WLOG \(j^2 = 1 \text{ or } \tau_F\). \text{ However, if } j^2 = 1 \text{ then by choosing } x \in E_0 \text{ st. } x^2 = x \text{ we see } x^2 j \oplus 0, (x^2 j)^2 = 0 \text{ so } D \text{ is split, formally a contradiction. Hence } j^2 \in \tau_F \text{ is the only choice.}
Lemma 8. There is only one non-split quaternion algebra over $F$. Any quadratic extension $F'/F$ embeds in $D$.

If $p \neq 2$, then $F(\sqrt[p]{r}) \hookrightarrow D$ for arbitrary $r$, & it's quite easy. (There's only 1 field ext's agreed)

If $p = 2$, then the same argument as the above line gives 3 embeddings. The unramified ext also embeds. For the rest, just repeat the above argument. It's a bit messy, but works. □

Lemma 9. $D/F$ not split $\Rightarrow D^2 \cong \text{spct}$ (note $D/F$ split $\Rightarrow D^2 \cong \text{SL}_2(F)$ not spct)

If $O_0$ is spct because $O_0^\times \cong \mathbb{Z}_{\geq 0}$

$D^2$ is closed in $O_0^\times : D^2 \cong \text{spct}$. □

Remark. For those of us who like concrete things, he'll tell us the non-split one explicitly:

$D = \left\{ (a + b \omega) \mid a, b \in E_0, \omega \right\}$

$s + t \in \text{Gal}(E_0/F), \omega$ is the Frobenius $j = (0, 1, 0, 0)$

Remark. $E/F$ split $D \cong [E:F]$ is even.

So over a local field, everything is rather simple.

Now say $D$ is any quaternion algebra / $F$, maybe split.

$O \subseteq D$ is called an order if $O/O_F$ is free of rank 4 & $O$ is an algebra.

Lemma 10. 1) If $D$ is split then any order is conjugate to a subset of $M_2(O_F)$

2) If $D$ is not split then $O \subseteq O_D$

Cor. 1) If $D$ is split then any maximal order is conjugate to $M_2(O_F)$

2) If $D$ is not split then $O_D$ is the unique maximal order.

Proof of lemma (sketch) 1) $O$ is spct - $O$ stabilizes a lattice

2) $x \in O \Rightarrow v(x) \geq 0$ as $x$ is integral.
13. Quaternion algebras over number fields

(Now I'll hardly even sketch the proofs.)

Say \( F / \Omega \) finite, \( D / F \) a quaternion algebra, \( v \) c. place of \( F \), \( D_v = D \otimes F_v / F_v \)

\[ S(\Omega) = \{ v \mid D_v \text{ is not split} \} \]

**Facts**
1) \( S(\Omega) \) is finite, it contains no complex places, \& \( \# S(\Omega) \) is even.
2) Any set \( S \) satisfying 1) comes from some quaternion algebra.
3) \( S(\Omega) \) determines \( D \).

There's proofs of this in Pierce & Wald.

**Facts** (not nearly as deep) about orders

\( O \subseteq D \) is an order if \( O \) is a \( \mathcal{O}_F \)-algebra, \( O \) is \( \mathcal{O}_F \)-finite as an \( \mathcal{O}_F \)-module and \( O \otimes_{\mathcal{O}_F} F = D \).

Eg. \( \mathcal{M}_2(\mathcal{O}_F) \subseteq \mathcal{M}_2(F) \).

Here are the facts.

Orders exist. In fact, maximal orders exist.

If \( O \) is a fixed maximal order, then an bijection

\[ O' \leftrightarrow \{ O'_v \} \text{ (localizations)} \]

where \( O' \) is the bijection is between the orders \( O'_v \) of \( D_v \) and the collections of orders \( O'_v \) of \( D_v \), st., \( O'_v \otimes_{\mathcal{O}_F} F_v = D_v \).

This is easy once you understand lattices.

\( O'_v \) is maximal for almost all \( v \).

\( O' \) is maximal \( \iff \) \( O'_v \) is maximal for all \( v \).

This is all an exercise if you understand lattices.
Now fix a maximal order $O_F$.

If $R/O_F$ is a commutative algebra, define $G(R) = (O_B \circ O_F)^x$

Eg if $O_B = M_n(O_F)$, $D = M_n(F)$, then $G(R) = GL_n(R)$.

We have a reduced norm map $\nu: G_B(R) \to R^x$.

Set $G'_B = \ker \nu$. Eg $G_2 = SL_2$ in above example.

These are the generalisations of $GL_n$. John will talk about $GL_n$ & this is how to generalise it.

$G(A)$ is locally split once you've given it the correct topology, which isn't the subspace topology on $O_B \circ A$.

The correct topology is the subspace topology under the map inclusion

$$G(A) \to D_B^2,$$

$$x \mapsto (x, x^2)$$

$G(A)$ is in fact a restricted direct product $\bigoplus G(F_i) \circ \text{restricted w.r.t. } G(O_F)$. Of course, $G(F_i) = D_{F_i}$, $G(A) = (D_B \circ A)^x$, $G(O_F) = O_B^x$.

$G(A)$ is just a generalisation of $GL_n(A)$.

Define $\| \cdot \|: G(A) \to R^+_0$

$$G(A) \xrightarrow{\nu} A^x \xrightarrow{\text{diag}} R^+_0$$

Define $G'(A) = \ker \| \cdot \|.$

Lemma II (cf. Martin Taylor lemmas on $A$, $A'$ case)

$$\begin{align*}
G(F) &\to G(A) \xrightarrow{\text{diag}} A^x \quad \begin{cases}
\nu \\
\| \cdot \|
\end{cases}
\end{align*}$$

& the image is discrete (product formula ensures $\|G(F)\| = 1$).

Proof To prove discreteness we replace $F$ by an extension $E/F$ which splits $D$, & we're reduced to the following problem...
We need $GL_n(F) \subset GL_n(A)$ to be discrete.

It will do to show $GL_n(O_F) \subset GL_n(F_o)$ is discrete

where $F_{oo} = \mathbb{T} F_o = M_n$.

If $x \in O_F$ and $|x|_F < \frac{1}{2}$ for all $x > 0$. 

Prop 1.2. If $G$ is not split then $G(F) \backslash G(\mathbb{A})$ is compact.

Remarks: i) Compactness fails in the split case.
ii) Thuey like finiteness of class group & unit theorem all are related to this result so there's some content to the proof.

Also: move on Haar measure

Say $G$ is a locally compact top gp. Martin Taylor told us about in $C_c(G) \rightarrow \mathbb{R}$ $m(gf) - m(f)$ $\forall g \in G$

$m$ had various properties & was unique up to scalar.

Richard wants to talk about measures of Borel sets.

Suffice it to say that the Riesz representation theorem gives us $m \mapsto \mu$

$\mu: (\text{Borel subsets of } G) \rightarrow [0, \infty]$ a measure.

The Borel subsets of $G$ are the $\sigma$-field generated by the open sets.
$\sigma$-fields are things closed under complement & countable unions roughly.

We have $\mu(gx) = \mu(x)$ for $g \in G$. Also, $K \in G \Rightarrow \mu(K) < \infty$.

We have $\Delta_g: G \rightarrow \mathbb{R}_{>0}$ $\forall g \in G$ $\mu(x) = \inf_{U_x} \mu(U)$

$\Delta_g$ is $\theta$ to $HM$, defined by

$\mu(xg^{-1}) = \Delta_g(g) \mu(x)$

$G$ is unimodular $\Leftrightarrow \Delta_g = 1$. (e.g. $G$ split, $G$ abelian, $G$ discrete $\Rightarrow (G = (\mathbb{R})^n$ a stream)

He's written:

$G$ is the points of a reductive algebraic group over a local field or the adele ring. I think that in this case this asserting $\Delta_g = 1$ again.

A reference is Bourbaki, chapter 7 for something Richard does not know. Why this I above statement is true.
Then if $H \leq G$ is a closed subgroup, & suppose $\Delta_H \mid \Delta = \Delta_H$.

\[ \begin{align*}
(\exists) & \quad G \cap H \text{ unimodular} \\
& \quad X = GL_r(H), \quad B = B_r(H)
\end{align*} \]

Then E! measure (on Borel sets) of $G/H$ s.t.

1) $\mu(gX) = \mu(X)$ \quad $\forall g \in G$

2) $\int_G \mu(g) \, dg = \int_{G/H} \mu(gh) \, dh \, dg$

Then $E!$ measure (on Borel sets) of $G/H$ s.t.

1) $\mu(G) \mid \mu = \mu(G)$.

\[ \begin{align*}
& \quad \int_G \mu(g) \, dg = \int_{G/H} \mu(gh) \, dh \, dg
\end{align*} \]

& also if $\int_H \mu(gh) \, dh$ exists for almost all $g$ & $\int_{G/H} \mu(gh) \, dh \, dg$ exists

then $\mu(G) / \mu = \mu(G)$.

\[ \begin{align*}
& \quad \mu(G) / \mu = \mu(G)
\end{align*} \]

Let $\mu(gh) \mid dh \, dg$ exist for almost all $g$ & $\int_{G/H} \mu(gh) \, dh \, dg$ exists

then $\mu(G) = \mu(G)$.

Then $E!$ measure (on Borel sets) of $G/H$ s.t.

1) $\mu(G) \mid \mu = \mu(G)$.

\[ \begin{align*}
& \quad \int_G \mu(g) \, dg = \int_{G/H} \mu(gh) \, dh \, dg
\end{align*} \]

& also if $\int_H \mu(gh) \, dh$ exists for almost all $g$ & $\int_{G/H} \mu(gh) \, dh \, dg$ exists

then $\mu(G) = \mu(G)$.

Thus Martin did the case where $H$ was split & normal, & the cover then as $f$'s with split support on $G/H$ can be pulled back to $f$'s with split support on $G$.

Thus all on Haar measure.

Say $D/F$ is a quart alg / $F$ a number field. Set $A = A_F$.

Pick a fixed max'ed order $O_D$

$G(D) \leq G(F) = D^\times$. \quad $G^\times = \ker(v: G \to GL_\ell)$

$G(A) = \ker(G(A) \to \mathbb{R}^\times \to \mathbb{R}_{>0}^{x_0})$

$G(F) \leq G(A)^\times$ discrete subgp.

Here comes that prop again.
Prop 1.2. If $D$ is not split then $G(F) \backslash G(A)^d$ is cpt.


Sublemma. $V/F$ a vector space. $V \subseteq V \otimes A$ is discrete & $V \backslash V \otimes A$ is cpt.

Proof. WLOG $F = Q$. WLOG $\dim_A V = 1$. $G(A) \cong \mathbb{Z} \times \mathbb{Z}^d$.

Cor. $\exists$ Haar measure $\mu^V$ on $V \otimes A$ s.t. $\mu^V(V \otimes A) = 1$.

Cor. If $C \subseteq V \otimes A$ is a closed set with $\mu^V(C) > 1$ then $\exists x, y \in C$ with $x \cdot y \in V \otimes A$.

Proof. $\mu^C \geq \int_{V \otimes A} \# \pi^{-1}(x) \, d\lambda$ where $\pi : V \otimes A \to V \otimes A$.

Pick $x \in V \otimes A$.

Rek. If $d \in \mathbb{R}$ then $\mu^V(dx) = \#(\dim A) \cdot \mu^V(A)$

Proof of prop 1.2. Choose $C \subseteq V \otimes A$ cpt with $\mu^V(C) > 1$.

Claim. $C \cap (V \otimes A) = \emptyset$.

Proof. Let $d \in G(A)^t$. Then (exercise) $\mu^V(dC) = \mu^V(C) > 1$ & so

(Lebesgue measure for addition, $\mu^V$ is additive) $\exists x, y \in C$ & $d \in V \otimes A$ s.t. $d(x, y) = \delta$ (intsplit).

Also, $\mu^V(C_{d^{-1}}) > 1$ & so $\exists x' \in C_{d^{-1}}$ s.t. $(x', y) d^{-1} = \delta$.

Then $\delta \delta = (x', y) (x, y) \in C \cap (V \otimes A)$

& $(x, y) d^{-1} \in C \cap V \otimes A$

$(d^{-1})(x, y) = \overline{d}(x, y)$ for small $t, 2, \ldots$ & we're home.

There's a proof, for what it's worth.
Exercise. Repeat the argument to show that $f^* \left( R^1 \right)^2$ is compact and deduce that the class $\mathcal{O}_D$ of $F$-finite & Dirichlet's unit theorem for $F$.

Exercise. $\mathcal{O}_D$. Invertible fractional ideal $I \subset D$ is

finite $O_F$-module that \[ I \mathcal{O}_D \text{ is finite, } \mathcal{O}_D \text{ is finite,} \text{ and } I \mathcal{O}_D \text{ is finite.} \]

Proposition: $I \sim J$ if $I = DJ, D \in \mathcal{O}_D$. 

Say $\text{RIC}(\mathcal{O}_D) = \sim$ classes.

Show $\# \text{RIC} < \infty$.

(Hint: Note $\text{RIC}(\mathcal{O}_D) \leftrightarrow D \setminus G(\mathcal{O}_D^+) / \mathcal{T} \mathcal{O}_D^+$. But of a heavy hint, actually.)

$\mathcal{O}_D \setminus \mathcal{O}_D^+$ $\leftrightarrow$ \[ \mathcal{O}_D \setminus \mathcal{O}_D^+ \leftrightarrow \mathcal{O}_D \setminus \mathcal{O}_D^+ \]

Exercise. $\{ \text{conj. classes of max order} \} \leftrightarrow D \setminus G(\mathcal{O}_D^+) / \mathcal{T} \mathcal{O}_D^+ \times \mathcal{T} \mathcal{O}_D^+$.

$\mathcal{O}_D \setminus \mathcal{O}_D^+$ $\leftrightarrow$ \[ \mathcal{O}_D \setminus \mathcal{O}_D^+ \leftrightarrow \mathcal{O}_D \setminus \mathcal{O}_D^+ \]

Exercise. Show $\# \text{conj. classes of max order} < \infty$.

Lemma. \[ \# \text{conj. classes of max order} < \infty \]

Whether or not $D$ is split, we have

1) $\mu (G(F) \setminus G(\mathcal{O}_D^+)^2) < \infty$

2) If $v$ is a place of $F$ then $\exists$cpt set $X \subset G(\mathcal{O}_D)$ s.t. $G(\mathcal{O}_D) = G(F) \times G(F_v)$

Proposition. $D$ not split: 1) $\sqrt{2}$. 

2) $\mathcal{O}_D \setminus \mathcal{O}_D^+$ $\leftrightarrow \mathcal{O}_D \setminus \mathcal{O}_D^+$ is cpt.

$D$ split: Use Iwasawa to reduce to Borel & then use normal results for reals & octals.

Note. 1) in case $F = \mathbb{Q}$, $D = M_2(\mathbb{Q})$ translates as $\mu \left( \mathbb{Q} \setminus \mathbb{Q} \setminus \mathbb{Q} \right) < \infty$. 

Note.
Prop. 13 (Strong Approx Thm) D/F. If \(v \in \text{place of } F, v \nleq S(D), \) then 
\[ G(F) G(F_v) = G^*(A) \text{ is dense.} \]

I think he said I think he said  
the meant that every ide did not  
be generator.  

Note John mentioned this in the SL_1(A) case, \(F = Q.\)  

There's some generalization to simply-connected alg gp, or sthg.  

Cor. Suppose \(D\) is split at \(v\) and that \(O_F\) has strict class no. one. Let \(U \in G(A_v)\) be an upper split subgp, \(Y \in \bigcap_{v \mid \mathfrak{m}} O_v.\) Then \(G(F) U G(F_v) = G(A).\)

Ex. Exercise. 

Cor. Suppose \(D\) is split at some \(v\) place, \(v \nleq \text{Of has strict class no. 1.} \)

Then \(a)\) all max orders in \(D\) are conjugate.  
\(b)\) \(\# \text{Ric}(O_0) = 1\)

Ex. Exercise. 

He will omit the pf of prop. 13 because "something's got to give". 
I think he said that there was a proof in Vigneras.

There will be no lunch tomorrow because 12 sandwiches isn't enough for the caterer.

2. Functional Analysis  He's changing tack totally today.

2.1 Hilbert spaces  Reference: Gelfand & Vilenkin - Generalised f; vol 4 chap 1 & 2.  
Wallach - Reductive gp 1 §8.9.1

Richard says that Wallach's book has served him well. There's lots of results by Harish-Chandra that aren't really written up anywhere,  
as far as Richard knows, except in Wallach (lovely) & Harish-Chandra's collected works (much less user-friendly).

\(H, \) a Hilbert space, is a v.s. \(\mathbb{C}\) with \(\langle, , \rangle\) \(H^{1} \rightarrow \mathbb{C}.\)
\(\langle x, y, z \rangle = x(x, z) + (y, z) - (x, y) - (z, x) = 0 \Leftrightarrow x = 0.\)

He will also assume His separable, i.e. has a dense dense subset, unless he explicitly says otherwise. This is to keep him out of trouble.
So say \( H \) is a (separable) Hilbert space.

If \( G \subseteq H \) set \( G^\perp = \{ x \in H \mid (x,y) = 0 \ \forall y \in G \} \).

\( G \) a closed subspace \( \Rightarrow H = G \oplus G^\perp \).

An orthogonal basis \( \{ e_\lambda \} \) is not really a basis, \( (e_\lambda,e_\mu) = 0 \) & \( \sum \epsilon e_\lambda \) is dense in \( H \).

Then \( \{ e_\lambda \} \) is necessarily countable.

Also, \( x \in H \Rightarrow \sum_{\lambda \in \mathbb{N}} (x,e_\lambda) e_\lambda \text{ converges } \in H \) and converges to \( x \).

Then \( (x,y) = \sum_{\lambda \in \mathbb{N}} (x,e_\lambda)(e_\lambda,y) \).

**Examples**

1) If \( X \) is a measure space, \( L^2(X) = \{ f \colon X \to \mathbb{C} \mid f \text{ measurable, } \int_X \lvert f(x) \rvert^2 \, dx < \infty \} \)

\[ (f_1,f_2) = \int_X f_1(x) \overline{f_2}(x) \, dx \]

\( L^2(X) \) is a Hilbert space, not necessarily separable!

2) \( H_1, \ldots, H_n \) Hilbert spaces.

\( H \) consists of vectors \( (x) \) s.t. \( \sum_{\lambda \in \mathbb{N}} \lVert x_\lambda \rVert^2 < \infty \).

Then \( (x_\lambda,y_\lambda) = \sum_{\lambda \in \mathbb{N}} (x_\lambda,y_\lambda) \).

Say \( T \colon H \to H \) linear.

Write 1) \( \mathfrak{L}_0(H) \) for the bounded linear maps,

\[ \{ T \mid \exists A > 0 \text{ s.t. } \lVert T \lVert \leq A \lVert \lVert x \rVert \lVert \forall x \in H \} \]

2) An off-putting thing about this stuff, to this number theorist (ie Richard) is there's lots of collections of linear maps, each having its own use. Here's another one:

**spectral linear maps**

\[ K(H) \subseteq \mathfrak{L}_0(H) = \{ T \mid \lVert T \lVert \leq 1 \text{ s.t. } \text{ has spectrum} \} \]

If you lean analysis in Cambridge then you there are precisely the kinds of maps you learn about.
3) \( L_2(\mathcal{H}) = \text{Hilbert-Schmidt operators. For some orthonormal basis } \{ e_i \} \text{ we have } \sum \| Te_i \|^2 < \infty. \)

Hence for all o.n. bases, \( \sum \| Te_i \|^2 < \infty. \)

These little remarks are exercises if you're brave. Richard tried to do them & got stuck. He then looked in Gelfand & everything got easier. He thinks they're exercises if you're slightly cleverer than him.

4) \( L_1(\mathcal{H}), \text{ Trace class} \) (nuclear: check defn if you're reading a book because things vary!)

\( \Rightarrow \text{true if } \mathcal{H}, \text{on } \mathcal{H} \)

5) \( \text{F.R.}(\mathcal{H}) = \text{finite-rank range}. \)

We have \( \| T \|_\infty \) on \( L_\infty \):

\[ \| T \|_\infty = \sup_{\| f \|_\infty = 1} \| Tf \| \]

\& \( \| T \|_1 \) on \( L_1 \):

\[ \| T \|_1 = \left( \sum \| Te_i \| \right)^{1/2} \text{ for any o.n. } \{ e_i \} \]

\( \| T \|_1 \) on \( L_1 \):

\[ \| T \|_1 = \sup \sum |(Te_i, f_i)| \text{ over all } \{ e_i, f_i \}. \]

They're all normed. \( L_\infty \) & \( K \) are complete w.r.t. \( \| \cdot \| \).

\( L_1 \) is complete w.r.t. \( \| \cdot \|_1 \)

\( L_2 \) is complete w.r.t. \( \| \cdot \|_2 \)

Also \( \text{F.R.} \subseteq \text{K} \) has \( \| \cdot \|_1 \)-closure \( L_2 \), \( \| \cdot \|_2 \)-closure \( L_2 \), \& \( \| \cdot \|_1 \)-closure \( K \).

Why we're doing all this is that you want to take the base of a rep & it's not clear how you should do this in the \( \infty \)-dim case.

\( L_2(\mathcal{H}) \) is another Hilbert space: \( (T,S) = \sum (Te_i, Se_i) \) \text{ inner of } \{ e_i \}.

Given \( T, T^* \) st. \( (T_e, y) = (x, T^*y) \) \forall x, y \in \mathcal{H}.

\( T^* \) is the adjoint of \( T \).

All spaces are preserved under \( T^* \). Norms don't change.
$T, S \in L_n \Rightarrow TS \in L_n$, & $\|TS\|_n \leq \|T\|_n \|S\|_n$

$T \in L_n, S \in K \Rightarrow TS \in K$

$T \in L_n, S \in L_2 \Rightarrow TS, ST \in L_2$, and $\|TS\|_2 \leq \|S\|_2 \|T\|_n$

$\|ST\|_2 \leq \|S\|_2 \|T\|_n$

$T, S \in L_2 \Rightarrow TS \in L_2$. NB this seems to be about the only way of checking that things are trace class.

If $T \in L_2(H)$, we define $\text{tr}_T = \sum_{v \in \mathcal{G}} (T_v, e_v)$

This sum is absolutely convergent, & indpt of $\{v; \}$.

NB some people define $T$ to be trace class if $\sum (T_v, e_v)$ is absolutely cgt. This seems to strictly contain $L_2$, but doesn't seem to have a sensible norm and.

\textbf{Def.} Note $\text{tr}: L_2(H) \rightarrow \mathbb{C}$ is cba.

If $T, S \in L_2(H), \text{tr}(TS) = (T, S^*)$

\textbf{Thm.} (Spectral thm. He's gonna state it in a slightly more general, slighty uglier than usual, form. A good reference is Dawidowicz's book.)

(Richard was rather impressed by the thm.)

Say $H$ is a Hilbert space, $V \subseteq H$ a dense subspace. (Usually, the theorem is stated for $V = H$).

Say $T \in K(H)$ with $T^* = T$ and $TV \subseteq V$.

Let the spectrum of $T = \sigma(T)$ be $\{ \lambda \in \mathbb{C} \mid T - \lambda I$ is not invertible on $H \}$

Define $\text{def. } V_\lambda = \{ v \in V \mid Tv = \lambda v \}$

Then $\sigma(T) \subseteq \mathbb{R}$; the only possible limit point is 0

If $\lambda \in \sigma(T)$ then dim $V_\lambda < \infty$

and $H = (\ker T) \oplus \bigoplus_{\lambda \in \sigma(T)} V_\lambda$ and $V_\lambda$ is dense in $\ker T$. (Recall $V \subset H$ dense)

\textbf{Def.} $T$ in the theorem if $\sigma(T) \subseteq \mathbb{R}$ is called \textit{positive}. If $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$. 
2.2 Kernel

\( X \) locally cpt, Hausdorff & with a slide basis of open sets.

\( \mu \) a measure \( \mathcal{B} \) on Borel sets of \( X \), s.t. A cpt \( \Rightarrow \mu(A) < \infty \).

Then \( L^2(X) \) is separable, & \( C_c(X) \subseteq L^2(X) \).

See Rudin: Real & Complex Analysis for this stuff.

If \( K \in L^2(X \times X) \) we get \( T_K: L^2(X) \to L^2(X) \)

\[ f \mapsto \int_X K(x,y) f(y) \, dy \]

Then \( T_Kf \in L^2(X) \) & \( \|T_Kf\|_2 \leq \|K\|_2 \|f\|_2 \)

Moreover, \( T_K \) is bounded & \( \|T_K\|_\infty \leq \|K\|_2 \).

In fact \( T_K \) will be Hilbert-Schmidt, I think he said.

If \( \{e_i\} \) is an orthonormal basis of \( L^2(X) \), then \( \{\hat{e}_i(x), e_j(y)\} \) is an orthonormal basis of \( L^2(X \times X) \).

Thus \( K = \sum e_i(x) \overline{e}_j(y) \) in \( L^2(X \times X) \)

& \( T_K e_x = \sum \hat{e}_i(x) \overline{e}_j(y) e_i(y) \) dy

\[ = \sum_i e_i(x) \hat{e}_i(x) \]

and \( \|T_K\|_2 = \|K\|_2 < \infty \).

Hence \( T_K \) is Hilbert-Schmidt, & hence \( T_K \) is cpct.

Also, \( T_K^* = T_{K^*} \), where \( K^*(x,y) = \overline{K(y,x)} \)

\[ T_{K^*} T_K = T_{K^* K}, \text{ where } (K^* K)(x,y) = \int_X K^*_x(z) K_z(y) \, dz \]

These are easy exercises.

Prop: This is the prop that there was such a claimer to prove last time. He'll prove it next time.

Suppose \( \mu(A) < \infty \).

1) \( K \in C_c(X \times X) \), \( K^* = K \), and \( T_K \) is positive. Then \( T_K \) is trace class.

and \( \|T_K\| = \int K(x,x) \, dx \).

This is where the work lies. The statement "\( T_K \) is positive" is difficult to check. We can get stiff more though.
2) Suppose $K_1, K_2 \in C_c(x \times x)$. Then $T_{K_1 K_2}$ is trace class & $\text{tr} T_{K_1 K_2} = \int (K_1 * K_2)(x) \, dx$.

All the work goes into 1) & 2) comment.

There's a short false pf. We'll tell it in as it may convince us it's true.

False pf. 1) $T_K : C_c(x) \to C_c(x)$.

Choose $\lambda_i \in \mathbb{R}_{\geq 0}$ s.t. $\lambda_i \leq \lambda_j$, $\lambda_i \in \sigma(T)$, & every elt of $\sigma(T) \setminus \{0\}$ occurs $k_i$ with multiplicity $d_{i,j} \leq L^2(x \times x)$.

Then $\exists \phi_i \in C_c(x)$ with $T_K \phi_i = \lambda_i \phi_i$

& $\psi_j \in \ker T_K$, s.t. $\{ \phi_i, \psi_j \}$ is an o.b. basis.

$L^2(x \times x)$ has basis $\{ \phi_i(x), \psi_j(y) \}$.

$K(x, y) = \sum_{i,j} \langle \psi_i(x), \psi_j(y) \rangle + \psi_i(x) \psi_j(y)$ with $\phi_i$'s in $L^2(x \times x)$

$T_K \phi_i = \lambda_i \phi_i \quad T_K \psi_j \psi_j = 0$

$\Rightarrow K(x, y) = \sum_{i \leq j} \lambda_i \phi_i(x) \phi_j(y)$ in $L^2(x \times x)$

(exercise)

The false line of the proof is the next one.

$\int K(x, x) \, dx = \sum_{i \leq j} \lambda_i \int \phi_i(x) \phi_i(x) \, dx$

$= \sum_{i \leq j} \lambda_i = \text{tr} T_K$.

The point is that phrase-phrase you haven't even got phrase-phrase:

$\sum_{i \leq j} \lambda_i \phi_i(x) \phi_j(y)$

may not be true for every $x$,

because $K = \sum \lambda_i \phi_i \psi_j$ is an equality of $\phi$'s in $L^2$.

He made a rather improper remark about analysts here, but I shall not record it.
Recall last time \(X\) was compact Hausdorff, \(\mu\) a measure on Borel subsets of \(X\), 
\(\mu([K])=\mu(K)\) of \(K\). 

\[
\begin{align*}
H &\subseteq L^2(X) & L_2(H) &\subseteq L^2(X \times X) \\
T_K &\subseteq K(x,y)
\end{align*}
\]

an iso of Hilbert spaces. \((T_K f)(x) = \int K(x,y) f(y) dy\)

\[
T_K^* = T_{K^*} \quad K^*(x,y) = \overline{K(y,x)}
\]

\[
T_{K_1}^* T_{K_2} = T_{K_2^* K_1^*} \quad (K_1 \ast K_2)(x,y) = \int K_1(x,z) K_2(z,y) dz
\]

**Prop. 2.** If \(K_1, K_2 \in L^2(X \times X)\) & \(K = K_1 \ast K_2\) then \(T_K\) is trace class, &

\[
\text{tr } T_K = \int_X \int_X K(x,y) dx dy
\]

Last time he said he was going to deduce it from a very complicated thing. But in fact this is easy to prove, as he now realizes.

\[
\text{tr } T_K = \text{tr } T_{K_1} T_{K_2} = (T_{K_1}^* T_{K_2}) = \int (T_{K_1} + T_{K_2}) = \int K_1(x,y) K_2(y,x) dx dy
\]

By Fubini Thm, \(\int (K_1 \ast K_2)(x,y) dx\) exists & equals this.

So we are done. \(\Box\)

2.3. Hilbert & admissible reps:

Reps of Hilbert spaces. Everything seems to be a mess in the literature. There don't seem to be any general theorems - you just prove what you need when you need it. He will try to do stuff in some generality.

\[
G = G(A), G(A) \cdots, G(F_\mu) \cdots \\
G = G^* G^*_\mu.
\]

Fix \(K_{\infty} \leq G^*\) max spc\(pt\), \(U \leq G^*\) fixed spc\(pt\) open.

\[
G_0 = \text{Lie}(G_{\infty}) \\
U = U(G_0), \quad \partial = \text{centre of } U.
\]
f : G \to C \text{ is called smooth if given } g = g^x g^s \in G \text{ find } U \text{ of } g^x \text{ s.t. } U \times G \to G \text{ is a smooth map.}

This may well just be "of loc. est @ finite places & smooth @ oo places".

1) \eta : G \to G_\mathcal{H}, \ H \text{ a Hilbert space } \& \eta : G \times H \to H \text{ etc.}

Wallach calls this a Hilbert rep.

If \im \eta \subseteq \text{unitary autos} \; \text{then } \eta \text{ is unitary.}

Also assume that \eta : G_\mathcal{H} \to \text{unitary autos} \text{ this can always be achieved by varying the inner product without varying the topology} \text{ see e.g. Wallach 1.4.8}

If \phi \in C_c(G) \text{ then we can define } \eta(\phi) : H \to H \text{ s.t. }

\( (\eta(\phi)w, v) = \int_G \phi(g) (\eta(g)w, v) \, dg \quad \forall w, v \in H \)

2) \( G_\mathcal{H} \times (g_\mathcal{H} ; \mathcal{K}_\mathcal{H}) \)-module

\( V \) a vector space. \eta : \left\{ \begin{array}{c}
G_\mathcal{H} \times \mathcal{K}_\mathcal{H} \to \text{Aut}(V) \\
g_\mathcal{H} \to \text{End}(V)
\end{array} \right. 

s.t.

\( a) \quad \langle gv | g \in U \times \mathcal{K}_\mathcal{H}, x \in \mathcal{K}_\mathcal{H} \rangle \text{ is s.t.}
\)

\( b) \quad X \in \text{Lie}(\mathcal{K}_\mathcal{H}) \Rightarrow \eta(X)v = \frac{d}{dt} (\eta(\exp(tx))v) \bigg|_{t=0}
\)

\( c) \quad x \in \mathcal{K}_\mathcal{H}, \ g \in G_\mathcal{H} \times \mathcal{K}_\mathcal{H} \text{ then }

\[ \eta(g) \eta(x) \eta(g^{-1}) = \eta(g^x X g^{-1}). \]

\textbf{Aside} \quad \text{If one had a Hilbert space rep. } U \text{ of } G(\mathbb{F}), \text{ finite.}

\text{then for } v \in H \text{ we say } v \text{ is smooth if } g \mapsto \eta(g)v \text{ is smooth}

\text{in finite } v \text{ finite } \eta \text{ need talk!}

There are the same thing if } v \text{ \& } \eta \text{ \& } U \text{ they might be different.}

\[ v \text{ smooth } \eta \text{ finite } \eta \text{ need talk!} \]
Sometimes you need to have just 1 or the other - eg v just smooth.

Smootheness is not under the group action of $g$, but it turns into $G^w$-smoothness.

3) Say $G \to \text{Aut}(V)$, $V$ a top. v.s., $G \times V \to V$ contr.

A smooth rep is one s.t. $\forall v \in V$, $g \to \rho(g)v$ is smooth.

People usually assume $V$ is a Fréchet space or something.

We don't really want to talk about all this.

4) If $(\pi, W)$ is a Hilbert rep, $H : \mathcal{D}(\omega)$, or $\mathfrak{h}$, unit rep of $K_w$.

$$H(\omega) = \sum_{\theta \in \mathcal{H}} \text{Im}(\theta)$$

$c$ : Wallach.

See e.g. [W] 1.4.7.

$\tau$ is admissible if $\dim H(\omega)^W < \infty \; \forall \omega, W \in G^w$ open opn.

5) $V$ a $G^w \times (\mathfrak{g}_w, K_w)$-module. $V = \oplus_{\theta} V(\omega)$ as above.

$V$ is admissible if $\dim V(\omega)^W < \infty \; \forall \omega, W$ as above.

NB we get $\tau \in G_0 \to \text{End}(V)$ in 3; $\tau(x)v = \frac{d}{dt}(\rho(\exp(tx))v)|_{t \to 0}$.

**Lemma 3**

1) $V$ a smooth rep $\to V^\circ = \{v \in V \mid \text{dim } \langle K_w v \rangle < \infty \}$ is a $G^w \times (\mathfrak{g}_w, K_w)$-module.

2) $H$ a Hilbert rep. Set

$$H^w = \{v \in H \mid g \to \rho(g)v \, \text{smooth} \}.$$ Then $H^w$ will be a Fréchet space (and you put some norm on it).

If $G = G_0$, then $H^w$ can be given the structure of a smooth rep.

3) If $H$ a Hilbert rep, define $H^w(H^w)^w$. Then $H^w = G^w \times (\mathfrak{g}_w, K_w)$-module.

$H^w$ is dense in $H$.

4) $H$ is admissible $\iff H^w$ is admissible.

In this case, $H^w = \{v \in H \mid \langle K_w v \rangle < \infty \}$.

He'll sketch some rts.
1) You just need to check that $K_0$-finiteness is preserved by $g_0$.

But $g_0 \circ <K_0, v> \to V$ has f.d. image, preserved by $K_0$.

2) [WI 1.6.4] You see $H^0$ will probably be dense in $H$ so you need a new norm & you take some sequence of norms somehow.

3) $H^0$ is dense in $H$ is the only tricky part.

Anyway, $H^0 \cong \oplus H(\sigma)$ (pointwise equality holds here?!) You will have to show $H(\sigma) \cap H^0$ is dense in $H(\sigma)$.

Choose $\varphi_0 \in C^\infty(G)$ s.t. $\text{supp}(\varphi_0) \supset \text{supp}(\varphi_{\alpha})$, $\varphi_0$ is real-valued.

\[ \bigcap \text{supp}(\varphi_0) = \{ e \} \]

\[ \int \varphi_0 = 1 \quad \varphi_0(1) = \varphi_0(x^2) \]

\[ \varphi_0(keK_0, g G) = \varphi_0(g) \quad \forall k \in K_0, g \in G \]

\[ \hat{\pi}(\varphi_0) H(\sigma) \cong H(\sigma) \]

\[ \hat{\pi}(\varphi_0) H \cong H^0 \]

If $v \in H$ then $\hat{\pi}(\varphi_0) v \to v$.

4) There's only the problem that some $H(\sigma)$ may grow.

Pick $W \subset G^\times$ open set, $\sigma$ f.d.

Then $H(\sigma)^W \geq H^0(\sigma)^W \geq H(\sigma) \cap H^0)^W$ dense

so $H^0(\sigma)^W$ must also be f.d. \[ \square \]

He hopes he's given us the idea.

**Lemma** (He won't prove this one) $V \leq H^0$ int subspace, $V$ is $G$-invariant \[ \Rightarrow \text{V is G-invariant} \]

**Proof** Waldh automatic 3.69 & 166. It's quite deep. Use the fact that there's some elliptic differential operator so all of $V$ are well-behaved. \[ \square \]
**Lemma 5** Suppose $H$ is unitary irreducible, $V \subset H$ dense $G$-module subspace.

T. $V \subset H$ commutes with $G$-action

\& suppose $f : V \to H$ st. $(Tx, y) = (x, Sy)$ \forall x, y \in V.$

Then $T$ is a scalar.

**Proof** [W] 1.22

There last 2 lemmas & the next one are due to Harish-Chandra.

**Lemma 6** $V$ a $fg. (G_0, K_0)$-module (ie $V_{11}, V_{12}$ st. only submodule $V_{11}$ is $V$)

consisting of $g$ finite vectors. Then $V$ is admissible. This is the case if $V$ is irreducible.

**Proof** [W] 3.49

Lecturer 6

Har 22 Feb 93

1:00 am

Last time Richard was talking about the relationship between reps of $G_1$

$G = \text{eg. } G_0, (M), G_0(F_r)$ - we wanted $G = G_0 \times G_0, K \subset G_0 \text{ and split, } G_0 \times \text{Lie}(G_0).$

We had Hilbert reps & smooth reps & sometimes admissible reps & stuff.

& we had a dictionary $H \to H^0.$ & with some $g$-finite we could get back or something.

We did some funky lemmas last time eg Schur's lemma. Note that in. Lemma 6 last bit we use the amazing fact that Schur's lemma holds if dim $V < 2^\infty.$

**Corollary 7** $H$ irreducible unitary $\Rightarrow g$ acts by a character $K_H$ on $H^0.$

Hilbert rep & for $G_1$ $\Rightarrow \text{Hilbert rep & for } G.$

$K_H : g \to c$ is the infinitesimal character.

**Proof** Use Lemma 5 applied to $H^0.$

\[ \begin{align*}
& y \in G(m) \\
\text{ie. } & z \in G(m) \\
& z = n(z) \\
& S = n(z) \\
\end{align*} \]

where $\star : U \to U$ & on $G(m) \star z = x \to -x.$

(Note $(xy)^* = y^* x^*.$)

(see eg Wallach 1.5.5)

NS Richard doesn't know what happens if you remove the unitary cond. He has no feeling for the subject really. I don't think he knows either but he does look deep in thought. There's no "lemmas for the algebraists" - all the results have nasty analysis warts in like unitary.

**Corollary 8** $H$ irreducible unitary $G_1$-module $\Rightarrow H$ admissible. (NS lecture says also true if for $G_0,$ but we're already for $G.$)

**Proof** $H^0$ is $g$-finite by Cor. 7. If $0 \neq H^0,$ let $V \subset H$ $\subset (K_0)^G \subset V^H.$

Lecture 6 $\Rightarrow V$ admissible

**Lemma 4** $V$ is $G_1$-invariant $\Rightarrow V \subset H, V(0) \subset H(0).$ & we're rep of $K_0.$

$\Rightarrow H(0) : V(0) \neq 0.$
He had hoped to get everything from \( \mathcal{V} \), lemma 4.5.6, all in Wallach. He now realizes he needs more than lemma 4 for the next bit, so he'll have to be reviled to Hanisch-Chandra. No, it may be too bad. He only realized all this this morning and he's just reviled to Hanisch-Chandra for completeness/observation.

**Lemma 9** \( \mathcal{V} \) is a smooth rep of \( \mathcal{G} \), \( \forall \mathcal{V} \in \mathcal{V} \) \( \mathcal{V} \) is \( \mathcal{g} \)-finite.

Then \( V \in \mathcal{C} \), \( \mathcal{G} \) s.t. \( V(Xp) = V \)

**Hansich-Chandra:** reps of semisimple Lie gops are Banach spaces \( \mathcal{V} \)

**Cor 10** Say \( H \) a Hilbert space rep of \( \mathcal{G} \). Then \( H \) \( \mathcal{g} \)-admissible \( \mathcal{H} \) \( \mathcal{g} \)-admissible. 

(\( \Rightarrow \) clear. (\( \Leftarrow \)) Suppose \( H \) \( \mathcal{g} \)-admissible. Use the trick that John used this morning.

If \( W \leq H \) is a proper submodule, then \( \exists \mathcal{V} \in \mathcal{W} \) an \( \mathcal{g} \)-admissible submodule.

The proof of this uses Zorn: for some \( \mathcal{g} \)-admissible rep of \( \mathcal{K}_0 \times \mathcal{K}_0 \), open \( \mathcal{K}_0 \cdot \mathcal{g}_0 \) (rep in \( \mathcal{G} \)) we have \( W(0) \leq \mathcal{Z} \) but do \( \mathcal{g} \)-admissible, so choose \( \mathcal{V} \leq \mathcal{W} \) min s.t. \( W(0) \leq \mathcal{Z} \).

(\( \mathcal{g} \) acts by scalars on \( V \) (Schur))

\[ V \leq \mathcal{H} \Rightarrow V(0) \leq \mathcal{H}(0) \Rightarrow V(0) \oplus \mathcal{H}(0) : V = \mathcal{H} = \mathcal{H} \oplus \mathcal{H}(0). \]

**Cor 11** If \( H \) is a Hilbert rep of \( \mathcal{G} \) and suppose \( \mathcal{H} \) \( \mathcal{g} \)-admissible \& \( \mathcal{g} \)-finite

\[ \{ \text{submodules} \} \xrightarrow{\text{closed unit subspaces}} \{ \mathcal{H} \leq \mathcal{H} \} \]

\[ V \xrightarrow{(\mathcal{H}^0)} \mathcal{V} \text{ (int by Lemma 6 or sth) \( \Rightarrow \) } \mathcal{H} \]

**Lemma 12** \( \mathcal{H}_1, \mathcal{H}_2 \) \( \mathcal{g} \)-admissible unitary. If \( \mathcal{H}_1 \) \( \mathcal{g} \)-admissible, \( \mathcal{H}_2 \geq \mathcal{H}_1 \) then \( \mathcal{H}_2 = \mathcal{H}_1 \), \& even the inner product is preserved.

(\( \Rightarrow \) easy - see e.g. Wallach chapter 3. Use the adjoint map; if \( \mathcal{H}_1 \to \mathcal{H}_2 \) we get \( \mathcal{H}_1(0) \to \mathcal{H}_2(0) \) \( \Rightarrow \) \( \mathcal{H}_1(0) \to \mathcal{H}_2(0) \)

(\( \mathcal{g} \cdot \mathcal{g}_0 \cdot \mathcal{H}_2 \not= \& \text{ Schur = scalar. } \text{ We did } \mathcal{g} \text{ preserve inner product; extend by continuity } \text{ & check the action preserved. We'll span us the details to same time.} \)
Lemma 13. Let $G = G(F_v), v$ finite or infinite, $H$ unitad, $\varphi \in C_c(G)$.

Then $\pi(\varphi)$ is trace class, and in fact $tr(\pi(\varphi))$ only depends on the corresponding rep $\pi^0$ (so we can write it $\pi(\varphi)$).

Proof. First observe (since we have assumed $H^0$ is unitary)

$$\forall \varphi, \psi \in C_c(G): tr(\varphi \psi) = tr(\varphi \psi) = \int_{G} \varphi(h) \psi(h^{-1}g) dh.$$ 

Lem ms 14. (Not really about reps but it's analytic so I'll throw it in here)

If $G = G(F_v), \varphi \in C_c(G)$, then $\exists \varphi_1, \varphi_2 \in C_c(G)$ for $v_1, v_2$.

where

$$\varphi = \sum_{i \in I} \varphi_i \varphi_i.$$ 

Proof. For any. Jean-Pierre did it so ask him

Lemma 15. Suppose $\pi$ is an irreducible unitary rep of $G(F_v)$. Let $\pi^0$ be the associated $G(F_v^0) \times (\mathfrak{g}_{F_v}, K_v)$-module. Then $\pi^0 = \otimes^r \pi$, with $\pi$, irreducible, uniquely determined, & unramified almost everywhere.

He'd rather hoped John would have told us about this bit this morning, but it appears still be this afternoon.

Then $\pi^0 = (\pi^0)^v$ for some (by determined) irreducible unitary rep $\pi_0 \in G(G(F_v^0))$. (Quick ref: embed $\pi^0$ in $\otimes \pi$ & take its closure, then use earlier facts)

Then $\pi = \otimes \pi_0$, & if $\varphi \in C_c(G(F_v^0), G(F_v))$, then

$$tr(\pi(\varphi)) = tr(\otimes \pi_0(\varphi)).$$ 

I think because $\varphi$ is core $K_v$ almost everywhere.

He'll tell us what $\otimes \pi$, is: $\pi$ has an or basis $\{e_i\}$, & for almost all $v$ we have $C_{\varnothing} = N_v G^{10^v}$. & 1.

Then $\pi_{\varnothing} = \{e_i\} \mapsto \mathbb{Z}_{20}, \delta : \mathbb{Z}_{20} \to G(F_v)$. Is an or basis for $\pi_{\varnothing}$.

That concludes the analytic results. Now we'll do stuff which, to them at least, seems more interesting.
3. Automorphic forms on $G_3(F) \backslash G_3(A)$

In this section, $F/k$ is a finite ext. & $D/F$ is always a nonsplit quad alg.

3.1. $L^2(G(E) \backslash G(A)^7)$

Set $X = G(E) \backslash G(A)^7$ & $H = L^2(X)$.

Then $G(A) = G(A)^7 \times R^*_\infty \times R^*_\infty$ (the embeddings diagonal at $\infty$)

$G(A)$ acts in a unitary way on $L^2(X)$.

(density) $(R(g) f)(h) = f(hg)$ (here $g \in G(A)^7$, $f \in L^2(X)$, $h \in X$)

& $R$ is trivial on $R^*_\infty$.

If $\varphi \in C_c^\infty(G(A))$ then define $(R(\varphi) f)(h) = \int_{G(A)^7} \varphi(g) f(hg) \, dg$

"Hecke operator"

Note $R(\varphi)^* = R(\varphi^*)^*$ where $\varphi^*(g) = \overline{\varphi(g^2)}$

$R(\varphi_1) R(\varphi_2) = R(\varphi_1 \varphi_2)$, $(\varphi_1 \star \varphi_2)(g) = \int_{G(A)^7} \varphi_1(h) \varphi_2(hg) \, dh$

Define $K_\varphi : X \times X \to C^*$ by

$K_\varphi(g, h) = \sum_{x \in G(A)^7} \varphi(g x h)$ (note defined for $X$ not just $G(A)^7$)

If $g, h$ lie in some cusp then the sum is finite :: $K_\varphi \in C_c(X \times X)$.

Lemmas: If $D$ is not split then $K_\varphi \in C_c(X \times X)$ (note this is false if $D$ split), &

that's what makes the whole theory harder)

and $T_{K_\varphi} = R(\varphi)$.

In particular, $R(\varphi)$ is Hilbert-Schmidt & hence cusp.

By $(R(\varphi) f)(g) = \int_{G(A)^7} \varphi(h) f(hg) \, dh$ (he's probably integrated out the $R^*_\infty$ fibre)

$= \int_{G(A)^7} f(h) \varphi(g^2 h) \, dh$ (change of variable)

He might be assuming $\varphi \in C_c^\infty(G(A)^7)$.
He thinks this is WLOG.
\[ (R(\psi)f)(h) = \int \sum_{\mathbb{G}(F)\backslash \mathbb{G}(A)^2} \hat{f}(gh^{-1}) \psi(h^2) \, dh \]
\[ = \int \hat{f}(h) K_p(h,h) \, dh \]

This afternoon he'll discuss the ramifications of this point.

Lecture 7
Nov 22nd 2016
3:45 pm

Recall F a finite field, D/F a non-split quad alg, \( \text{g.s.t.} \ G(F) = D^2 \)

We're looking at \( L^2(\mathbb{G}(F)\backslash \mathbb{G}(A)^2) \)

He should remark that this all ties up with what John's dang

\( \psi: \mathbb{G}(A)\backslash \mathbb{G}(F) \rightarrow C \) given in \( S_k \in L^2(\mathbb{G}(A)\backslash \mathbb{G}(A)^2) \)

& the analysis he was doing earlier relates the analytic \( L \) with the more algebraic \( S_k \).

Now if \( \psi \in C_c(G(A)) \), we can contract \( \hat{\psi} \) by \( \mathbb{G}(g) := \int \hat{\psi}(gh) \, dh \)

so because of this remark which is a slightly more rigorous remark version of a remark he made earlier, we reduce ourselves to the case \( \psi \in C_c(G(A)^2) \).

Then define \( (R(\psi)f)(h) = \int \psi(gh) \hat{f}(h^2) \, dg \)

\[ K_p := \sum_{T \in \mathbb{G}(F)} \psi(gh^2) \epsilon(T) \in C(X \times X) \]

**Lemma 1:** (D not split) \( T_{K_p} = R(\psi) \) & so \( R(\psi) \) is H-S & c.p.

We will try to understand the \( L^1 \) space like John's done \( S_k \) questions to ask are one is it ss? Can we decompose into local bits?

**Prop 2:** (D not split) \( L^2(\mathbb{G}(F)\backslash \mathbb{G}(A)^2) = \bigoplus_{\pi} \pi \otimes \chi^* \)

\( \pi \) distinct, \( m_{\pi} \in \mathbb{Z}_{\geq 0} \), \( \pi \) with multiplicity \( \chi \). Well just shown, will be a variant on the pf John gave this morning, we don't have admissibility, but he thinks there is a variant of the pf which will prove admissibility.
Step 1. Suppose $0 \notin H$. Then $f \in C_c(G(\mathbb{R})^2)$ s.t. $\varphi(\varphi^*)$ and $R(\varphi)f \neq 0$.

Let $U$ be an open nbhd of 1 in $G(\mathbb{R})^2$ s.t. $\forall \epsilon > 0 \exists U \ni \|r(w) - f\| < \epsilon \|f\|$

Choose $\varphi \in C_c(G(\mathbb{R})^2)$ s.t.

- $\text{supp } \varphi \subseteq U$
- $\varphi^* = \varphi$
- $\varphi$ non-real-valued
- $\varphi^* \cdot 1$

Choose $U_1, U_2$ s.t. $U_1, U_2 \subseteq U$, s.t. $\varphi_1, \varphi_2$ s.t. $\text{supp } \varphi_1 \subseteq U_1$, $\text{supp } \varphi_2 \subseteq U_2$.

Let $\varphi_1, \varphi_2$ be real-valued, and set $\varphi = \varphi_1^* \varphi_2$

Now $\|R(\varphi)f - f\| = \| \int_{G(\mathbb{R})^2} \varphi(g) R(g)f - f \|$

$= \| \int_{G(\mathbb{R})^2} \varphi(g) (R(g)f - f) \|$

$\leq \int_{U} \| \varphi(g) \| \| R(g)f - f \| \, dg \leq 1 \cdot \|f\|$

$\therefore R(\varphi)f \neq 0$. \hfill \Box$

Step 2. Suppose $0 \notin H \subseteq H$ is a closed unit subspace, then $H_1$ contains a closed unit subspace.

Choose $0 \neq f \in H_1$ & $\varphi$ as in step 1. Then $R(\varphi)|_{H_1}$ is non-zero & self-adjoint.

Let $V \subseteq H_1$ be an eigenspace for $R(\varphi)$ with a non-zero eigenvalue. Then by the spectral theorem, $\text{dim } V = \infty$.

Now $H_0 \subseteq H_1$ be a mini-closed unit subspace with $H_0 \cap V = \{0\}$. (Zorn's lemma)

He claims $H_0$ is closed. If not then $H_0 = H_0 \oplus H_2$ with $H_2$ closed unit.

Then $H_0 \cap V = (H_0 \cap V) \oplus (H_2 \cap V)$ as $R(\varphi)$ preserves $H_2 \& H_1$

for $i = 2, 3$ we have $H_i \cap V = H_i \cap V$ so $H_i = H_i$. \hfill \Box$

Now things are rather easy.
Step 3. \( H_0 \oplus \pi, \pi \) irreducible

\[ \text{If } \exists \text{ a maximal subset of } \{ W_i \mid W_i \text{ closed subspaces s.t. } \oplus W_i \subseteq H, \text{ (Zorn)} \] \[ \text{Then } H = H_0 \oplus (\oplus W_i). \text{ If } H_0 \cap (0) \text{ then } \exists H_0 \subseteq H, \text{ closed, s.t. } \] \[ W_i \subseteq \{ W_i \} \supseteq \{ W_i \} \neq \{ \} \]

Step 4. No \( \pi \) occurs with infinite multiplicity.

\[ \text{If it did, then choose } O \neq 0 \to H. \]

Choose \( \pi \in C_c(G(A^{\infty})) \) s.t. \( R(\pi) \neq 0 \)

Let \( X \) be a non-zero eigenvalue of \( R(\pi) \). If \( \pi \) occurs infinitely often, then the \( X \)-eigenspace of \( R(\pi) \) is \( \infty \)-dual. \( \square \)

Note there was some content in all of this that \( R(\pi) \) was split. In the split case \( \pi \) it's false. In fact

Prop. The result is false for \( L^1(G(F) \setminus GL(n,F)) \).

However, \( L^1(G(F) \setminus GL(n,F)) \) is irreducible, so in this case where \( L^1(F) \) have the added property that \( \int gl((b, g)) \text{ d} \mu = 0 \) for almost all \( (b, g) \) \( \text{(NB, he doesn't put "for all" on these } L^1 \text{-f.s. are only defined up to some } f \text{ which is } 0 \text{ a.e. }). \)

\[ \text{The error is the theory of } \]

\[ \text{core } L^1(G(F) \setminus GL(n,F)) = \hat{H}^\infty, \text{ where } x : y \to c, & y \text{ acts on } (H^\infty) \text{ via } x. \]

Here \( y \) is the centre of \( U \). (Inc A)

There's some debate now as to whether \( \int gl((b, g)) \text{ d} \mu = 0 \) makes sense for almost all \( g \).

Richard has never really thought about it before. He thinks Fulvio then kills it. Brian Bush thinks everything so. OK. Gle is, of course, really John's problem.

Prop. 3. Suppose \( D \) is not split. Then \( H^\infty \) is admissible. Yes.

Say \( U \subseteq G(F) \) an open split subgroup. Look at \( H^\infty \) as a \( U \times GL(F) \)-module.

Apply the argument of prop. 2.
Then $H^u = \hat{\otimes} \pi^{m_u}$, \( m_u < \infty \)

\[ (H^u)^u = \hat{\otimes} \pi^{m_u} \]
\( m_u < \infty \).

I think the idea is that unitary \( \to \) K-finite \( \Rightarrow \) admissible \( \to \) only finitely many choices (\( \delta \approx \delta_g \)).

If only finitely many \( \pi \) with infinitesimal char \( \chi \).

The sum is finite.

Each \( \pi \) is admissible.

\[ (H^u)^u \] is f.d., for all irreps of \( K_u \).

\( H^u \) admissible.

\[ \text{Coh: Any irreducible constituent } \pi \text{ of } L^2(G(F) \backslash G(A))^2 \text{ is admissible.} \]

\[ \text{Rmk: Prop 3 can remain true for } L^2(G_u(F) \backslash G_u(A))^2 \]

So we've shown precisely what John is doing in the holomorphic case form.

Prop 4 (Trace formula)

If \( D \) is not split, $L^2(G(F) \backslash G(A))^2 = \hat{\otimes} \pi^{m_u}$

\( m_u < \infty \) (distinct).

Then \( \sum \pi m_u \text{ tr } \varphi(\pi) = \text{tr } R(\varphi) \)

\( \text{group via together} \)

(Here \( [\varphi] \) = conjugacy class of \( \varphi \) in \( G(F) \)).

\( \sum \pi \text{ vol}(G(F) \backslash G(A)^2) O_\varphi(\varphi) \)

\( \text{Here } G_y = \text{centraliser of } \varphi \text{ in } G, \& \)

\[ O_\varphi(\varphi) = \int \varphi(g'\varphi g) dg \text{ is an orbital integral.} \]

\( G_y(A) \backslash G(A)^2 \text{ (are normalized?) } \)

We've got to make sure our measures/correspond to get an equality. Fix a Haar measure on \( G(A)^2 \). Pts of \( G_y(F) \) have measure 1. Use the same Haar measure twice for \( G_y(A)^2 \).
So \( \text{tr} \, m(p) = \prod m_r(p_r) \), almost all \( L \)

\[
U_p(p) = \prod \int \theta(g^2 g) \, dg, \text{ almost all } L
\]

\[
G_0(F) \backslash G(F)
\]

The way to think about this equality is

\[
\text{Trace of Hecke operator} = \sum \text{global integrals}
\]

\[
\prod m_r(p_r) = \prod \text{vol} \left( G_0(F) \backslash G_0(A)^r \right) \, \text{tr} \, m(p)
\]

\[
G_0(F) \backslash G(F)
\]

& both sides abs. cgt.

\[
\prod m_r(p_r) = \prod \text{vol} \left( G_0(F) \backslash G_0(A)^r \right) \, \text{tr} \, m(p)
\]

& both sides abs. cgt.

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& both sides abs. cgt.

\[
\prod m_r(p_r) = \prod \text{vol} \left( G_0(F) \backslash G_0(A)^r \right) \, \text{tr} \, m(p)
\]

& both sides abs. cgt.
Tony will be explaining an application of the trace formula later in the week.

3.2 Automorphic Forms

He stresses again that $\mathcal{D}$ is not split.

For $\chi: R_{x_0}^+ \to \mathbb{C}$. Then

$$\text{Def}: \quad \mathcal{A}_\chi = \left\{ \eta \in G(F) \backslash G(\mathbb{A}) \to \mathbb{C} \mid \begin{array}{l}
\eta \text{ smooth}, \\
\eta_{|U} \text{ U-infinite} \quad \text{ if } U \subset G(F) \text{ open flat,} \\
\eta \text{ } J \text{-finite} \quad \text{ here } J \text{ the } J \text{-finite set (of } \mathfrak{a} \text{) in } G(F) \\
\text{ and } \eta \in \text{ center of } \mathbb{A} \\
\text{ and } \eta \text{ restricted to } \eta_{|R_{x_0}^+} \to F_\mathbb{C} \in G(F_\mathbb{C})
\end{array} \right\}$$

If you take the sum over all unitary char $\chi$, you get something analogous to $\mathcal{A}^0$ or $\mathcal{A}$ or something. Boundedness & stuff all come from compactness. $\eta$ will be bounded in $G(\mathbb{A})^2$ & slowly increasing in $G(\mathbb{A})$ if $\chi$ non-unitary. (Bounded in $G(\mathbb{A})$: if $\chi$ is unitary.)

Also if compact & cocharacter $\eta_{|U}$ has no analogue in $G(F)$.

Set $\mathcal{A} = \mathcal{A}^0$-local. Then $\mathcal{A}_\chi = \mathcal{A} \otimes \chi_0 \| \chi \|^1/2$. Let $L: [F: Q]$

$$\text{Prop}:
\begin{align*}
\mathcal{A} = \mathcal{A}^0 \otimes \mathcal{A}^0, \\
L^2(G(F) \backslash G(\mathbb{A}))^{0,J\text{-finite}} = \mathcal{A}^0
\end{align*}
$$

& $\mathcal{A}^0 \simeq \bigoplus_{\mathfrak{a}} \mathbb{C}(\mathfrak{a})^{\mathfrak{a}}$ U-infinite vectors (no smoothing necessary, recall)

$\Rightarrow \mathcal{A} = \bigoplus (\mathfrak{a})^{\mathfrak{a}}$ (a big direct sum)

Also, $\mathcal{A}^0$ unit adjoint $\Rightarrow \mathcal{A}^0 = \bigoplus \mathcal{A}^0$. 

$\mathcal{A}^0$ admissible.

Write Aut$(G_0(\mathbb{A})) =$ the automorphic reps of $G_0(\mathbb{A})$ for the set of automorphic reps $\mathcal{A}$ occurring in any $\mathcal{A}_\chi$. 

\[ \alpha: \gamma \to \mathbb{C}; \quad \mathcal{A} = \bigoplus \mathcal{A}_\chi, \quad \mathcal{A}_\chi \text{ admissible} \]
In the GSpin case, we can define \( \Phi^0_x \) - cuspidal automorphic forms on \( GL_n(F) \) & all the remarks above remain true.

To the costs in our def of \( \Phi^0_x \) above you would add (1) if cuspidal (2) if slowly increasing.

Then \( \Phi^0_x \otimes \Phi^0_x \) unramy.

Write \( \text{Aut}^0(GL_n(A)) \).

**Theorem 5 (Jacquet-Langlands)**

They only sketched the details of the trace formula they needed.

Arthur completed the proof while he was doing this, a chap called something like Monizo also proved the thing in another way.

The thing says that it's really rather similar whether you use \( D' \) or \( GL_n \).

1. If \( F \) a local field, \( D, D' \) the non-split quad alg/F; then JL define an injection from indec. adms. reps of \( D' \) to indec. adms. reps of \( GL_n(F) \); st.
   a) the image consists of all discrete series reps:
      \[ \psi \mid \frac{1}{2} \] only get special or supercuspidal
   b) \( \psi \langle \chi \rangle \quad JL(\chi, \nu) = \sigma(\chi \cdot \nu^k, \chi \cdot \nu^{-k}) \)
   c) \( \psi \langle \chi \rangle \quad (H \hookrightarrow M_{1}(C) \text{ standard}) \)

\[ JL(\text{symm}^k(\nu) \otimes \chi^\gamma) = \sigma(\chi^{1+h^2}, \chi^{1-h^2}, (\text{sgn} \chi)^k) \]

\[ \nu \in \mathbb{Z}_{\geq 0}, \sigma \mathbb{F}, \text{ these are all the indec. adms. reps of } D' \]

Note that JL hasn't defined \( D' \) completely yet if \( \psi \langle \chi \rangle \), but see d) below.

It leads easier if you use JL local Langlands, not JL local rep of \( WD_x \) instead.

1. The global part of the thing holds as well.
   (This tells us why about the supercuspidal case, \( \psi \langle \chi \rangle \)
2) If $F$ is a global field, $D/F$ a non-split quad alg.

Then a) $\pi_D = \pi' \in \text{Aut}(G_p(M))$.

b) either $\pi \cong \chi \circ \text{det}$ for some $\chi$, or

$$\exists J_L(\pi) \in \text{Aut}^*(GL_v(M)) \quad \text{(note - cuspidal only for $\text{rep}$s)}$$

s.t. $J_L(\pi) = \left( \otimes_{v \in S} \pi_v \right) \otimes \left( \otimes_{v \in \text{es}(S)} J_L(\pi_v) \right)$

c) The image of $J_L$ is all $\chi$ of $\text{Aut}^*(GL_v(M))$ s.t.

$\rho$ is discrete since $\forall v \in S(\chi)$.

**Thm 6 (Jacquet-Shalika)**

Suppose $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \in \text{Aut}(G_p(M))$ are co-dim $1$.

Suppose $\exists$ finite set $S$ of bad primes containing all $\infty$ primes &

all $v$ s.t. $\pi_v$ or $\pi_v$ ramifies, & all bad primes of $D$ too, but maybe

Then $\forall v \in S$, $\pi_{i,v} = \{ \alpha_{i,v}, \beta_{i,v} \} \in \text{C}^x$

$\pi_{i,v} = \{ \alpha_{i,v}, \beta_{i,v} \} \in \text{C}^x$

If also $\forall v \in S$, diag$(\alpha_{i,v}, \beta_{i,v})$ & $\text{diag}(\alpha_{i,v}, \beta_{i,v}, \omega_v, \rho_v)$

are conjugate in $\text{GL}(\text{C}^x)$

Then $\pi_{1,v}, \pi_{2,v}$ is a permutation of $\pi_{4,v}, \pi_{5,v}$.

The same is true for $\text{Aut}^*(GL_v(M))$ $\text{rep}$ is Stow Multiplicity 1.

I think he said silly about proving it for $GL_v$ then using $JL$ coprop.

3. Examples

This is just a $h$ excuse. If you understand things it shouldn't be too hard.

$F = \mathbb{Q}$, $\sigma = 3$

$S(2) = \{ \infty, p \}$

$$\oplus_{\pi \in \text{Aut}(G_p(M))} \pi = \{ \xi: \mathbb{D}^\infty(\mathcal{D}(A)^{\infty}) \to \text{C} \mid \xi \text{ right invert by some open neighborhood} \}$$

Also $S = \oplus_{\pi \in S} \pi^{\infty} = \{ \xi: \mathbb{D}^\infty(\mathcal{D}(A)^{\infty})/\mathcal{D}(A)^{\infty}_p \to \text{C} \mid \xi \text{ right invert by some open neighborhood} \}$

$\pi_p$ is trivial on $\mathcal{D}(A)^{\infty}_p$, $\epsilon$ is even root in $D^{(1)}_p$. 
If $x \in D^+$ s.t. $v_{x,y}(x) = 1$, then $R(x) = 1$ (so $x$ & $(R(x))$ has even val.)

\[ S = S^+ \oplus S^- \]

If $U = \bigoplus_{q \to q^+} U_q^+$ then:

\[ S^U = \left\{ f : RIC(O_q) \to \mathbb{C} \right\} \]

& \[ (S^+)^U = \left\{ f : MO(D) \to \mathbb{C} \right\} \]

\[ S = \text{conjugacy classes of } \text{max}^+ \text{ ideals} \]

After some discussion with Friedl, R conduct now believe that for $I, J \in RIC(O_p)$, even if $I \neq J$, we have $I \otimes O_p \cong J \otimes O_p$.

Use the Jacquet-Langlands then to deduce

\[ S^U \cong S_1(\Gamma_0(p)) \cong \mathbb{T}_p : \mathbb{T}_p^* = 1 \text{ & Hecke equivariant, } (S^U)^U = S_1(\Gamma_0(p))^U. \]

These maps are Hecke equivariant: $T_q$, $S_q$ for $q \mid p$ $q \mid p$, $S_q$ acts trivially on both sides.

\[ (T_q f)([J]) = \sum_{[I]} J \in \Gamma_0(p) \left\{ f([J]) \right\} \]

This sum is our qth things.

It may well be (Brandt mainly) that this stuff above was known before JL. JL is really rather beautiful way of seeing it.

Ex. $p = 11$; dem $S^U = Z$- get $[O_{11}]$ & $[I_{11}]$.

$[O_{11}] - [I_{11}]$ is an eigenclass & it's the one that survives when you mod out by it. $T_q$ has eigenvalue $a_q = \# \{ J \in O_{11} : \left[ \begin{array}{c} J \\ O_{11} \end{array} \right] \in \varphi^q \} - \# \{ J \in O_{11} : \left[ \begin{array}{c} J \\ O_{11} \end{array} \right] \not\in \varphi^q \}$

So we get the eigenvalues explicitly in terms of the arithmetic

We also know...
$y^2 + y = x^2 - x^1$ is related to all this.

So, $y^2 + y = x^2 - x^1$ has $q - q_1$ pts / #.

\[ \left| \frac{q^{1/2}}{2} - \# \left\{ J \in \mathcal{O}_2 \mid C_{O_2}J = q_2 J \right\} \right| < \sqrt{q} \]

He had intended to say sthg about the indefinite case, but he's got no time so he'll stop now.
Tomorrow at 9:30 by Richard XV

Wed
11:00 VII Peter Schneider
2:30 IV
4:00 III (Tony)

Thu
9:30 VI
11:00 VI
1:30 VIII (Labor)
3:00 VII (Tony)

Fri
9:30 I (prelim)
14:00 II (find Tony)
2:30 VIII
4:00 IV

Sat
9:30 VIII
11:00 IX
2:00 VIII
3:30 IX

Peter Schneider has lost a lecture.

Thursday @ 4:30 Hill Lane m 9. Francis Greene Kirwan will be talking about quotients of varieties or sth & Abiyah-Jones conjecture.

In these final lectures we will be understanding how the trace formula helps us understand base change.

\[ \pi \text{ of } G_{L}(F_{E}) \rightarrow \text{ rep } \rho \text{ of } G_{L}(F/F) \]
\[ \pi_{E} \text{ } \leftarrow \text{ } \rho_{|_{G_{L}(F/E)}^{G_{L}(F/E)}} \text{ of } E/F. \]

The construction \( \pi \rightarrow \pi_{E} \) is base change.

It has implications for the Artin conjecture (M. Harrisson).

Matis Taylor did enough for \( G_{L} \) to convince us that it is true in this case.
a gross-char of $\chi_r$ of type $A_c$ of $\text{Gal}(F/F)$

$\chi_r|_{\text{Gal}(F/F)}$

$\chi = \chi_{E/F} \text{ a gc of } \chi^e$ of $\text{Gal}(F/E)$

To understand the case $n=2$ & maybe general $n$? we use this trace formula.

**Orbital Integrals**

4 lectures:
1) Norms & $\sigma$-conjugacy
2) Matching orbital integrals
3) $\phi$
4) geometry of orbits

*Heil* "look after the bad places."

**Norms & $\sigma$-conjugacy**

Say $F$ is a finite ext. of $\mathbb{Q}$ or $\mathbb{Q}_p$ or $\mathbb{R}$

$E/F$ finite; $[E:F] = l > 2$ prime (Langlands also treats $l=2$)

$E = F^l, \text{not a field!} \text{ or } E$ a field; $E/F$ Galois, $G(E/F) = \langle \sigma \rangle$, $\sigma^l = 1$.

If $E = F^l$ then $\sigma$ cyclically permutes the coordinates.

He wants to consider $D/F$ a quaternion algebra, split or non-split.

for $D \rightarrow F$, $\psi: D^* \rightarrow F^*$

**Def:** 1) $x \in D$ is central $\iff x \in F$

2) $x \in D$ is regular $\iff T^2 - (\epsilon x)T + x^2$ has distinct roots. ($= F[x]$ is a quadratic field ext. of $F$, or $\mathbb{R}$)

3) $x \in D$ is semisimple $\iff x \text{ central or regular}$

Note that $D$ not split $\Rightarrow$ all elt. of $D$ are so.

**3)** $D_E = D \circ E$ & $M_E = M \circ E$ if $M \circ D$.
4) If $x, y \in D$, then $x, y$ are conjugate $\iff \exists \alpha \in \text{D} \text{ s.t. } \alpha x \alpha^{-1} = y$.

**Lemma 1**

1) $x, y \in D$ are conjugate in $D_E \iff x \sim y$ in $D$
2) If $x \in D_E$ & $x \sim \alpha x \Rightarrow x \sim y, y \in D$.

**Proof**

This is easy but important.

1) i) $x$ central $\Rightarrow x \sim y$
   ii) $x$ not ss $\Rightarrow x \sim (\alpha x)$ for $\alpha$
   iii) $x$ regular $\Rightarrow x \sim y$ by the Norther-Skolem thm.

For $D$

Norther-Skolem $\Rightarrow x \sim y$.

2) i) $x$ central $\Rightarrow x \sim x \Rightarrow x \in F$
   ii) $x$ not ss $\Rightarrow x \sim (\alpha x)$ for $\alpha$
   iii) $x$ regular $\Rightarrow x \sim y$.

Then $E(x)$ splits $D_E$. $E(x)$ splits $D$. $F(x)$ splits $D$.

\[ F(x) \sim D. \quad [\]

This is not at all as bad as it seems. We have to do many things.

**Def**

1) If $x \in D^*$ then $N_x = x^{\sigma_x} \cdots \sigma^{t_x}$ \[\text{Note: } \sigma(N_x) = x^{t_x}(N_x) x. \text{ Thus by the lemma above, } N_x \text{ is}
\text{conjugate to a unique conjugacy class } [N_x] \text{ in } D^*\]

2) $x \in D^*$ is $\sigma$-regular $\iff [N_x]$ regular

$\sigma$-ss

3) $x, y \in D^*$. Say $x, y$ are $\sigma$-conjugate, $x \sim y$. If $\exists g \in D_E$ s.t.

\[ x = g^{-1} \sigma(x g) \]

4) If $x \in D_E$ we define the $\sigma$-centralizer

\[ C_x^* = \{ g \in D^* \mid g^x \sigma(x g) = x \} \]

**Exercise**

1) $C_{h^k x \sigma h^k} = h^k C_x^* h^k$

2) $N(g^{x \sigma g}) = g^{x (N x) g}$ $\iff [N(g^{x \sigma g})] = [N x].$
Lemma 2

1) Suppose $x \in D^*_E$ is $\sigma$-regular. Then $\exists M \in D$ a max subfield s.t. $x$ is $\sigma$-conjugate to an elt of $M_E$, & $M$ is unique up to conjugacy. The norm defined above, restricted to $M_E^*$, is the usual field norm.

Let $x \in M_E^* \setminus E^*$, then $C_\omega^M = M^*$ (not $M_E^*$!)

2) Suppose $[N_x]$ is central. Then $x$ is $\sigma$-conjugate to an elt of $E^*$. The norm defined above on $E^*$ is the usual field norm. If $x \in E^*$ then $C_\omega = D^*$

Proof: For 2) we have $\sum_{l=1}^n : $ we use the facts: $l$ is odd.

Let $l = 2$ - maybe $[N_x]$ is central & $x$ isn't $\sigma$-conjugate to an elt of $E^*$.

Proof: 1) $\exists g \in D^*_E$ s.t. $g^{-1}(N_x)g \in D^*$

Let $M = F(N_x)$. Then $x^{\sigma}(N_x) = N_x$ as $N_x \in D^*$

i.e. $x$ & $N_x$ commute.

$x \in N_E$ by results on quaternion algebras.

Now say $g \in C^*_x$ i.e. $g^\sigma x^\omega g = x$.

Then $N(g^\sigma x^\omega g) = g^\sigma (N_x)g$

$x \in N_E$ s.t. $g \in M^*_E$.

Hence $g$ & $x$ commute.

Thus $g^\sigma x^\omega g = x \Rightarrow g^\sigma g = g \in M^*$

2) Slightly more complicated. Define a new action of $<\sigma>$ on $D_E$ on $D^*_E$ as $\alpha$.

$\alpha_\delta = x(\sigma) x^{-1}$: Need to check $\alpha_\delta^\tau \in D^*_E$

$\alpha_\delta : (x^{-1})^\tau (\alpha_\delta)(x^{-1})^{-1} = (N_x) \delta (N_x)^{-1} \in D^*_E$

$\alpha_\delta$ is $\ell$ & we have an action i.e. $\sigma$-linear.

Let $D^* = \{ \delta \in D_E | \alpha_\delta \cdot \delta \}$. Hilbert 90 $\Rightarrow D^* \otimes E = D_E$

Check $D^*$ is aquat alg. Then $D^* \subseteq D$. (This may be Noetherian.

In fact $g \in D^*_E$ s.t. $g^\sigma g = e$. 

Thus let has assumed $\text{odd}$ for simplicity. If so, the first time we've assumed $\text{odd}$.}
Hence $g x g$ is $\sigma$-centralized by $D$ & hence centralized by $D$ as $\sigma$ acts trivially on $D$. Hence it is centralized by $D_E$ by linearity.

Hence $g x g \in E^x$. The rest is an exercise. □

Thus leaves bare what's going on in this case.

Cor x & y are o.s.s. Then $[Nx] = [Ny] \Rightarrow x^\sigma y$

$\Rightarrow \Rightarrow (\Rightarrow): Nx = g^d(Ny)g = N(g^d y g)$

WLOG $Nx = Ny$.

Also WLOG $N = Ny \in E^x$.

2 cases: i) $N = Ny \in E^x$ is central.

Then WLOG $x,y \in E^x$. Then $\text{Hilbert } 90 \Rightarrow \exists \gamma \in k^x$ for some $x \in E^x$

$\Rightarrow x = \alpha x^\sigma y^\sigma$.

ii) almost the same. $N = Ny$ regular, the Hilbert 90 on max subfield; say $N = Ny \in M; \text{Hilbert } 90 \text{ on } M^x$.

If $D$ splits there is a third case but its all easy. □

Cor (special case of lemma) $x \in D_E^x$, x o.s.s, $Nx \in E^x; \text{always true here after conjugation}$

then $C_N^x = C_N x$.

in $D^x$. This was a odd. □

NB the penultimate corollary is just Hilbert 90 in a non-abelian case.

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Lecture 2
Wed 24th Feb '93
9:30am

2. Matching Orbital Integrals

Need to understand orbital integrals, to understand the trace formula. Global orbital integrals will feature, so let's do the local situation.

$id$

$F$ a finite ext of $Q_p$, $E/F$ cyclic, degree $l > 2$, prime. $<\sigma> = \text{Gal}(E/F)$.

$E$ may be $F^x$ & then $\sigma$ acts by permutation.

$D/F$ qual alg. (split or not). Fix Haar measure on $D$ & on $D_E$ st.

$\mu(O_D^c) = 1$ & $\mu(O_E^c) = 1$. 
Say $x \in \mathfrak{s}_e \& \gamma \in C_c^\infty (\mathfrak{g})$

Set $O_\gamma (\varphi) = \int_{C^\mathfrak{g}_e} \varphi (g^{-1}hg) \, dg$

Note $O_\gamma (\psi) = O_\gamma (\varphi)$, of $C^\mathfrak{g}_e \triangleright \mathfrak{g}$.

Note $C^\mathfrak{g}_e \triangleright \mathfrak{g}$.

Now say $\delta \in \mathfrak{d}_e \& \gamma \in C_c^\infty (\mathfrak{d}_e)$

Set $TO_\delta (\varphi) = TO_\delta (\gamma, \mu_{\mathfrak{d}_e}) = \int_{C_c^\mathfrak{d}_e} \gamma (g^{-1}\delta g) \, dg$

Note $C^\mathfrak{d}_e \triangleright \mathfrak{g}$.

The $\int$ exists, as $[\delta]_\sigma$ is closed. It is homeo (justification later) to $C_c^\mathfrak{d}_e$.

Recall we're trying to understand $tr_{\mathfrak{d}} (\gamma)$. Here's a good way of looking at things.
Def. We say that functions \( \psi \in C_c^\infty(D^*_E) \) & \( \chi \in C_c^\infty(D_E^*) \) are **associated** if

A regular \( \gamma \in \Omega \), \( \gamma \in \mathbb{N} \)**

\[
O_{\gamma}(\psi) = \begin{cases} \ 0 & \text{if } [\gamma] \neq [N_\delta] \text{ some } \delta \in D_E^* \\ 
T_{O_{\delta}(\psi)} & \text{if } [\gamma] \neq [N_\delta], \delta \in D_E^* 
\end{cases}
\]

We say they are **strongly associated** (highly non-standard notation) if true \( \forall \delta \geq \gamma \).

\( \psi \) associated \( \Leftrightarrow \) strongly associated (use the theory of germs, etc.)

Ya, see Tony was going to prove some theorem under an "associated" assumption. However, Richard & Peter Schneider get a "strongly associated" result, so if Latane does too then it makes Tony's life easier.

A word on measures: if \( \gamma \in N_\delta \) then \( C_\gamma = C_\delta \) so take the same measure.

Fix \( M_1, \ldots, M_r \) representatives of the conjugacy classes of maximal subfields (or POFs) in \( D^*_E \) (NB local \( \Rightarrow \) only finitely many). Fix Haar measures on each \( M_i^* \& \) extend to all centralizers \& \( \alpha \)-centralizers by conjugacy.

**Theorem 3** (clearly the best we can do)

1) If \( \psi \in C_c^\infty(D_E^*) \) then \( \exists \psi \in C_c^\infty(D^*_E) \) associated to \( \psi \)

2) If \( \psi \in C_c^\infty(D^*_E) \& O_{\gamma}(\psi) = 0 \) whenever \( \gamma \) is regular semisimple and \( [\gamma] \) is not a norm then \( \exists \psi \in C_c^\infty(D_E^*) \) associated to \( \psi \).

**Ex.** It doesn't set up a bijection in many-to-many.

**Proof:***

**Lemma 5**: If \( \psi \in C_c^\infty(A \times B) \) then \( \int A(x,y)dy \in C_c^\infty(A) \)

**Ex.** exercise - for products core reduce to \( \Theta \)-char.

**Cor.** If \( \psi \in C_c^\infty(D \times B) \) then \( \int A(x,y) dy \in C_c^\infty(D) \)

**Proof**: from now. There's 2 cases. \( E = k \& E \) is a field.
Df of then

Case 4. $E = 1$. Then $\Delta_E = (\delta_0, \ldots, \delta_1)$; $\sigma(\delta_0, \ldots, \delta_1) = (\delta_2, \delta_1, \delta_1)$

Then $\delta \sim_{\sigma} (\delta_2, \ldots, \delta_1, 1, \ldots, 1)$

via $(1, \delta_2, \delta_1, \delta_1, \delta_1, \ldots, \delta_1)$

Consider only then $\delta = (\gamma_1, \ldots, 1)$

Then $N\delta = (\gamma, \gamma, \ldots, \gamma) \in D^+$, say $\gamma$ is regular. Set $M = F(\gamma)$.

Then $T_{\delta}(\psi) = \int_{M^+} \psi(g_1^2 g_2, g_2^2 g_3, \ldots, g_l^2 g_1)\, dg_1 \cdot dg_l$

Now set $h_i = g_i^2 g_1, i = 2, \ldots, l$

Then $T_{\delta}(\psi) = \int_{M^+} \psi(h_2, h_3^2 h_4, \ldots, h_l h_1^2, h_1)\, dh_1 \cdot dh_l \cdot dh_2 \cdot dh_3 \cdot \cdots \cdot dh_l$

Set $\varphi(\lambda) = \int_{M^+} \psi(h_2^2 h_3, h_3 h_4^2, \ldots, h_l h_1^2, h_1)\, dh_1 \cdot dh_l \cdot dh_2 \cdot dh_3 \cdot \cdots \cdot dh_l \in C_c^\infty(D^+)$

Then $T_{\delta}(\psi) = \varphi(\lambda)$ if $\varphi$ is associated to $\psi$.

NB: If $\gamma$ was central then write $D^+$ instead of $M^+$ & $\int_{M^+}$ as "evaluate" & so we're in fact proved they're strongly associated.

Remark. If $\gamma = \gamma_1 \times \cdots \times \gamma_k$ then $\varphi = \varphi_1 \times \cdots \times \varphi_k$.

Now do

Case 2. Everything is a norm here so "[8.1] is not a norm" doesn't ever apply.

Given $\varphi \in C_c^\infty(D^+)$ write $\varphi = \sum_{i=1}^{\infty} \varphi_i(\psi_1^{(i)} \cdots \psi_k^{(i)})$ with $\varphi_i^{(i)} \in C_c^\infty(D^+)$

$\psi_1^{(i)}$ is an exercise. If $\varphi = \psi_1^{(i)} \cdots \psi_k^{(i)}$ then $\varphi$ is a character of small compact subgp (normalised).


We don't really even need $\psi_0$ but it makes the exposition cleaner.
Case 2. Let $F$ be a field.

Assume $F$ is pro-$p$ (as $l = 0$ odd).

Reduction. It suffices to prove that if $x \in \mathbb{G}_F^p$ is order $p$ and $F = N_F^p \mathbb{G}_F^p$ then $x$ open & closed. W.h.o. $W(x) \subseteq W(x)$ st.

a) $V$ is unit (under any) $x$ & $W(x)$ is order $p$.

b) $x \in W(x) \Rightarrow [N_F]^p \subseteq V$

$y \in V \Rightarrow \exists$ closed $x \in W(x)$ with $[N_F]^p \subseteq [y]$.

c) If $y \in C^p_c(W)$ then $\exists z \in C^p_c(V)$ associated to $y$.

d) If $y \in C^p_c(V)$ then $\exists w \in C^p_c(W)$ associated to $y$.

Note that all of $V$ is an open set, so the non-norm cond. has gone.
Note also that $x$ is sum of $\sigma$ is order regular, because if $x$ be threw away all central elt's, what's left wouldn't be closed, & he wants a openness argument to finish it.

Proof that reduction $\Rightarrow$ this.

1) Show that $y \in C^p_c(D)$ then $\exists y \in C^p_c(D)$ associated to $y$.

Let $x$ be image of $\text{supp}(y)$ under the map $(tr, v)_* N : \mathbb{G}_F^p \rightarrow F \cdot F$.

Then $x$ is open. Recall $H_x$ - $x$ represent max subfields of $D$.$\text{supp}(y)$

If $y \in M \cdot \mathbb{G}_F^p$ then choose rho $W_N$ of $\sigma$ & $V_N$ of $N_N$ as above.

Choose $\sigma_1$, $\sigma_2$, st. $W_{\sigma_i}$ cover $\text{supp}(y) \cdot N^t$.

Choose $\sigma \in \sigma_i$, st. $(tr, v)^* V_{\sigma_i}$ covers $(tr, v)^* M \cdot \text{supp}(y) = U_{W_{\sigma_i}}$.

(use the fact that $(tr, v)$ is open away from central elt's (exercise))

We may assume $W_5$ are all disjoint (i.e. $W_5$ is the $R$ case, of course, we can). Same position of unity trick will probably do it though.

By the reduction $\exists y$ on $V_{N_5}$ associated to $y|_{W_5}$. Let $y = \sum q_i$.

It's an exercise to show $\sum y$ associated to $y$.

2) $y \Rightarrow y$ is an exercise (do exactly the same).

He's sorry, this was a bit nasty. He hadn't realized $(tr, v)$ wasn't open at the central elt's 'till this morning, so had to patch a part up.
Recall we're trying to show for the reduction of the problem.

We're matching an orbital integral with a twisted orbital integral.

Recall for $\delta \in D^* \cap N(\delta \in W^*)$ we're trying to show $F$ open & closed:

$W \delta^e \in W \delta \in W^* \cap N(\delta \in W^*)$ s.t.

1) $W \delta^e$ is an inst. $W \delta^e$ is o-inst.
2) $x \in W \Rightarrow x W \delta^e = x \in W \delta^e$ s.t. $N_x = y$
3) $y \in C^\infty(V) \Rightarrow 3 y \in C^\infty(V)$ around to $y$
4) $y \in C^\infty(V) \Rightarrow 3 y \in C^\infty(V)$ around to $y$

This is a local condition. Note that everything is a norm.

There's no an attack that works, but we'll do a case - regular & control. For $G = d$ there's many cases but hey it's small.

Case 2a. $G$ regular. Set $M = F(d)$, $\delta \in H^*.$

We need some geometric facts about orbits which I'll just state.

Prop 4. $F$ open & closed: $\delta^e \in M^* \cap N(\delta \in W^*)$ s.t.

1. $U \rightarrow U^e$, $t \rightarrow t^e$ is a homeo (log & exp converge nr.1)
2. $W \delta^e$ consists of regular orbits
3. $\mu \rightarrow \mu^e \rightarrow V = \delta$
   $$(x, t) \rightarrow x^\delta(\delta t) x$$ is a homeo onto an open closed set $V = \delta$
4. $\mu \rightarrow \mu^e \rightarrow W \in D^e \cap W \delta^e$ is a homeo onto an open closed set $W \in D^e$

I'll prove this & things like it on Friday.

The $V$ & $W$ are the $V$ & $W$ we need.

So let's check 3). 4) is just the same. $\delta$ commutes.

Say $y \in C^\infty(V)$ we want $y$. Note that $y \delta^e U^e$,

$$\int_{D^e} y \delta^e \in C^\infty(U) \text{ (as we're near a regular elt)}$$
Choose \( \theta \in C_c^\infty (\mathbb{R}^m) \) st. \( \int \theta = 1 \). Define \( \varphi \in C_c^\infty (V) \) by
\[
\varphi(x^{-1} (\theta t)x) = \theta(x) TO_{xt} (\varphi)
\]
\( x \) smooth of speed \( t \). 
\[\text{smooth of speed support.}\]

Then \( \int \varphi(x^{-1} (\theta t)x) \, dx = TO_{xt} (\varphi) \).

Hence \( \varphi \) is associated to \( \varphi \).

4) works in exactly the same way: integrate at the fibres in the product structure & define everything how you'd expect.

That's the easier case. (near a regular elt the orbital \( \int \) works with any reasonable \( \varphi \) you like)

The only thing left is to prove the reduction in the case \( \gamma = Ns, s \in E^\times \)

Case 2): \( s = Ns, s \in E^\times \)

Similar but more complicated.

Proof: 2) Open+closed unit nhds \( U, \bar{U} \) of 1 in \( D^\times \) st.

\[ U \to \bar{U} \]
\[ t \to t^x \]

is a homeo (note \( D^\times \) not ab but still have log. exp)

6) \( \exists \) section of the map \( \tilde{D} \to \bigwedge \tilde{D} \).

\[ \gamma \in \bigwedge \tilde{D} \to \tilde{W} = \tilde{D} \]

\[ (s(t), x) \to \gamma(x) = \tilde{s}(t) \tilde{s}(x) \]

is a homeo onto \( \tilde{W} \) which is open & closed. Take \( \gamma = \gamma(t) \).

Then again the reduction follows with this \( V \& W \).
Recall 8.9 are central. 1), 2) of red. are easy

3) If \( \varphi \in C_c^\infty (W) \) then let's get \( \varphi \).

Say \( x \in U \) regular, \( M = E(1) \)

\[ TO_{xt} (\varphi) = \int_{\tilde{W}} \varphi(x^{-1} \tilde{s}t^{-1} x) \, dx = \ldots \text{see next page.} \]
\[
T_{\delta t}(\psi) = \int_{\mathbb{R}^2} \psi(x+\delta t y) \, dy - \int_{\mathbb{R}^2} \psi(x^2 y^2 + \delta t y^2 x) \, dx
\]
\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi \left( (s(x))^2 y^2 + \delta t y^2 \psi(s(x)) \right) \, dy \, dx
\]

Note that because \( s \) we can swap \( \int \) around.
Note everything is old style.
\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \psi \left( (s(x))^2 y^2 + \delta t y^2 \psi(s(x)) \right) \, dy \, dx
\]

make sense
by prop.

\[
= \int_{\mathbb{R}^2} \psi(y^2 + \delta t y) \, dy
\]

where \( \psi(z) = \int_{\mathbb{R}^2} \psi \left( (s(x))^2 y^2 + \delta t y^2 \psi(s(x)) \right) \, dx \in C(V) \) & \( \text{inf} \in C^\infty_c(V) \)

by lemma.

So the related \( \psi \)'s match those of \( \psi \).

Let's do 4) for completeness.

We have \( \psi \in C^\infty_c(V) \). Choose \( \theta \in C^\infty_c(\mathbb{R}^2) \) with \( \int \theta = 1 \).

Define \( \eta \in C^\infty_c(W) \) by \( \eta \left( (s(x))^2 y^2 + \psi(s(x)) \right) = \psi(\delta t \eta) \theta(x) \).

This works: \( T_{\delta t}(\psi) = \partial_{\delta t} \eta(\psi) \).

We have done very well today.

It remains to do 5)853 pop 4 & 6.

Let's do some geometry. The core of 5)853 both props is the openness of the maps. We need some pade's analysis. We need a pade's inverse function theorem.

Say. F/G. Check out [Reference: Lie Algebras & Lie Groups] (Benjamin Lecture Note series, 1965.)

The key is that these maps are (locally) analytic, so I power series expansions.
If \( U \subseteq F^n \) is open (\( F(\mathbb{C}) \)),

then \( \varphi: U \rightarrow F^n \) is called \textit{analytic} at \( x \in U \) if \( \exists \) power series

\[
\sum_{j=0}^{\infty} a_j z^j \in F^n[[T_{x1}, \ldots, T_{xn}]]
\]

\( a_j \in F^n \)

where \( z \) runs thru \( z = (i_{x1}, \ldots, i_{xn}) \in \mathbb{Z}_{\geq 0}^n \)

\[ T^j = \prod_{i=1}^{n} T_{x_i}^{i_j} \]

st. for all \( h \) in some \( \text{nhd of} \ 0 \) in \( F^n \), the power series converges at \( h \) and

\( \varphi(z+h) = \sum_{j=0}^{\infty} a_j z^j \)

NB the power series converges in some \( \text{nhd of} \ 0 \) if \( \exists \) \( \mathbb{R} \rightarrow \mathbb{R} \) s.t.

\[ a_j \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty, \quad |j| = \mathcal{O}(j) \]

\textbf{Rk.} \( \varphi \) analytic at \( z \Rightarrow \varphi \) analytic in a \( \text{nhd of} \ z \) (See page 12:4)

\( \forall \psi \text{ analytic at } z \in V \forall \varphi \text{ analytic on } U \) we can differentiate:

\( D\varphi: U \times F^n \rightarrow F^n \)

\( D\varphi \in \text{Hom}(F^n, F^n) \) given by \( (D\varphi)(y) = \sum_{i=1}^{n} a_i z^{i-1} y^i \quad y = (y_1, \ldots, y_n) \)

where \( a_i \) = \text{coeff. of power series of } \varphi \text{ at } z \)

Note that \( D(\varphi \circ \psi) = D\psi \circ D\varphi \)

What we need is

\textbf{Page 23.} If \( U \subseteq F^n \) is open \& \( \varphi: U \rightarrow F^n \), \& \( \varphi \) is analytic at \( x \) with

\( D\varphi \) an \( \text{iso.} \), then \( \exists \) nhds \( V \subseteq U \) of \( x \) \& \( W \) of \( \varphi(x) \) s.t. \( \varphi \) is a bijection from \( V \) to \( W \), \& the inverse \((\varphi | W)^{-1}\) is analytic

( this implies \( \varphi: V \rightarrow W \) is a homeo)

\( \square \)

See e.g. Same LG 2:10. Alternatively try it as an exercise. Not and too bad. Believe some if you have an idea of common sense.
If \( \phi: U \to \mathbb{F}^n \) is analytic at \( z \) & \( D\phi_z \) has rank \( n \), then

\[
\exists \text{ neighborhood } V \subseteq U \text{ of } z \text{ & } W \subseteq \mathbb{F}^n \text{ st.}
\]

1) \( \phi|_V \) is open

2) Analytic \( s: W \to V \) st. \( \phi \circ s = id \)

**Proof**: Usual analytic trick. Consider \( \phi: U \to \mathbb{F}^n \)

\[
y \mapsto (\psi y, Ay)
\]

where \( A \in \text{Hom}(\mathbb{F}^n, \mathbb{F}^m) \) is chosen s.t.

\[
D\phi_z \circ A \text{ is invertible as a linear map } \mathbb{F}^n \to \mathbb{F}^m.
\]

That's all the general nonsense we need. On Friday he'll prove

papers 4 & 6.

Today he's gonna talk about things like.

**Page 4**: 3 open nhds \( U \& U \) of 1 in \( M^s \) (regular, \( M: F(\mathbb{R}), x: N^s \)) s.t.

a) \( U \to \overline{U} \) homeo.

b) \( \overline{U} \) is only regular elt.

c) \( M \setminus U \to V \subseteq \mathbb{R}^x \) is a homeo onto clopen \( V \)

\[
x, t \mapsto x^t e^t
\]

d) \( M \setminus \overline{U} \to W \subseteq \mathbb{R}^x \) is a homeo onto clopen \( W \).

\[
x, t \mapsto x^t e^t
\]

**Proof**: Choose \( U, \overline{U} \) as in a) \& \( \overline{U} \) contains no central elt. \( x \in x \) s.t. \( x \notin \overline{U} \).

Use log & exp to compose a) \& the fact that \( x \) is a poly with distinct roots. See ch. 6, v.s.t. the coffee.

Now move, say, d) (a bit harder than c)

\[
d) M \setminus \overline{U} \to \mathbb{R}^x
\]

\[
(x, t) \mapsto x^t e^t
\]

is cto \& injective (by taking norms, I guess).

\[
x^t e^t \in M \implies x^t e^t \in M
\]

\[
x \neq 0 \implies x^t e^t
\]

We need to show it's open.

\[
x^t e^t
\]
It will do to show \( \mathbb{D}^e_x U \rightarrow \mathbb{D}^e_x \) is open.

It will do to show that the change of variable derivative \( \mathbb{D}^e_x M \rightarrow \mathbb{D}^e_x \) is surjective at all pts \((x, t)\)

\[
(0, 0) \rightarrow x^t (\psi(z, t) \circ (x^t) - (x^t) \circ z) = 0
\]

Hence we just need to show that \( \mathbb{D}^e_x \rightarrow \mathbb{D}^e_x \)

\[
\beta \mapsto m ( \beta \cdot \psi ) , \quad m: \mathbb{D}^e_x \mathbb{D}^e_x [0, 1]
\]

This is an exercise.

\( \) Exercise \( NS \) also need to check maps are closed but there is an easy reason why this is so.

\( e \) End up needing to show \( \beta \rightarrow [m, \beta] \) is surjective.

He now wants to talk about prop 6. Happy Birthday Danny. Is Danny here?

Prop 6 \( \mathcal{E} \in \mathbb{E}^x, \in \mathcal{C}, \in \mathcal{D}, \in \mathcal{E}^x \). Then \( \exists \) clopen chart whose \( U, \tilde{U} \) of \( 1 \in \mathbb{E}^x \) s.t.

\[
U \rightarrow \tilde{U} \quad \text{is a homeo.}
\]

Also \( \exists \) clopen chart to the map \( \mathbb{D}^e_x \rightarrow \mathbb{D}^e_x \) s.t. the map

\[
\gamma \tilde{U} \times \gamma \beta \rightarrow W \subseteq \mathbb{D}^e_x
\]

\[
\beta \in \gamma \beta \rightarrow s(x)^t \circ \beta = \beta
\]

\( \) is a homeo onto clopen \( W \)

(1) Select \( \mathcal{A} = \mathcal{A} \) open & closed subset of \( \mathbb{D}^e_x \) on which \( \exp \) converges with inverse \( \log \). We can replace them with the union of all their conjugates so \( \log \) \( \mathcal{A} \in \mathcal{F} \) are conjugation-invt.

Let \( U_{\mathcal{E}} = \exp \mathcal{A}, \quad \tilde{U}_{\mathcal{E}} = \exp (\mathcal{A}) \)

\[
U = U_{\mathcal{E}} \cap \mathbb{E}^x, \quad \tilde{U} = \tilde{U}_{\mathcal{E}} \cap \mathbb{E}^x
\]

Then \( U, \tilde{U} \) have the first property.

(b) Things are easier once you can't stay in a product analytic manifold but Richard will try to skirt round this.

Find chart \( Y \) of 1 in \( \mathbb{D}^e_x \) \& \( B \) of 0 in \( \mathbb{D}^e_x \) \& a decomposition

\[
\mathbb{D}^e_x = \mathbb{D}^e_x \mathbb{D}^e_x \quad \text{s.t.}
\]

\[
B \rightarrow \mathbb{D}^e_x \mathbb{D}^e_x \quad \text{tend}
\]

\[
\tilde{u} \mapsto \tilde{u} \exp (t) \quad \text{is a homeo with inverse } t
\]

\( \) See next page.
Consider $D^* \setminus \partial D^* \to D^*$

\[ \sigma \cdot b \mapsto \exp b \]

Let $(x,y) : Y \to A \times B$ denote the inverse. Then $(y,dy) \in \mathcal{A} \times Y \times Y$. Thus $l : D^* \times Y \to Y$ with inverse $l^{-1} : D^* \exp b \to A \times B$ are both homeo.

We want $\exists U \times D^* \to W \subseteq D^*$ to be a homeo onto closed image.

We've not sure he would have spotted it if he hadn't seen it in Arthur-Clozel. He's all quito (and though one you've said) $\sqrt{dw}$

1) injective. Say $s(x) \circ s(y) = s(x) \circ s(y')$. Then we get

\[ s(x) \circ s(y) = s(x') \circ s(y') \]

\[ \Rightarrow \sigma(s(x) \otimes s(y)) = \sigma(s(x') \otimes s(y')) \]

\[ \Rightarrow s(x) \otimes s(y) = s(x') \otimes s(y') \]

\[ \Rightarrow s(x) \circ s(y) = s(x') \circ s(y') \]

\[ \Rightarrow s(x) \circ s(y) = s(x') \circ s(y') \]

\[ \Rightarrow s(x) \circ s(y) = s(x') \circ s(y') \]

(Recall (c) a central)

\[ s(x) \circ s(y) \in D^* \Rightarrow x \cdot y \equiv b \cdot e \]

2) open. It will thereby to check

\[ s \left( (\exp b) \eta \right) \Rightarrow s(\exp \eta) \eta \equiv s(\exp b) \eta \]

\[ U \times B \to D^* \]

\[ \eta \cdot b \mapsto \exp(\eta \cdot b) \equiv \exp(b) \eta \]

\[ s(\exp(b) \eta) \equiv s(\exp(b) \eta) \]

(Use wrong then it all boils down to...
showing (the idea is we differentiate \( \mathcal{E} \) & we \( \mathcal{E} \) \( p \)-adic thing)

\[ D_E \rightarrow D_N \mathcal{E} \]

\[ \beta \rightarrow \beta_{t-r} \text{ is surjective.} \]

Well, \( NT \) couldn't, but he could show it was surjective at \( t-1 \), & surjectivity is an open cond. & an exact cond. We have to shrink our ratio if necessary.

(Richard feels it ought to be surjective.)

Surjectivity \& \( t \) is enough though. \( \square \)

Richard has done these because he feels it's the heart of the \( \mathcal{E} \) that things are associated.

\( NT \) in general these \( p \)-adic things have singularities & then things are more difficult. We were a bit lucky with our centralizers.
In these 4 lectures Tony will explain a lot of the ideas behind base change for $G_{L2}$.

He'll spend a lot of this lecture explaining the statement of the theorem, whether base change is $1$ and $G_L$.

He'll finally spend a while talking about $G_{L2}$.

5.1: Exact statement of results

"Base change" for $G_{L2}$ is rather easy:

If $E/F$ is a finite set of local fields or number fields, write

$$C_E = \left\{ \begin{array}{ll}
E^* & 	ext{E local} \\
\mathbb{A}_E/E^* & 	ext{E global}
\end{array} \right.$$ 

Similarly $C_F$.

Recall we have

$$\theta_F: C_F \rightarrow \text{Gal}(\overline{F}/F)^{ab}$$

$$\theta_E: C_E \rightarrow \text{Gal}(\overline{F}/E)^{ab}$$

commuting.

This is essentially base change. Here are the details. We replace it in a more representation theoretical way.

Suppose $\rho: \text{Gal}(\overline{F}/F) \rightarrow C^*$. Identify this with $\chi: C_F \rightarrow C^*$ by $\chi = \rho\circ \theta_F$.

We have $\rho \mapsto \rho' = \rho \circ \text{Gal}(\overline{F}/E)$

"Base change" is the corresponding assignment

$$\chi \mapsto \chi' = \chi \circ N_{E/F} = \rho' \circ \theta_E$$

Now restrict to $E/F$ cyclic, $\sigma : \text{Gal}(E/F)$

Then if $\chi': C_E \rightarrow C^*$, $\chi'$ is of the form $\chi' \circ N_{E/F} \Leftrightarrow \chi'^E = \chi$.

Hint for this on next page.
Use the exact sequence

\[ 1 \rightarrow \left(1, \sigma\right)C_E \rightarrow C_E \xrightarrow{N_{E/F}} C_F \rightarrow \left\langle \sigma \right\rangle \rightarrow 1 \]

since \( E/F \) is cyclic.

Hence \( \chi \cong \chi' \iff \chi' \) factors thru quotient \( C_E \rightarrow N(C_E) \)

Then extend this HM arbitrarily from \( N(C_E) \) to \( C_F \).

Now by

\[ \text{GL}_n, \ n \geq 2 \]

Local base change: Say \( F \) is a local \((p\text{-adic})\) field, \( E/F \) a finite ext.

The local Langlands conjecture

\[
\begin{pmatrix}
\text{(Conjugacy classes of)} & \text{???} & \text{(adm irr)} \\
\text{ss, HMs} & \text{of} & \text{rep}\ 	ext{of} \\
\text{WD}_E & \Rightarrow & \text{GL}_n(F) \\
\end{pmatrix}
\]

\[ \text{Hom}_G(WD, GL_n(C)) \]

Everyone believes this exists.

Now we have the restriction map \( \text{Hom}_G(WD, GL_n(C)) \rightarrow \text{Hom}_G(WD, GL_n(F)) \)

& the local Langlands conjecture \# for \( E \) gives us a map

\[
\begin{cases}
\text{adm irr} & \rightarrow & \text{adm irr} \\
\text{rep of } GL_n(F) & \rightarrow & \text{rep of } GL_n(E) \\
\end{cases}
\]

So the local Langlands conjecture suggests the existence of a local base change map from \( \text{rep of } GL_n(F) \) to \( \text{rep of } GL_n(E) \), which we should be able to understand representation theoretically.

Peter Schneider talked about the unramified case.

\[
\begin{pmatrix}
\text{unram irr} & \rightarrow & \left\{ \begin{array}{c} \alpha_1, 0 \\ 0, \alpha_0 \end{array} \right\} \\
\text{rep of } GL_n(F) & \rightarrow & \text{ss, conj. classes} \\
\text{rep of } GL_n(F) & \rightarrow & \text{ss, GL}_n(E) \\
\end{pmatrix}
\]

& we understand Eulerian.
Def: If $\pi$ is an unramified irreducible rep of $GL_n(F)$ with parameter $(\alpha_1, \ldots, \alpha_n)$, the local lifting $\tilde{\pi} = \pi_F$ of $\pi$ is the unramified rep of $GL_n(E)$ with parameter $(\alpha_1^*, 0, \ldots, 0)$, $\alpha_1^*$ being the residue class degree of $E/F$.

We can interpret this in terms of base change map on unramified Hecke algebras:

$$\pi \mapsto \pi_F: H \mapsto H(G(F), K^{max}) \rightarrow C$$

(given by action on $K$-fixed vectors).

Then $\pi_{F,E} \mapsto C$ is given by $\pi_F = \pi_{E,F} \circ (E/F)$

for $\pi_{E,F}: H_F \rightarrow H_E$ the base change $HM$.

NB this only assuming $E/F$ is unramified. Have to keep track of $\mathfrak{c}$.

In the global case, we have the local ext. unramified $\alpha_i, \beta_j$ thus we only need.

$L$-functions

$$L(\pi, s) = \prod_i (1 - \alpha_i q_i^{-s})^{-1}, \quad L(\pi, s) = \prod_i (1 - \alpha_i q_i^{-s})^{-1}$$

Enough local stuff.

Global base change: Say $E/F$ is a finite ext. of number fields.

Example: Say $X/F$ is an elliptic curve, $S$ a finite set of places.

$$\forall v \in S \exists \ell \text{ ell curve } X/\mathbb{F}_v, \quad \exists x(x_v) = 1 + q_v - q_v^\alpha, \alpha, \beta, \gamma \in \mathbb{F}_v$$

$\pi_v^\alpha$, unramified rep of $GL_1(F_v)$ with parameter $(\alpha, \beta, \gamma)$

Taniyama-Weil: (this is probably Weil) $\Rightarrow \exists (\text{cuspidal if no CM})$ auto rep $\pi_v \cong \pi_v \circ GL_2(A_F)$

s.t. $\pi_v \cong \pi_v \forall v \in S$.

$X/E \mapsto \forall w | v, v \in S, \exists \pi_w$ unram rep of $GL_1(E_w)$ with parameter $(\alpha^w, \beta^w, \gamma^w)$

$\alpha_v^w = \alpha^w, \beta_v^w = \beta^w, \gamma_v^w = \gamma^w$.

$\Rightarrow \exists \pi_w = \pi_{w,v} \circ GL_2(A_F)$

s.t. $\pi_w \cong \pi_w$ for almost all $w$.

$\pi_w$ would be a local lifting of $\pi_v$ for all $w, v \in S$.

The relation $\pi_v \rightarrow \pi_{w,v}$ is part of any conjectured relationship with elliptic curves.
Def: \( \pi \otimes \pi_w \), \( \mathcal{T}_w \) inred auto reps of \( \text{GL}_n(A_F), \text{GL}_n(A_E) \) resp.

Say \( \mathcal{T} \) is a weak basechange of \( \pi \) if for almost all \( v, \theta \) with \( \mathcal{T}_w \) is the (unramified) local basechange of \( \pi_w \).

Conjecture (Langlands): Any \( \pi \) has a basechange.

(No He thinks this is what Langlands said. He hasn’t put in the word "cuspidal".)

This is an extremely strong conjecture e.g. gives you lots of analytic-like properties of non-abelian \( L \)-functors.

Eg. \( n=2 \) gives us \( X/F \) ell. curve \( \sim L(X, \zeta_2) \otimes L(x/E, \zeta_3) \) or sth.

Then (Langlands): (Based upon important ideas of Saito, Shintani)

If \( E/F \) is cyclic of prime degree \( l \), \( \chi_{E/F} \) a non-trivial char of \( \text{Gal}(E/F) \). \(< l^2 \)

(i) Any inred. auto rep of \( \pi \) of \( \text{GL}_l(A_F) \) has a (strong) basechange to \( E \), call it \( \mathcal{T} \).

(ii) If \( \pi \) is cuspidal then so is \( \mathcal{T} \) except when \( l=2 \) \& \( \pi \) is obtained from \( \theta: A_E^* / E^* \rightarrow C^* \), \( \theta = \theta^0 \).

If \( \pi \) is not cuspidal then neither is \( \mathcal{T} \).

(iii) Conversely, if \( \mathcal{T} \) is a cuspidal inred. automorphic rep of \( \text{GL}_l(A_E) \) then \( \mathcal{T} \) is a basechange of some \( \pi \), provided that \( \mathcal{T} \otimes l^2 \mathbb{C} \).

(iv) If \( \pi, \nu \) are cuspidal, then they have the same basechange \( \Rightarrow \pi \simeq \nu \otimes (\chi_{E/F} \det) \) for some \( j \).

The meaning of (iv) is that given \( \theta: W_{E,w} : A_{E,w}^* / E_w^* \rightarrow C_w^* \), let \( S \) include all infinite places & all places of \( v \) ramified, for \( E/F \) or for \( \theta \).

Then if \( \theta \) wts \( S \), \( G_w(\omega) = \{ \omega \} \) (unramified).

\[
\mathcal{G}_w \leftrightarrow \text{f.d. rep. of } \mathcal{W}_{E,w} \rightarrow \text{Hom}_{W_{E,w}} \mathcal{W}_{E,w} \rightarrow C_w^*
\]

\[
\mathcal{G}_w \leftarrow \text{Maps } W_{F_v} \rightarrow \mathcal{G}_l(\mathbb{C})
\]
Take \( \tau \) with parameter \((0, w^0 \omega, v=ww, v=vw, v=vw \text{ inert})\).

There are automorphic, local \( \tau \), at \( v \neq 5 \).

That was an explanation of the \( l \neq 2 \) bit of (ii).

Lots of ingredients are necessary for this proof. One that we have not got is a trace formula for \( \text{GL}_2 \) - we need Eisenstein series for that. We will do a version of the thing for \( D \) a non-split quaternion alg.

We'll also avoid the tricky case \( l = 2 \).

Here is the thing that we will prove.

**Theorem:** Let \( G \) be the gp of elts in a quaternion division algebra \( D/F \), & let \( E/F \) be cyclic of prime degree \( l > 2 \). We have \( G(A_E) \cong G(A_F) \).

Then every irreducible auto rep \( \tau \) of \( G(A_F) \) has a (weak) base change to \( E \), & if \( \tau \) & \( \tau' \) have the same base change \( \Leftrightarrow \tau \cong \tau \otimes (\chi_{E/F} \circ N_{E/K}) \) for some \( j \).

Moreover, every \( \tau \) with \( \overline{\tau} \cong \tau \) is a base change.

**Remark:** Since \( G(F_v) = \text{GL}_2(F_v) \) for all but finitely many \( v \), the notion of weak base change makes sense for \( G \).

Recall \( E/F \) a cyclic ext. of no. fields, \( D \) quaternion division algebra / \( F \)

\( \langle \sigma \rangle = \text{Gal}(E/F) \) of order \( l \) prime \( > 2 \)

\( G = \text{gp of invertible elts of } D, G(F) = D^*, G(E) = (D \otimes E)^* \) (\( G \) is an alg gp)

\( AR(F) = \{ \text{isom. classes of irreducible auto reps of } G(A_F) \} \)

\( AR(E) = \{ \text{isom. classes of irreducible auto reps of } G(A_E) \} \)

Here comes a thm. It's the one we had earlier, I guess.

**Theorem:** If \( \tau \in AR(F) \), then it has a weak base change \( \overline{\tau} \in AR(E) \).

(ii) \( \tau, \tau' \) have the same base change \( \Leftrightarrow \tau \cong \tau \otimes (\chi_{E/F} \circ N_{E/K}) \)

(iii) Any \( \overline{\tau} \in AR(E) \) is a base change of some \( \tau \) \( \Leftrightarrow \overline{\tau} \cong \overline{\tau'} \).

We will prove this.
Theorem 4. We need the strong multiplicity 1 theorem, (which has 2 parts)

If \( \Pi = \oplus_w \Pi_w \) & \( \Pi' = \oplus_w \Pi'_w \) are in \( AR(E) \) & \( \Pi_w = \Pi'_w \) for almost all \( w \), then \( \Pi \cong \Pi' \). (\( \iff \Pi_w = \Pi'_w \) for all \( w \))

- the proof is a reduction to \( GL_2 \) using J-L correspondence.
- The J-L correspondence uses the trace formula for \( J \) & \( GL_2 \)
- The trace formula for \( GL_2 \) uses the theory of Eisenstein series.

So it's a lot of work.

Anyway, it shows that if \( \Pi, \Pi' \) are weak liftings of \( \tau \), then \( \Pi \cong \Pi' \).

(2) \( G(E) \cong \mathbb{Q}(E) \times \mathbb{Q} \); so we get a semidirect product \( G(E) = G(E) \rtimes \mathbb{Q} \).

\[
G(E_v) = \Pi \otimes G(E_v) \rtimes \mathbb{Q} \quad \text{for} \quad E_v = E_{w_f} = \Pi \otimes \mathbb{Q}
\]

If \( \Pi = \oplus_w \Pi_w \in AR(E) \), write \( \Pi_v = \oplus_w \Pi_w \); which is a rep. of \( G(E_v) \).

Notation: \( \Pi_v, \Pi_v \).

[ If \( \Pi \) is a ~holomorphic ~of some \( \tau \), then \( \Pi_v \cong \Pi_v \) for almost all \( v \), & hence \( \Pi \cong \Pi' \). By strong multiplicity 1 ]

(3) If \( S \) is some finite set of primes of \( F \), including all the ones ramified in \( E \) or \( D \), set

\[
\mathcal{H}_F^S = \bigotimes_{v \in S} \mathcal{H}(G(F)_v, K^{m_v})
\]

for all finite \( v \) and unramified Hecke algebras. For all \( v \) finite \( v \).

Similarly \( \mathcal{H}_E^S \).

If \( v \in AR(F) \), unramified at all finite \( v \)& \( s \),

\[
\Pi_v = \Pi_v
\]

then for each \( v \& s \) we get a character of the unramified Hecke algebra (on \( \Pi_v \)) & hence a HM

\[
\omega^S : \mathcal{H}_F^S \rightarrow \mathbb{C}
\]

Take any non-zero vector space of \( \tau \), fixed by \( \Pi V_w \); then

\[
\omega^S(g) = \omega^S(g) \quad \text{for} \quad g \in \mathcal{H}_F^S
\]

By strong multiplicity 1, \( \omega^S \) determines \( \tau \) up to \( w \) & \( \Pi \) is a branching change. 

\[
\implies \omega^S : \mathcal{H}_E^S \rightarrow \mathbb{C}
\]

where \( \mathcal{H}_E^S \rightarrow \mathbb{C} \) is, of course, the base change HM, which is, of course, the \( \omega \) of all the local \( E_s \)s.
This is the form of the thing which we'll attack.

We need some twisted trace formulas.

52. Trace Formulas

Recall $L^2 = L^2\left( G(F) \backslash G(A_E)^2 \right) = L^2\left( \mathbb{R}_{>0}^* G(F) \backslash G(A_F) \right)$

Recall $G(A_F)^2 = \left\{ x \in G(A_F) \text{ s.t. } \mathcal{N}_{F/E}(x) = 1 \right\}$

It's a unitary rep of $G(A_F)$. Let

$L^2 = \hat{\otimes}_\mathcal{R}$, summing over pairwise non-isomorphic unitary reps $\mathcal{R}$. The $\mathcal{R}$ that occur are precisely the unitary reps corresponding to those Elts of $AR(F)$ whose central char. is trivial on $\mathbb{R}_{>0}^*$.

$a$ is Richard with the $K$-charts.

vector stuff

Note that we have to pass to the $L^2$ way of thinking to get the trace formula.

NB of course any $\mathcal{R} \in AR(F)$ can be twisted to make its central char. trivial on $\mathbb{R}_{>0}^*$.

$\hat{\otimes}_\mathcal{R} = L^2\left( G(E) \backslash G(A_E)^2 \right) = \hat{\otimes}_\mathcal{R}$

$f \in C_c^\infty(G(A_F))$; $r(f)$ the associated operator on $L^2$

This next stuff was all done in Richard's lectures.

$r(f)$ is of trace class, represented by kernel $K(x,y) = \sum_{\mathcal{R}\in\text{Elts}(F)} f(x) \mathcal{R}(y)$, and (formula:)

$$tr\ r(f) = \sum_{\mathcal{R}} \text{vol}\ G_F(F) \backslash G_F(A_F)^2 \times O_\mathcal{R}(f)$$

sum over conj. classes in $G(F)$.

where $O_\mathcal{R}(f) = \int f(g^{-1}xg) \text{ d}g = \hat{T} O_\mathcal{R}(f)$ where $f = \otimes f_v$.

$G_F(A_E) \backslash G_F(A_F)$
Analogous formula for $\tilde{L}$

Acting on $\tilde{L}$ we have $\sigma$, acting by $(\sigma y)(x) = \psi(x^* y)

This gives an action of the semidirect product $G'(\mathbb{R}^n)$ on $\tilde{E}$.

Let $R$ be this action. Study $R(\psi \sigma^\ast) = R(\psi) R(\sigma)$ for $\psi \in C_0^\infty(G(\mathbb{R}^n))$.

The functional analysis is identical to the case of $f(y)$ so he will just stick to the formal details.

The kernel of $R(\psi)$ is $\sum_{\delta \in G(\mathbb{R}^n)} \psi(x \delta \sigma y)$, so $R(\psi \sigma^\ast)$ has kernel $\sum_{\delta \in G(\mathbb{R}^n)} \psi(x^* \delta^\ast \sigma^\ast y) = R(x, y)$, and

$$\text{tr } R(\psi \sigma^\ast) = \int x \in G(\mathbb{R}^n) \text{ d}x = \int \sum_{\delta \in G(\mathbb{R}^n)} \psi(x^* \delta^\ast \sigma^\ast y) \text{ d}x$$

$$= \sum_{\delta \in G(\mathbb{R}^n)} \int \psi(x^* \delta^\ast \sigma^\ast y) \text{ d}x$$

where $G_\delta^\sigma(\mathbb{R}^n)$ is the stabilizer of $\delta$ in $G(\mathbb{R}^n)$.

Now this implies

$$\text{tr } (R(\psi \sigma^\ast)) = \sum_{\delta \in G_\delta^\sigma(\mathbb{R}^n)} \text{vol } (G_\delta^\sigma(\mathbb{R}^n) \backslash G_\delta^\sigma(\mathbb{R}^n)) \int \psi(g^* \delta \sigma^\ast y) \text{ d}g$$

$$= \sum_{\delta \in G_\delta^\sigma(\mathbb{R}^n)} \text{vol } (G_\delta^\sigma(\mathbb{R}^n) \backslash G_\delta^\sigma(\mathbb{R}^n)^2) \int \psi(g^* \delta \sigma^\ast y) \text{ d}g$$

This nasty thing is the twisted trace formula.

It's so nasty we'll discard it.
Recall from Richards lectures, that $\varphi, \psi$ are associated if
\[ TO_\xi(\varphi_v) = O_\xi(\psi_v) \] whenever $[\xi] = [NS]$
\[ O_\xi(\psi_v) = 0 \] if $[\xi]$ is not a norm.
(for all regular elt $\xi$)

So if $\varphi_v, \psi_v$ are associated for all $v$, the corresponding global orbital integrals are equal (for all regular elt $\xi$)

If $\xi = NS$ then $G_\xi(E) = G_{\xi^*}(E)$

& also for adelic pts, so volumes are equal.
NS there's an important technicality about picking sensible Haar measure normalisations to make the fundamental lemma work, or something. He may come back to this later. But he may not.

**Thm 2.** If $\varphi_v$ is associated to $\psi_v$ for all $v$ \[ \text{[and for some } \xi, O_\xi(\varphi_v) = TO_\xi(\varphi_v) = 0 \text{ whenever } \xi \& NS \text{ are central]} \]
then $tr r(\varphi) = tr R(\psi_\infty)$

Note that the bit in brackets is not necessary but we have to put it in because we haven't analyzed central elt $\xi$ enough. It is not that difficult to judge $\varphi_v \& \psi_v$ so that we use no info & st. the bracketed statement holds.

Note also that the statement is vacuous if we can't form $O_\psi$. Fortunately, Peter proved this morning that \[ E \ni \varphi_v \perp _E \psi_v \& \xi^* \text{, are associated for } v \text{ unramified in } E/F \& \xi. \] So we can take $\varphi_v, \psi_v$ to be unit elts for almost all $v$.

I think he said that now Thm 2 was content-free by the base formula but now it's well into §3 so I'd best start that. It's a bit of functional analysis that we need but fortunately it's not too difficult.
3. Spectral decomposition at \( \infty \)

**Lemma:** Let \( \{ (\rho, V_\rho) \} \) be a family of pairwise non-isomorphic unitary reps of \( G \), with kernel \( \rho_0 \).

Let \( B \subseteq L^1(G) \) be a dense subalgebra. Suppose \( \exists \rho \in C \) s.t.

\[
(\sum \|\rho(f)\|^2 = 0 \} \text{ abs cgt)}
\]

for all \( f \) in \( B \). (Here \( \|\rho(f)\| = \text{ Hilbert-Schmidt norm} \)

Then \( \rho = 0 \) for all \( \rho \).

Recall \( \|\rho(f)\|^2 = \text{tr} \rho(f) \rho(f)^* = \text{tr} \rho(f g f^*) \). If \( g^* = g(f^*) \).

---

**Proof:** Pick \( \rho_0 \) and \( V_{\rho_0} \) in \( V \). This next bit (interwining operators) is a tech found in Jacquet-Langlands.

Define \( W = \text{End}_{rs}(V_{\rho_0}) = V_{\rho_0} \otimes V_{\rho_0}^* \). Acting by left composition with \( \rho(f) \).

Define \( \theta: B \to W \) by

\[
\theta(f) = (1_{C^1} \rho(f)) \text{ s.t. } \theta(f) \text{ is } \text{in}\ W
\]

Note that by hypothesis \( \sum \|\rho(f)\|^2 < \infty \).

Let \( W \subseteq W \) be the closure of the image of \( \theta \).

Suppose there was \( C > 0 \) s.t. \( \forall \rho \in V \), \( \|\rho \rho_0(f)\|^2 < C \sum \|\rho(f)\|^2 \) \( \text{ (2)} \)

Then there is a well-defined \( C^* \) map

\[
W \to V_{\rho_0} \text{ s.t. } \text{on the image of } B \text{ it is given by }
\]

\[
\theta(f) \mapsto \rho(f \rho_0).
\]

We must check \( \theta(f) = 0 \Rightarrow \rho(f \rho_0) = 0 \). But this is clear because \( \text{Tr} \).

It's also clear for the same reason.

It's also surjective because \( B \subseteq L^1(G) \) & \( \rho_0 \neq 0 \).

\( \theta \) covers the image in \( G \). Indeed, I guess.

Composing with the orthog proj \( W \to W \) we get a \( G \)-equivariant \( C^* \) linear map \( W \to V_{\rho_0} \) which is impossible as no \( V_{\rho} \) is \( \neq V_{\rho_0} \). \( \rho \neq \rho_0 \).
Hence no $C$ exists st. (*) holds.

However, $|c_{j_0}| \leq \sum_{j \neq j_0} |c_j| \|p_j\|_1^1$ for all $j \neq j_0$, so $c_{j_0} = 0$. \(\square\)

**Elementary facts about traces**

\(\text{tr}(A \otimes B) = \text{tr} A \cdot \text{tr} B\)

If $V = \oplus V_n$, $A = \oplus A_n$, $A_n$'s are diagonal operators on $V_n$,

\[A_n = \text{projection onto the distinguished 1-dim subspace, for all } n\]

Then, $\text{tr}(\oplus A_n) = \prod \text{tr} A_n$

$V, A_1, \ldots, A_l$ endomorphisms of $V$, $A_1 \otimes \cdots \otimes A_l \in \text{End}(\otimes^l V)$

$\sigma : \chi_1 \otimes \cdots \otimes \chi_l \mapsto \chi_2 \otimes \cdots \otimes \chi_l \otimes \chi_1$.

Then $\text{tr}(A_1 \otimes \cdots \otimes A_l \circ \sigma) = \text{tr}(A_1 A_2 \cdots A_l)$

Recall also Thm 2:

\[\text{tr} r_j^*(g) = \text{tr} R(p x \sigma); \qquad (\sigma)_x = \int e \in C_c^\infty(G(A_{F})) \cdot \int_{s, \psi \sigma \epsilon x} \text{assoc. } \psi_{x, \sigma}, \psi_{x, \sigma}\]

He never really told us what $r_j^*(g)$ was though. There’s actually 2 choices:

1. Can act on $L^2(G(A_F)^{\frac{1}{2}}) = L^2$ in 2 ways. Say $\psi \epsilon L^2$

   \[(i) (r_j^*(g) \psi)(x) = \int_{G(A_{F})^2} f(g) \psi(xg) dg\]

   \[(ii) (r_j^*(f) \psi)(x) = \int_{G(A_{F})} f(g) \psi(xg) dg \text{regarding } \psi \epsilon L^2(G(A_{F}), G(A_{F}))\]

2. $r_0^*(g) = r_1^*(g^4)$ where $\int_{\mathbb{R}^+} f(g^4) = \int f((s_{\sigma} g)^4) ds = 2$.

   (Assume we’ve been sensible with our Haar measures)

We proved them 2 for $r_0$. The association $f \rightarrow f_1^4$ changes the matching at infinity, unfortunately.
$f \rightarrow f^t$ change matching at $\infty$ by a multiple of $t$.

So the formula for the $\chi$-action is $tr(f) = tr(\rho \circ \chi)$. (of associated conjugate)

Note that the $\chi$ $\rho$-action is factorizable, the $\chi$ action unitary.

Now let's decompose the formula according to the decomposition of $L^1$, $\hat{L}^1$,
then
$tr(f) = \sum_{\rho} tr(\rho \circ \chi) \quad (\chi$ occurring in $\hat{L}^1$) (note we're using multiplicity 1 here)

$R(\rho \circ \chi)$. Say $\hat{L}^1 = \bigoplus T\chi$. If $T \chi \neq T\chi^\sigma$ then $R(\rho \circ \chi)$ cyclically
permutes the spaces $T\chi^{\rho \circ \chi}$ & so these spaces do not contribute to
the trace.

Now suppose $T\chi = T\chi^\sigma$. Then $T\chi$ is stable under $\sigma$, & therefore we can
regard it as a rep $T\chi_\rho$ of $G(E_{\chi}) \cong \langle \sigma \rangle$.

Note $T\chi = T\chi^\sigma \Rightarrow T\chi \otimes T\chi^\sigma \Rightarrow T\chi \otimes T\chi^\sigma$, so we can extend the
action of $G(E_{\chi})$ (in a non-unique way*) to a rep $T\chi_\rho$ of $G(E_{\chi})$ on the
space of $T\chi_\rho$ but any 2 extensions differ by an $\rho$-root of $1$ in $\sigma$.

For almost all $\chi$, choose $T\chi \rho$ st. $\rho$ is $\chi$ on the spherical vector, & adjust
$T\chi \rho$ for the other $\chi$ st. $T\chi \rho \cong T\chi^\sigma$ (this is a bit silly as $G(A) = T\chi G(E_{\chi})$
but $G(A) = T\chi G(E_{\chi})$. I think this is what's going on).

Now look at $\infty$.

If $T\chi$ is in $L^1$, $T\chi = T\chi_{\rho \circ \chi}^\sigma = T\chi_{\rho \circ \chi}^\sigma$ a rep of $G(F_\chi)$ & $T\chi^\sigma$ a rep of
$G(F_\chi^\sigma)$.

If $T\chi$ is in $\hat{L}^1$ st. $T\chi = T\chi^\sigma$, then $T\chi_{\rho \circ \chi} = \rho \circ \chi$, for some $\rho \circ \chi$ & we can
take $T\chi_{\rho \circ \chi}$ st. $\sigma$ acts by

$$X_{\chi} \circ X_{\chi} \circ X_{\chi} \rightarrow X_{\chi} \circ X_{\chi} \circ X_{\chi} \circ X_{\chi} \circ X_{\chi} \circ X_{\chi} = G(E_{\chi}) = G(F_\chi)_{\rho \circ \chi}$$
Let $\{ p \}$ be all unitary irreducible reps of $G(F_\chi)$ which occur either
as a $\pi_{\chi}$ or as a factor $\chi$ of a $\pi_{\chi}$.

The trace identity then becomes...
\[ \text{tr} \, r(y) = \sum_p \text{tr} \rho(f \circ \sigma) \circ \alpha_p, \quad \alpha_p = \sum_{n \in \mathbb{Z}} \text{tr} \pi(n) (y^n) \]

\[ \text{tr} \, R(\psi \circ \sigma) = \sum_p \text{tr} (\rho \circ \sigma \circ \rho)(\psi \circ \sigma) \circ b_p \]

\[ \text{with} \quad b_p = \sum_{I} \text{tr} \, T^{|I|} (\psi \circ \sigma) \]

Also recall \( \text{tr} \, r(y) = \ell \, \text{tr} \, R(\psi \circ \sigma) \).

Choose \( \psi_1 = \psi_{1,1} \otimes \ldots \otimes \psi_{1,1} \), \( \psi_{1,i} \in C_c^\infty (G(F_v)) \)

\[ f_v = \psi_{1,1} \otimes \ldots \otimes \psi_{1,1} \]

These are associated, like in Richards' case. (So is a split place?)

\[ \text{tr} (\rho \circ \sigma \circ \rho)(\psi \circ \sigma) = \text{tr} \rho(\psi_{1,1}) \rho(\psi_{1,2}) \ldots \rho(\psi_{1,1}) \]

\[ = \text{tr} \rho(\psi_{1,1} \ldots \psi_{1,1}) = \text{tr} \rho(f_v) \]

\[ \{ f_v \} \subseteq C_c^\infty (\ldots \otimes C_c^\infty (G(F_v))) \]

\[ \text{by e.g. Dixmier-Malliavin (although you can probably get away with much less)} \]

\[ \in L^2(G(F_v)) \]

So apply lemma 1, using \( f_v = f \circ f^{*} \).

\[ \Rightarrow \alpha_p, \, b_p \text{ for all } p, \, \& \text{ hence for all } p \text{ we have} \]

\[ \sum_{n \in \mathbb{Z}} \text{tr} \pi(n) (y) = \ell \sum_{\ell \in \mathbb{Z}} \sum_{I} \text{tr} \, T^{|I|} (\psi \circ \sigma) \]

\[ \text{for all } f = \otimes f_v \in C_c^\infty (G(F_v)) \]

\[ \psi = \otimes \psi_v \text{ associated everywhere.} \]

Note that this sum is finite. If \( f \) is bi-int by \( U \subseteq G(\mathbb{A}_F^\infty) \) & \( y \) is bi-int by \( U \subseteq G(\mathbb{A}_E^\infty) \), then \( \pi, \, \Pi \) don't contribute to the same unless \( (\Pi^{|I|}) U \ast (0), \, (\Pi^{|I|}) U \ast (0), \) & the set of such \( \pi, \, \Pi \) with \( f \) fixed cpt at infinity \( = \) finite (see e.g. Richards' case, probably).
§4. Spectral decomposition: finite places

We want to decompose $\Theta$ over $\mathbb{Q}$ even further.

Let $S$ be a finite set of finite places of $F$, including all primes ramified in $E$ or $D$.

$\Theta^S_E : = \Theta (G(A_F^S), \max \text{ opt } ) = \Theta$ unramified Hecke alg at all finite $v \notin S$.

$\Theta^S_\pi$ is the corresponding char of $\Theta^S_E$ if $\pi$ is unramified at all $v \notin S$.

Theorem 3. This theorem involves a sum over things which have so many cancelations in them that there's hardly any at all. In fact, both sums are typically empty, & the RHS sum has $0 = \ell$.

Let $\psi : \Theta^S_E \to \mathbb{C}$ be a character. Fix $\ell$.

Then $\sum_{\pi \in \mathbb{P}} \text{tr} \, \psi_\pi (f_v) = \frac{1}{2} \sum_{\pi = \pi_0} \text{tr} \psi_\pi \left( \varphi_{3 \times \varphi} \right)$.

Here, of course, $f_v = \otimes_{v \in S} f_v ; \varphi_v ; \Theta \otimes \Theta_v$ associated.

By next time.

Remark. Since $\Theta^S_\pi$ determines $\pi$ by strong mul, the RH expression has $0 = \ell$ has just 1 term.

Now suppose we're given $\pi$. Choose $S$ "large enough" & $\psi = \Theta^S_\pi$. Then RHS is 1 term only.

Choose $\varphi_3$, s.t. $\text{tr} \, \psi_\pi \left( \varphi_{3 \times \varphi} \right) = 0$.

Then RHS $= 0$; LHS $= 0$; LHS is not a sum over 0 elt(s).

Choose $\varphi_3$, s.t. $\Theta^S_\pi \cdot b_{E/F} = \Theta^S_\tau$.

But this just asserts that $\pi$ is a basechange of $\tau$. Hence

Cor. If $\pi = \pi^S_{\tau}$ then $\pi$ is the basechange of some $\tau \in \mathbb{P}$.

Unfortunately we can't use the same trick going the other way, as typically the LH sum has 1 elt.
Last time, Tony defined \( \mathcal{H}_e^S, \mathcal{H}_e^S \) to be the \( \circ \) of the unramified Hecke algebras at all finite \( v \neq S \).

Recall we're gonna prove

**Thm 3** Let \( \mathcal{H}_e^S \to \mathcal{C}, P \) a rep of \( G(F_w) \). Then

\[
\sum_{\nu_0 \in L^0} \text{tr} \tau_{\nu_0}(y_0) = \sum_{\nu_0 \in L^0} \text{tr} \tau'_{\nu_0}(y_{\nu_0})
\]

where \( y_0 \) are associated.

Recall \( \tau'= \tau'' \to \text{RHS} \to \sum \nu \text{br} (\nu) \to \exists \nu \tau'' \to \exists \nu'' \)

**Rk** It's easy to find \( y_0 \) s.t. \( \text{tr} \tau'_{\nu_0}(y_{\nu_0}) \neq 0 \)

- since if \( (\tau''_{\nu_0})^{K_S} = 0 \) (\( K_S \) suff small open cpt \( \subseteq \tau''_{G(E_v)} \))

then the image of \( \mathcal{H}(G_S,K_S) \) is the full endo alg of \( \tau''_{G(S)} \) (since it's unramed).

So e.g. can take \( y_0 \in \mathcal{H}(G_S,K_S) \) s.t. \( \text{tr} \tau'_{\nu_0}(y_{\nu_0}) = \text{tr} \tau'_{\nu_0} (y^{* -1}) \).

Then \( \tau'_{\nu_0}(y_{\nu_0}) \to \text{proj onto} (\tau''_{\nu_0})^{K_S} \), so trace \( \neq 0 \).

\[
\sum_{\nu_0 \in L^0} \text{tr} \tau_{\nu_0}(y_0) = \sum_{\nu_0 \in L^0} \text{tr} \tau'_{\nu_0}(y_{\nu_0})
\]

We need a first lemma analogue for pf. These things are traditionally called lemmas but they're really theorems.

**Lemma** (See Lafforgue's lectures) Fix \( v \) unramified in \( D E ; G(E_v) \cong GL_2(E_v) \)

\( G(E) \cong GL_n(F_v) \)

Suppose \( \{ \tau_{\nu_0} \}, \{ \tau'_{\nu_0} \} \) are finite collections of reps (unit, admn) of \( G(F_v), G(E_v) \).

Suppose we have the identity

\[
\sum_{\nu_0} c(\tau_{\nu_0}) \text{tr} \tau_{\nu_0}(y_0) = \sum_{\nu_0} d(\tau'_{\nu_0}) \text{tr} \tau'_{\nu_0}(y_{\nu_0})
\]

for all associated \( (y_0, y_{\nu_0}) \), under Iwahori subgroups of \( G(E_v), G(E_v) \) (cf. above) \( c(\tau_{\nu_0}), d(\tau'_{\nu_0}) \in \mathcal{C} \).

Then \( \bigcirc \) holds for all \( (\nu_0 \in D E, y_{\nu_0}) \) where \( y_{\nu_0} \) unramified Hecke algebra of \( G(E_v) \). Lafforgue will prove this this afternoon. \( \square \)
It's analogous to the Fundamental lemma, which is tricky to prove in this context.

Apply this lemma as follows: pick some associated \((\psi_3, \psi_5, \phi)\) \((f_s, y_s) \ (\& \rho)\).

Pick \(\psi_5 \in S\). Define \(f\) to be

\[ f = f_s \otimes f_5 \otimes (\text{unit elt at all } v \not\in S \cup v^\text{tr}) \]

\& \( \psi = y_s \otimes y_5 \otimes (\text{unit elt}) \)

where \(f_5, y_5\) are as in the lemma (associated & Iwahori-int)

Then \(f, \psi\) are associated everywhere (hypothesis + fund. lemma for unit elt).

\[ \sum_{\pi \in \text{rop}} \text{tr}_{\psi_3}(f_s) \text{tr}_{\psi_5}(\psi) = \sum_{\pi \in \text{rop}} \text{tr}_{\psi_5}(\psi_{E, \pi}) \text{tr}_{\psi_3}(\psi) = \text{C} \]

The sets of \(\{\pi\}, \{\pi_{E, \pi}\} \) occurring in both sides have non-zero trace are finite (unramified all \( v \not\in S \cup S^\text{tr} \), but by fixed \( v \) open at \( S^\text{tr} \)).

Applying lemma, get that \(\text{C} = \text{C} \) holds also for \(\psi_5 \) unramified Hecke algebra.

Now take another place \( v \not\in S \cup S^\text{tr} \)... (to be continued)

We end up with the identity

\[ \sum_{\pi \in \text{rop}} \text{tr}_{\psi_3}(f_s) \text{tr}_{\psi_5}(\psi_{E, \pi}) = \sum_{\pi \in \text{rop}} \text{tr}_{\psi_5}(\psi_{E, \pi}) \text{tr}_{\psi_3}(\psi) \]

for all \( \psi_5 \in \mathfrak{H}_{E}^{\text{un}} \), \( \lim_{\psi_5 \to \psi_5} \mathfrak{H}(G(E), \text{Knew}) \)

\[ \text{is} \sum_{\pi \in \text{rop}} \text{tr}_{\psi_5}(f_s) \text{tr}_{\psi_5}(\psi_{E, \pi}(\psi_5)) = \sum_{\pi \in \text{rop}} \text{tr}_{\psi_5}(\psi_{E, \pi}(\psi_5)) \text{tr}_{\psi_3}(\psi) \]

(\( \H_{\text{un}, E} \), \( \pi \approx \text{known} \). The character of \( \mathfrak{H}_{E}^{\text{un}} \) are unramified, so can decompose the last identity corresponding to char \( \mathfrak{H}_{E}^{\text{un}} \rightarrow \mathbb{C} \)

\[ \Rightarrow \text{Thm 3...} \]
This is the fundamental lemma. It would be a bit
simpler, but we use the 3 to prove II. lemma.

This is nearly all the ingredients. We do not
mention the rest.

\[ \text{§5 Application of L-factors} \]

For all $v \in S$ (finite set) $\pi_v$ is unramified, so it corresponds to
\[ \pi_v = (\alpha, \beta) \in \text{GL}_2(C) \] (up to conjugacy)

\[ L(\pi_v, s) = \det(I - q^{-s} \pi_v)^{-1} = 1 + q_c q_v s (I - \beta v q_v^{-1})^{-1} \]

\[ \det(I - q^{-s} \pi_v (F_v))^{-1} \]

where $\sigma_v : W_v \to \text{GL}_2(C)$ is the unramified HM st. $\sigma_v(F_v) = \pi_v$

\[ L_s(\pi_v, s) = \prod_{v \in S} L(\pi_v, s) \] (the incomplete L-factor)

$\pi$ cuspidal $\Rightarrow$ an entire $f^*$ of $s$ (Hecke theory as applied
by Jacquet, Langlands)

$\pi, \pi'$ are unram for $v \not\in S \Rightarrow$ "Rankin convolution".

\[ L_s(\pi \times \pi', s) = \prod_{v \in S} L(\pi_v \times \pi'_v, s) \]

where $L(\pi_v \times \pi'_v, s) = L(\sigma_v \sigma'_v, s) = \det(I - q^{-s} \pi_v \pi'_v)^{-1}$

\[ \mathcal{L} = (\alpha, \beta) \]

\[ (\alpha, \beta) = \left( \begin{array}{cc} \alpha \alpha & \beta \beta \\ \beta \alpha & \alpha \beta \end{array} \right) \]

\[ \text{Theorem 4 (Jacquet, Shalika) (true for $GL_m, GL_n$). Assume $\pi, \pi'$ are unramified unary & cuspidal. Then $L_s(\pi \times \pi', s)$ is holomorphic for } s \in C \text{ except if } \pi \sim \pi', \text{ when } s \text{ has a simple pole.} \]

\[ (\pi, \pi' \to \exists s \text{-finite } \text{ in } L^2, \text{ with boundedness of } \text{gives } \text{ an } L^2 \text{ on } \text{the group in } GL_2). \]
Suppose $\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \ldots, \pi^{(n)}$ are $2r$ irreducible auto reps of $GL_r(\mathbb{R})$, & $S = \{a \text{ (large enough)}\}$ set of primes.

Suppose that for all $v \in S$, $\pi_v^{(1)}, \pi_v^{(2)}, \pi_v^{(3)}, \ldots$ are unramified, & also that

$(\dagger) \quad \sigma_v^{(1)} \otimes \cdots \sigma_v^{(n)} \cong \sigma_v^{(n_1)} \otimes \cdots \sigma_v^{(n)}$.

Then $[\pi^{(1)}], [\pi^{(2)}]$ are the same (up to reordering).

Notes $(\dagger) \iff$ the matrices $(e_v, 0) \& \left( \begin{array}{cc} e_v & 0 \\ 0 & e_v \end{array} \right)$ are conjugate in $GL_{2r}(\mathbb{C})$.

Proof Let $\Lambda^r(s) = \prod_{j=1}^{\infty} L(s, \pi^{(j)} \otimes \pi^{(j)})$.

The local factor $\Lambda_v^r(s)$ (obvious notation)

$$\Lambda_v^r(s) = \prod_{j} \det (1 - q_v^{-s} L_v^{(j)} \otimes L_v^{(j)})^{-1}$$

$$\otimes \sigma_v^{(n_1)}(F_{\mathbb{R}}) \otimes \cdots \sigma_v^{(n)}(F_{\mathbb{R}})$$

$$\Lambda_v^r(s) = \prod_{j} \det (1 - q_v^{-s} \sigma_v^{(j)}(F_{\mathbb{R}}) \otimes \sigma_v^{(n_1)}(F_{\mathbb{R}}))^{-1}$$

$$= L(\sigma_v^{(n_1)} \otimes \sigma_v^{(j)}(\pi^{(j)}), s) = \Lambda_v^r(s)$$

(Note: $\Lambda^{(1)} \otimes \cdots \otimes \Lambda^{(1)}$)

$\Lambda(s)$ has a pole at $s=1$ (from $\pi^{(1)} \times \pi^{(1)}$), by Thm 4.

So $\Lambda^r(s)$ has a pole at $s=1$, hence $\pi^{(1)} \cong \pi^{(1)}$ for some $j$ (Thm 4).

Result by induction.

NB Tony has no clue, he has to confess, so to, why the thm is true (the $40$ let). Classically Rankin proved stuff about the complete $L$-f. so maybe you have to understand that prime.

This will not stop Tony drawing further corollaries.

Cor. 2

$E/F$, $G$ as before, $\pi$, $\pi'$ irreducible auto reps of $GL(\mathbb{R})$. Assume $\pi, \pi'$ each have place change to $E$. Then the place changes are iso. Then

$$\pi'_v \cong \pi_v \oplus N_{v_1}(\pi_v) \otimes N_{v_2}(\pi_v)$$

for some $j$.

Proof If $\pi, \pi'$ are $1$-dim, this is just CFT. If $\pi$ is so, then $\pi' \cong \pi \otimes \phi$ for only many $\nu$. $\pi$ is associated to a cuspidal auto rep by $J$-$L$ correspondence.
As $\pi, \pi'$ have the same branching for almost all $v, \pi'$ is also a direct sum of cuspidal reps of $GL_n(A_F)$. Apply the colliding to

$$\{ \pi \otimes (\chi_{E/F} \circ N_{\text{red}}) \}, \{ \pi' \otimes (\chi_{E/F} \circ N_{\text{red}}) \}$$

It's enough to check $\boxplus$. (cor 2)

$$\pi_v \mapsto \sigma_v,$$
$$\pi'_v \mapsto \sigma'_v.$$

Then $\sigma_v \big|_{W_{E_v}} \sim \sigma'_v \big|_{W_{E_v}}.$

$$\sum_{j=1}^l \sigma_v \otimes \chi_{E_v/F_v}^j \equiv \bigoplus_{j=1}^l \sigma_v \otimes \chi_{E_v/F_v}^j \quad \Box$$

The same pf shows that the LHS of thm 3, namely

$$\sum_{\text{triv}, \oplus \pi, \Theta, \text{g} \in \mathbf{E}} \text{tr} \Theta(\bar{\chi}(s))$$

is either empty, or a sum over reps of the form

$$\pi \otimes (\chi_{E/F} \circ N_{\text{red}}), \text{ some fixed } \pi.$$ Given $\pi$, choose $g$s s.t. $\text{tr} (\pi \circ \chi^j)(g) = \text{tr} \pi(g) \neq 0$.

Choose $g$s with support a suff small nhbd of 1. Then LHS $\neq 0$ ( $\equiv 0 \text{ tr} \pi(g)$)

So RHS $\neq 0$ is $\exists T$. \[\square\]
This lecture is really about

Unramified local base change

& it'll be putting together lots of things we've heard before. (Scholl etc.)

\( \mathbb{O} \) = integers
\( \mathbb{F} \) a local field, \( \mathbb{O} \)-adherence
\( \mathfrak{m} \) a uniformizer, \( \mathbb{K} \) residue class field, \( q = \mathbb{K}^* \), \( d \) = normalized dir. val. of \( \mathbb{F}^* \).

\( G = \text{GL}_2(\mathbb{F}) \) (hardly any less of generality here)
\( K = \text{GL}_2(\mathbb{O}) = \text{max} \text{cpt subgp} \)
Norm an \( dg \) s.t. \( \int_K dg = 1 \)
A general convention: if \( H \leq G \) is a closed subgp then \( H \) normalize \( dh \) s.t. \( \int_{H/K} dh = 1 \).

\( H = \text{Hecke algebra} = K\text{-bi-int \( \mathfrak{f} \)'s on} \ G \text{ with cpt support} \)

\( \phi \ast \psi = \int_G \phi(g) \psi(g^{-1}) \ dg \)

\( 1 = \text{clap of} \ K. \)

Later \( E/F \) will be an unramified ext.

1) Say \( S \leq G \) is the diagonal matrices.

\( \text{An unramified char of} \ S \to \chi: S \to \mathbb{C}^\times \) s.t. \( \chi_{(0)^d} = 1 \)

We have a bijection

unram char of \( S \) \( \leftrightarrow \mathbb{C}^d \)

via \( \xi = (\xi_1, \ldots, \xi_d) \mapsto (\chi_{\xi_1}; (a_1, a_d) \mapsto \prod \xi_i \omega(a)) \)

\( W = \text{subgp of permutation matrices in} \ G \cong S_d \)

\( W \) acts on \( S \) by conjugation.
We also act on unram. char of $S$, & this corresponds to the permutation action on $C^d$.

Think of $C^d$ as being the diagonal matrices in $GL_d(k)$.

We have the Jordan normal form: any $a \in GL_d(k)$ is conjugate to a diagonal elt

- 2 diag elts are conjugate if they're a permutation of each other.

We get a bijection

\[
\left\{ \text{W-orbits in the unramified char of } S \right\} \leftrightarrow \left\{ \text{semisimple conj. classes in } GL_d(k) \right\}
\]

There's bijection (1). Here comes another one.

(2) $W_F$, the Wedge of $F$.

An unramified parameter $\mathfrak{a}$ of $W_F$ is (the isom. class of) a semisimple rep,

$\mathfrak{a} : W_F \rightarrow GL_d(k)$, s.t. $\mathfrak{a} \mid_{\text{inertia subgroup}} \neq 1$.

However, $W_F/\text{inertia} \cong \mathbb{Z} = \langle \text{Frobenius} \rangle$.

So $\mathfrak{a}$ is determined by $\mathfrak{a}(\text{Frob}) \in \left\{ \text{any classes in } GL_d(k) \right\}$.

Hence

\[
\left\{ \text{unramified parameters of } W_F \right\} \leftrightarrow \left\{ \text{semisimple conj. classes in } GL_d(k) \right\}
\]

So we get from (1) & (2), the following bijection:
\[
\{ \text{W-orbits in the unramified chaos of } S \} \leftrightarrow \{ \text{unramified param of } W_f \}
\]

\[
\chi \longrightarrow \left( \text{Frob} \mapsto \begin{pmatrix} \chi(1) & 0 \\ 0 & \chi(\omega) \end{pmatrix} \right)
\]

where \( s_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \)

What has this got to do with Hecke operators?

3. If \( \Lambda = S/\text{Sk}_K = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_d \), \( \lambda_i = s_i(\text{Sk}_K) \).

Then we get yet another bijection:

\[
\{ \text{W-orbits in unram. chaos} \} \leftrightarrow \text{Hom}_{\text{alg}} \left( \mathbb{C}[\Lambda]^w, \mathbb{C} \right)
\]

\[
\chi \longrightarrow \alpha_\chi : \alpha_\chi (\Sigma c_{i,k}) = \Sigma c_{i,k} \chi(\lambda)
\]

Note: \( C[\Lambda]^w \subseteq C[\Lambda] \) is finite

\[
\max_{\text{alg}} ( C[\Lambda] ) \rightarrow \max_{\text{alg}} ( C[\Lambda]^w ) \quad \text{is surjective (this is some kind of going-up thing)}
\]

We also recall the Satake iso:

\[
\mathcal{H} \rightarrow C[\Lambda]^w
\]

\[
\varphi \mapsto ( S \naturel \mapsto \sum_N \varphi(s_\infty) du \times \delta(s/4)); \quad N = \left\{ \left( \begin{smallmatrix} 1 & \ast \\ 0 & 4 \end{smallmatrix} \right) \right\} \text{ is not unimodular, & so you put } S \text{ in to make it work.}
\]

Then (Satake) \( S \) is an algebra isomorphism... We've seen a pf for GL_2.
Consequences

(a) Set $\sigma = \lambda_1 + \ldots + \lambda_d$ be the elementary symmetric poly.

Then $C[\lambda] = C[\lambda_1^\pm 1, \ldots, \lambda_d^\pm 1] = C[\lambda_1, \ldots, \lambda_d, \sigma_2]$.

Hence $C[\lambda]^w = C[\sigma, \sigma_2, \sigma_3]$.

So this ring is explicit. Let's make the map explicit too.

Define $t_i = \left( \begin{array}{ccc} 1 & & \\ \sigma_i & & \\ & & 1 \end{array} \right)$.

Define $\tau_i = \text{char } \Phi$ of $K \in K = K(G, K)$.

Fact: $\tau_i(\tau_i) = q^{\frac{1}{2}(d+1)} \sigma_i$ (easy for $d=2$) (Tony did it).

So the Satake map is now explicit.

Hence

(b) $\{W- orbits \} \leftarrow \text{Hom}_G \left( C[\lambda]^w, C \right) \rightarrow \text{Hom}_G (K, C)$.

We understand $\otimes$. If $\alpha \in \text{Hom}_G (K, C)$ then $\otimes$ sends $\alpha$ to the rep. sending Frobenius to $\left( \begin{array}{cc} \tau & 0 \\ 0 & \tau \end{array} \right)$.

where $\tau^i = \sum_{\lambda} (1)^d \alpha \left( \tau^{i}_{d-1} \right) X^i$.

This is rather obvious.

There is more...
Clearly \(\mathbb{Z}\) is commutative & e.g. \(\mathbb{Z}/N\).

So by Schur's lemma, any simple \(\mathbb{F}\)-module is 1-dim & given by a char in \(\text{Hom}_\mathbb{F}(\mathbb{F}, \mathbb{C})\).

He's not so sure about any more. He'll postpone it.

4. The unramified rep-theory of \(G\)

A smooth & irred rep of \(V\) of \(G\) is unramified if \(V^K = \{0\}\).

Fact. \(V^K\) is a simple \(\mathbb{F}\)-module via \(g \ast v = \sum \chi(g)gv\)

Hence we have a map

\[
\begin{align*}
\text{unramified reps of } G & \hookrightarrow \text{Hom}_\mathbb{F}(\mathbb{F}, \mathbb{C}) \\
V \cong V_\omega & \leftrightarrow \omega \text{ s.t. } g \ast v = \chi(g)v \text{ for } g \in K, v \in V^K
\end{align*}
\]

We've done this when \(d=2\) but he wants to talk about

Realisation of \(V_\omega\)

Choose the unramified char \(\chi\) of \(S\) corresponding to \(\omega\). Then

\(\rho\) via principal series: \(\text{I}(\chi) = \text{space of all } f:G \rightarrow \mathbb{C} \text{ s.t.} \)

\[
f(gns) = s(s)^{-1/2} \chi(s^{-2}) f(g)
\]

He's addicted to left actions so it's all the other way round. \(G\) acts by left translations. Boo.

Fact. \(\text{I}(\chi)\) has a 1-irred subquotient \(V_\chi\) which is unramified \(\cong V_\omega\).

There's another way

\(\rho\) via spherical \(f\)'s. It uses 4) above which was an explanation of spherical \(f\)'s so he'll have to and 6) too.

The spherical \(f\) would have been written \(F_\chi\).
So we have a bijection (the unramified local Langlands correspondence):

\[
\left\{ \text{Isom classes of unramified reps of } G \right\} \leftrightarrow \text{Hom}_{\text{alg}}(\mathbb{K}, \mathbb{C}) \leftrightarrow \left\{ \text{unramified params of } W_F \right\}
\]

this is the correspondence.

Now say $E/F$ is a finite unramified ext. of degree $l$.

It's all easy:

\[
\left\{ \text{Isom classes of unram reps of } G = \text{GL}_d(F) \right\} \leftrightarrow \left\{ \text{unram params of } W_E \right\}
\]

\[
\text{Hom}_{\text{alg}}(\text{GL}_d(F), \text{GL}_d(O_E)) \quad \text{restriction}
\]

\[
\left\{ \text{Isom classes of unram reps of } G_E = \text{GL}_d(F) \circ E : \text{GL}_d(E) \right\} \leftrightarrow \left\{ \text{unram params of } W_E \right\}
\]

\[
\text{Hom}_{\text{alg}}(K_E, E)
\]

We get a base change map for Hecke algebras $b : H_E \rightarrow H$ characterized by:

$W$-orbit in $\text{unram. reps of } S \quad \rightarrow \quad W$-orbit in $\text{unram. reps of } S_E$

$\chi \quad \rightarrow \quad \chi^{10}$

(note new Frobenius field $E$)
Thus comes from the map
\[
\begin{align*}
\Sigma C[\lambda] & \leftarrow \Sigma C[\lambda] \\
\Sigma C[\lambda] & \rightarrow \Sigma C[\lambda] \\
\mathfrak{H}_E & \leftarrow \mathfrak{H}_E
\end{align*}
\]

He has \( E/F \) unramified & \( \langle \sigma \rangle = \text{Gal}(E/F) \) although he may call it \( \Theta \) by accident because Kottwitz calls it \( \theta \) & his cribbed this off Kottwitz.

We have \( G \) & \( G_E \), & if \( g \in G_E \) define \( N_g = g^{-1} \sigma g \cdot \sigma^t g \)

This \( N \) induces an injection
\[
N: \{ \sigma \text{-conjugacy classes in } G_e \} \rightarrow \{ \text{conjugacy classes in } G \}
\]

& also recall we have \( \mathfrak{H}_E \rightarrow \mathfrak{H} \).

We also have the orbital integrals:
\[
\forall g \in G, \sigma \in H \rightarrow O_g(\chi) = \int_{G} \chi(x^{-1}y) \, d\lambda 
\]

( Pick a fixed Haar measure)

\[
ge \in G, \sigma \in H \rightarrow O_{ge}(\chi). \text{ This is our twisted orbital integral but he will call it } O, \text{ not } O_T \text{, as } \sigma \text{-orbits are really just orbits in some semidirect product of } G_E \text{ & } \langle \sigma \rangle \text{ or so.}
\]

\[
O_{ge}(\chi) = \int_{G_E \backslash G} \chi(y^{-1}g^{-1}y) \, dy
\]

We only defined this stuff for \( \sigma \) elts. I think he's only going to use it for \( \sigma \) elts.

He's now in a par to state the fundamental lemma.
Fundamental Lemma

1) If the orbit $O_y \leq G$ is not a norm, then $O_y(by) = O$ for any $y \in H_E$.

2) If $O_y = N(O_{go})$ then $\exists c \in K^+, \text{ st. } O_y(by) = c.O_{go}(y) \forall y \in H_E$.

His job is to prove this for $y = 1_E$. Then $by = 1$.

Define:
Put $X = G/K \hookrightarrow X_E = G_E/K_E$

& put $X^g = \text{fixed pts of } g \text{ on } X$, $(G_g)_x = \text{stabilizer of } x \in X^g$ in $G_g$

$X^g_{1_E} = \text{ fixed pts of } g_0 \text{ on } X_E$.

$(G_{g_0})_x = \text{ fixed of } x \in X^g_{1_E} \text{ in } G_{g_0}$

He needs

Lemma.

1) $O_y(1) = \sum_{x \in O_y \setminus X^g} \text{vol}((G_g)_x)^{-1}$

( may be infinite sum; he thinks)

2) $O_{go}(1_E) = \sum_{x \in G_{go} \setminus X^g} \text{vol}((G_{go})_x)^{-1}$

( nb: both sides of both eqn depend on a choice of Haar measure... so take the same one!)

$\therefore 2) \Rightarrow 1) (E = F)$

2) $O_{go}(1_E) = \text{vol}((G_{go}/\bigcup_{y \in Y} G_{go} y)_{1_E}) = \sum_{y \in Y} \text{vol}(G_{go}/G_{go} y K_E)_{1_E}$

$I = y K_E \setminus X^g$

$= \sum_{x \in G_{go} \setminus X^g} \text{vol}(G_{go}/G_{go} y K_E)$

$= (y^{-1} G_{go} y K_E)_{y K_E}$

$= (G_{go} \cdot y K_E y^t)_{y K_E y^t}$

volume $1$
Now we need

Prop. Assume $X^g = \emptyset$; then there exists $g \in G_{\mathbb{E}}$ s.t.

a) $N_g \gamma$

b) $\gamma$ lies in the centre of $G_{\mathbb{E}}$

c) $X_g = X_{\mathbb{E}}$

Moreover, we then have $X^\gamma \rightarrow X^g_{\mathbb{E}}$ & $G_{\mathbb{E}} \gamma : G_{\mathbb{E}}$

We'll prove this in a sec...
We'll now prove the fundamental lemma

ii) If of fund. lemma

i) $O_{\mathbb{E}}$ is not a norm $\Rightarrow X^T = \emptyset \Rightarrow O_{\mathbb{E}}(T) = 0$

ii) Assume $O_{\mathbb{E}} = N(O_{3\mathbb{E}})$

Case 1) $X^T = \emptyset$. Apply prop

Case 2) $X^3 = \emptyset \Rightarrow X^3 = \emptyset \Rightarrow O_{\mathbb{E}}(T) = 0$

$N(h^2 g^2 h) = h^2 N(g) h \gamma$

$\Rightarrow$ can assume $N_g = \gamma \Rightarrow X^g_{\mathbb{E}} \subseteq X^g_{\mathbb{E}}$

Proof of prop

Think of $X_{\mathbb{E}}$ - set of all $O_{\mathbb{E}}$ lattice in $E_d$

$K_{\mathbb{E}} = GL_d(O_{\mathbb{E}})$ = stabilizer of $O_{\mathbb{E}} \subseteq E_d$

Put $C_{\mathbb{E}} = O_{\mathbb{E}}$ subalgebra in $M_d(E)$ of all $E \in$ s.t.

$-y \in$ belongs to the centre of the centralizer of $y$ in $M_d(E)$

$-y \in \Lambda \subseteq \Lambda$ for any lattice $\Lambda \in X_{\mathbb{E}}$

$C_{\mathbb{E}}$ is commutative; $\exists G \Rightarrow C_{\mathbb{E}}$ is $G$-stable

$X_{\mathbb{E}} \neq \emptyset \Rightarrow C_{\mathbb{E}}$ is as an $O_{\mathbb{E}}$ module finitely generated & free.

$(C_{\mathbb{E}}$ is some conjugate of $M_d(O_{\mathbb{E}}))$

Put $C = C_{\mathbb{E}}$, an $O_{\mathbb{E}}$ subalgebra. It's somehow clear that $C \otimes O_{\mathbb{E}} \rightarrow \otimes C_{\mathbb{E}}$
Claim: $C^\sigma \rightarrow C^\tau$ is surjective

Clearly $\forall x \in C^\sigma$, & hence $\exists g \in C^\sigma$ s.t. $Ng = g$. (a) \( \forall \)
and also (b) \( \forall \) because of def of $C^\sigma$.

Moreover, $g^\sigma \in C^\sigma$ too so $g^\sigma \in C^\sigma$, $g^\sigma \in C^\sigma$, $\forall \lambda \in C^\sigma$

$L = g\lambda \ \& \ \text{we get e} \ \forall$

Note $\sigma^{-1}(g) \in C^\sigma \Rightarrow x_E^\sigma \in X_E^\sigma \Rightarrow g\sigma^{-1}(x) = \sigma^{-1}(\sigma g(x)) = \sigma x$ for $x \in X_E^\sigma$

But $N g = g \Rightarrow x_E^{g\sigma} \in X_E^\sigma$, a general fact (easy direct proof)

$\therefore x_E^{g\sigma} = (x_E^\sigma)^{\sigma} \Rightarrow X_E^\sigma$

Finally we need to show $G_{g\sigma} = G_g$

Now $N(h^* g^* h) = h^* N(g) h \Rightarrow G_{g\sigma} \subseteq G_g(E)$

(b) $g$ lies in the centre of $G_g(E)$

h $\in G_{g\sigma}$; $h^* g^* h = g$

$gh^* g = G_{g\sigma} \subseteq G_g(E) \Rightarrow G_{g\sigma} = G_g(E)^{g\sigma} \Rightarrow G_{g\sigma}$ \( \square \) of prop.

So we just have to prove the claim now.

Pf of claim: (It's just some generalisation of the CFT norm map being mg)

Consider the filtration $C^\sigma \supseteq 1 + \tau C^\sigma \supseteq 1 + \tau^2 C^\sigma \supseteq \ldots$

$C^\sigma \supseteq 1 + \tau C^\sigma \supseteq 1 + \tau^2 C^\sigma \supseteq \ldots$

It suffices to prove the assertion for each subquotient.

But $\otimes (1 + \tau^m C_E) / (1 + \tau^{m+1} C_E) \cong \tau^m C_E / \tau^{m+1} C_E$

$N$

$\tau^m C / \tau^{m+1} C \cong \text{trace}$

$(1 + \tau^m C) / (1 + \tau^{m+1} C) \cong \text{trace}$

So we just have to deal with the first step.
But here we can use a general fact (proved in e.g. Serre's book) if $A$ is a finite commutative $K$-algebra, then

$$(A \otimes K_E)^x \xrightarrow{N} A^*$$

is surjective.

So we're home.
Jean-Pierre is going to chat a bit about the fundamental lemma.

If things don't match then you can't even start - e.g. \( \psi \otimes \phi \), etc - you absolutely needed the matching result for \( 1_E \) & \( 1 \).

The fundamental lemma in its full generality may not be needed. He will discuss (a variant of) it anyway.

We have a local field \( F \) with unif. par to so same confusion with \( \Omega \).

We have \( H \), \( H = G \), \( K = G \) of \( \Omega \).

We want to do some easy harmonic analysis, but hey - we're beginners!

Say \( \chi(k) = \text{char}(K \times K) \)

We have \( K \cdot K \times K = \sum c(t_1, t_2; c) K \cdot K \).

\( K \cdot K = K \cdot K, \quad t = (B_{n+1}, 0) \).

Set \( s = (0; 1) \in K \). The map \( g \mapsto s(g) s^2 \) is an antiautomorphism & it preserves \( K \cdot K \).

Hence the commutative \( (?) \).

We have \( S \), the Satake transform: \( \psi \) is \( H \), \( K \), \( s \).

\[ (S \psi)(m) = \int \psi(h) \delta(m) \psi(d) \mathrm{d}h \]

where \( m = (\text{diag}) \), \( N = (0; 1) \).

Now assume \( m \) is regular, \( m = (m_1, 0) \), \( m_1 \neq m_2 \).
\[ (\hat{\text{Sh}})(m) = \int h(g^* \text{mg}) \Delta(m)^n \, dg \quad ; \quad \Delta(m) = \det(1 - \text{Ad} m | \mathfrak{g} / \mathfrak{m}) \]

He will prove this. Now to slow him down.

We have the Iwasawa decomposition \( G = MNK ; m \mathfrak{g} \rightarrow m \mathfrak{d} \mathfrak{n} \mathfrak{k} \).

Then \( (\hat{\text{Sh}})(m) = \int \int h(n^m \mathfrak{n}) \Delta(m)^n \, d\mathfrak{n} \, d\mathfrak{m} \) - here vol(K) = 1.

Note that \( n^m \mathfrak{n} = m^{-1} n^m \mathfrak{n} \) so \( d\mathfrak{n} = c(m) \, dn \).

\[ \Delta(m) = |D(m)| = |(1 - m^2)(1 - m^3)| ; \quad m^2 = m^3 / m_2 \quad \text{and} \quad \delta(m) = |m^3| \]

\[ \delta(m)^2 \Delta(m) = |1 - m^2| \]

That's enough of that nonsense. It's very calculations. You should see them once in your life.

We have \( \text{H}(G, K) \rightarrow \text{H}(M, MNK) \).

If \( \chi \) is an unramified char then a char of \( M / MNK \),

we get a rep of \( \text{H}(G, K) \):

\[ h \rightarrow \hat{\text{Sh}}(\chi) = \int \text{Sh}(m) \chi(m) \, dm \]

\[ \text{H}(G, K) \rightarrow \mathbb{C} \]

whose so \( \hat{\text{Sh}}(\chi) = \int \int h(g^* \text{mg}) \Delta(m)^n \chi(m) \, dg \, dm \)

Now given \( \chi \) we can of course form the principal series \( I\chi \)

\( I\chi = \rho(\mu_1, \mu_2) \) in Tani's notation.

We have \( \hat{\text{Sh}}(\chi) = \mathbb{L} \chi \chi(h) = \mathbb{L} \chi \text{tr} \pi_\chi(h) \)

\( \pi_\chi = \sigma(\mu_1, \mu_2) \) spherical subquadratics (ie contains a K-inert vector).
Now \( f \mapsto \hat{f} \) we get \((\hat{f})(m) = \int \hat{f}(g^2mg) \Delta(m)^x dg\).

\[ (\hat{f})(m) = \int \int f(k^2mnk) \Delta(m)^x dk dc \]

It turns out that \( tr I_X f = \hat{f} \)

but the trace of the subquotient may not be equal in general:

\( tr I_X f \neq tr R_X f \) in general. That's life.

Now \( I_X \) acts by right translations by \( G \) on the space of \( f \)'s on \( G \)

\[ (I_X f)(x) = \int f(xy) g(x) dy \]

\( I_X \) has a kernel on \( K \), \( L^1(K) \) (note \( f \) is determined \( (y, f|_K) \)

\( I_X \) not unitary \( \implies \) rep. unitary, probably.

\[ \int K_{I_X}(y)(k) f(x) dk = (I_X f)(x) \]

\[ (I_X f)(x) = \int f(xy) g(k^2xy) dy, \quad y = \text{simple mnk, careful about Haar measure} \]

\[ K_{I_X}(y)(k) = \int \int f(k^2mnk) \Delta(m) \delta(m)^x dk dc \]

\( (\hat{f})(x) = \int \int f(k^2mnk) \Delta(m) \delta(m)^x dk dc \)

Hence trace of \( I_X f \) = Mellin transform of Zucke transform.

Note \( tr I_X f \cdot I_X f = tr I_X f \cdot I_X f \)

However \( tr I_X f \neq tr R_X f \) in general - semisimplification of \( I_X \) \( \neq R_X \) (other stuff).

He's still not quite at the end of his introduction.
Say $E/F$ is a cyclic unramified ext.

$\langle \sigma \rangle = \text{Gal}(E/F) ; \ G_E = \text{Gal}(E)$. 

We can form $G_E \times \langle \sigma \rangle = G'_E \& K \times \langle \sigma \rangle = K'_E$ 

If we go again, $\varphi(\alpha m) = \varphi(x) \chi(m) \delta(m)^v , m \in M'_E$ 

$\varphi(x) = 1 \text{ for } k \in K$ 

$\varphi(x) = 1 \text{ for } k \in \mathbb{Z}/12$ 

Now let's look at $H(G_E, K_E)$ (no $\sigma$ an $K_E$) 

$X$ extended from $M_E$ to $N'_E$, by making it trivial at $\sigma$, is gonna be called $X$.

$I_X(\sigma) ; \sigma = \text{sum of double coset classes translated by } \sigma$

$(Sh')(x) = \int_{M'_E} h'(g', m') \chi(m) dg'$

You have to be a bit careful to with the centralizer

We have $b_{E/F} : H(G, K) \rightarrow H(G_E, K_E)$, & $x \mapsto H(G_E, K_E)$,

$\text{tr } R_{x_E}(h) = \text{tr } R_{x_E}(h) \text{ when } x \cdot x = N_{E/F}$

This tells you a lot. But not everything. (w.r.t the Fundamental Lemma)

$\text{tr } R_{x_E}(h) = \text{tr } R_{x_E}(h)$ implies immediately that

$TO_{x'}(h) = O_x(h) \text{ if } x \in M_E, x_E M_E \& x' \in N_x$

$O_x(h) = \int_{M_E} f(x', y) dx'$ ;

$O_x(h) = \Delta(x)^h S(h)(x)$

Similarly $TO_{x'}(h) = \Delta(x)^h S(h)(x)$

$\Delta(x) = \Delta^*(x) \text{ if } x \in N_x$
\[
\det \left( \begin{bmatrix} 1 - t \sigma & \mathcal{G}_E / \mathfrak{m}_E \\ \mathcal{G}_E / \mathfrak{m}_E & \mathfrak{m}_E \end{bmatrix} \right) = \det \left( \begin{bmatrix} 1 - N \sigma \\ \mathfrak{m}_E \end{bmatrix} \right) \\
\mathfrak{m}_E / \mathfrak{m}_E \otimes F
\]

If we had to work only with split elements then, we would be done.
However we have to deal with elliptic els.

There are 2 strategies: the first due to Langlands is to compute explicitly or the somethings (all the cases?)
- do it for all cases. \( G_2(h) \rightarrow O_6(h) \)
  
Larglands did it for \( G_2 \)  
Kottwitz did it (a tour de force) for \( G_2 \)
  
I'm doing so discovered the trick that Peter Schneider told us about this morning.

The 2nd trick is due to Clozel et al & is not to compute but to find enough els which you know is true for \( \mathfrak{m} \) then deduce the general case.

Tomorrow he'll talk about this.

Technique: work with a subalgebra of the Iwahori-Hecke algebra

\[
\mathfrak{B} / \mathfrak{B}_1 = \text{Iwahori:} \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) & \text{we have} \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) / \text{we have fundamental lemma for this (we will only have split objects then) & then deduce the general fundamental lemma.} 
\]
Recall $B(t) = \frac{B(\delta Q_n \circ Q_n^*) B}{\text{Vol}(B)}$, $n > n'$ integers.

Let $\delta = (\delta_t^Q_0 \circ Q_n), \delta' = (\delta_t^Q_0 \circ Q_{n'})$, $n > n'$.

Note $B(t) B(t') = B(t') B(t)$ & the $B(t)$ generate a commutative subalgebra of the double coset algebra. (Here $B$ is the Iwahori $\left( \begin{smallmatrix} 0 & \mathbb{C}^n \\ 0 & 0 \end{smallmatrix} \right)$)

We have $E/F$ a cyclic unramified ext^r, $l = \deg E/F$.

Set $f_\delta^E = B_E(t)$, $f_\delta^F = B_F(t')$.

Prop 1: $f_\delta^E$ & $f_\delta^F$ are associated. He will prove this.

\text{ie. they satisfy the fundamental lemma.}

Prop 2: If $\chi$ is an unram. adj. rep of $G_E$, then

$$\text{tr}(f_\delta) = \begin{cases} 0 & \text{unless} \\ \chi \text{ is a subquotient of an unramified principal series} & 
\end{cases}$$

2 is easier than 1, so maybe we'll have a crack at 2 first.

Set $I_\chi = \text{princ. series} (\chi \text{ unramified})$.

$$\text{tr}(I_\chi) = \Delta(t)^r (\chi(t) + \chi(E))$$

Of course $f_\delta(G, \chi) \leq \mathcal{H}(G, B)$, $\not\equiv$-not commutative.

Prop 2 (Twisted version of 2): $T$ rep $f_\delta$ of $G_E$.

$$\text{tr}(f_\delta^E T(\sigma)) = \begin{cases} 0 & \text{unless} \\ T \text{ is a subquotient of an unramified p.s.} & 
\end{cases}$$

$$\text{tr}(I_\chi(f_\delta^E) I_\chi(\sigma)) = \Delta_e(t)^r (\chi(t) + \chi(E))$$

Notes: 1) $\chi$ is next to do.

with $f_\delta^E$.

2) Work assuming $T$ acts trivially in spherical vectors.
There are various other sublemmas we'll need. It's gonna prove those props.

There's some kind of key that relates everything. This might be

**Technical lemma**

\[(b, m) \to b^{-1}m b\]

\[b \in B, m \in M \iff b^{-1}m b = (b_0, e, s) \iff \text{id} \text{ is } 1 \text{ in } B.\]

This tells us how you can analyse things. If we have this, then tackling orbital integrals can be done by small kids.

**Note that** \(B(t) = \frac{1}{\text{Vol}(B)} \cdot \text{char } \{ b^{-1}m b : b \in M, m \in B \}\)**

\[O_g(f_t) = \int\int f_t(x^t y) \, dx \, dy\]

\[x^t y = b^{-1}m b : b \& m \text{ are conjugate.}\]

So to compute this orbital integral, we see it's

\[O_g(f_t) = \begin{cases} 0 & \text{unless } x \sim t m \text{ for some } m \in B \text{ or below if } x \sim t m. \end{cases}\]

**Assume** \(x = t m.\)

\[\int\int f_t(x^t m) \, dx \, dy = 1; \quad x^t m x = b^{-1} m b\]

\[\text{vol}(B) \quad \text{vol}(B) \quad x^t m x = t m\]

\[x^t m = t^m \quad & \text{and this is true in } \mathbb{C} \quad \text{and the eigenvalues of } t \text{ have distinct values.}\]

\[x \in \mathbb{C} \quad \text{otherwise we get unbounded orbit things.}\]

So **our integral** = \[\int f_t(b^{-1} m b) \, db \quad \text{and we've proved his statement about orbital integrals.}\]

**Note that** they are "just stupid."
Hi an exercise to do the twisted prop now.

We can now get prop 1 (modulo the technical lemma again)

- note that we do know the fundamental lemma for $n$.

So just use the identity $\eta e \rightarrow \eta \eta^{-1}$

He's now proved prop 1, B.

It's an exercise to prove this for GL. $E = \begin{pmatrix} \mathbb{C}^n \times O \\ O \times \mathbb{C}^n \end{pmatrix}$, $n \geq 2$

Use Kottwitz' pf that 1 is associated to 1. (Peter Schneider's lecture)

This is also true for $B$.

To do prop 2 we recall $\xi_e = B^{-1}B$

$$tr \pi(\xi_e) = (\text{scalars}) tr \pi(e_B) \oplus \pi(\xi_e) \pi(e_B) \neq 0$$

$$\Rightarrow \pi B = 0$$

This characterizes the subquotients of the unramified principal series.

Recall Tors did this ($V'' \neq 0 \Rightarrow$).

It's in fact an example of a much more general thing, which he may will be about to tell us.

If $(\pi,V)$ admissible rep of $G$ a quasi-split gp eg GL

Need this to define $B$

Then $V^B \rightarrow V_N = V/V(N)$ $(\forall \pi(V(N) \rightarrow \mathbb{A}(\mathbb{C}))$

$$\pi(V) = \{ (\pi(n)-1) V \mid \nu \in V \}$$ in the Lang module,

If $V'$ and $V'' \neq 0$ this is $\nu = 0$.

He just needs an injection. (Your answer)

He quite clearly knows the proof of this & verbally sketched it.

He did it "to lose time".
\[ \text{tr } I(x) = \int \int_s f_t(x^{a(x)}) \Delta(u) \Delta(v) \, dx \, dv \]

\[ f_t(x^{a(x)}) = 0 \quad \Rightarrow x \in W_B \]

\[ \text{tr } \Omega(f_t) = \Delta(t) = \text{vol}(B(t)) \]

\[ B(t) = \frac{1}{2} B + B , \quad \Delta(t) = \frac{1}{2 B_n B B^*} \]

\[ \Delta(t) \delta(t)^{-1} + \Delta(t) \delta(t)^X ; \text{ note } \Delta(t) = \delta(t)^{-1} = \delta(t) \]

\[ \text{char of } 1 \quad \text{char of } \delta t \]

\[ \Delta(1) = 1 \]

\[ \text{tr } \Omega(f_t) = \Delta(t) = \text{vol}(B(t)) \]

\[ \text{tr } St(f_t) = 1 \]

Replacing \( \sigma \)-conjugacy by conjugacy etc. gives you lots more formulae.

Use \( n_s > n_i \).

This proves prop 2. He's now going to talk about the tricky proposition.

\[ b \in B_n B_n , \quad b = (1 \rho 0), \quad r = n_s n_i \quad t = (1 0 0) \]

\[ m = (1 0 0) \quad b^{-1} m b \]

\[ = \frac{1}{1-R \rho \tilde{w}} \left( \begin{array}{ccc} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tilde{s} \end{array} \right) \left( \begin{array}{ccc} \omega - R \rho \tilde{w} & 0 & 0 \\ 0 & \tilde{s} - \rho \omega & 0 \\ 0 & 0 & \tilde{w} \end{array} \right) \]
\[ \kappa \cdot \text{M}(H_6 \times H_{10}) \rightarrow 0 \in B \mathfrak{g} \]

\[ H_6 \bmod \mathfrak{g} \]

\[ \text{M}(\mathfrak{g}) / B \rightarrow B \mathfrak{g} \mathfrak{b} \]

\[ (\mathfrak{g}, \mathfrak{m}) \rightarrow \mathfrak{t} \}

Want volumes preserved by this \( p \)-adic analytic morphism.

Compute volume, Jacobian

\[ \Delta_{(t_0)} = \Delta(t) \]

This proves the technical lemma.

With this computation you're now home. \( \mathfrak{g} / \mathfrak{m} \). A no-computation proof.

Here is going to spend the last 2 minutes of the lecture proving the lemma that Tony needed this morning.

\[ \sum c(t) \mathfrak{v}_{i_{\mathfrak{g}}} (\mathfrak{g}) = \sum d(t) \mathfrak{v}_{i_{\mathfrak{b}}} I \mathfrak{p} \times \sigma \]

He's changed \( f \) to \( f_{\mathfrak{b}} \) & \( g \) to \( f_{\mathfrak{b}} \).

Everything is a twist of the trivial or the twist of the Steinberg.

Everything must compensate exactly.

\[ \sum c(t) \mathfrak{v}_{i_{\mathfrak{g}}} \quad = \quad \sum d(t) \mathfrak{v}_{i_{\mathfrak{b}}} \]

Spherical, substitute the form Hilbert algebra, def of base change to \( \Pi_{\mathfrak{f}} \mathfrak{b} \) \( \rightarrow \mathfrak{f}_{\mathfrak{b}} \mathfrak{b} \) trace in full place series = that of subcharacter ... establishes lemma.

This finishes his talk on the Fundamental lemma.
Thanks to Kansten for taking notes for this one, which I couldn't attend unfortunately.

Lab: "Louder!"
Mike: "Why don't you move a bit closer?"

**Artin L-functions**

Let $E/F$ be a finite Galois extension of number fields.

$$\sigma: \text{Gal}(E/F) \to \text{GL}(v), \quad V/C \text{ odd vs.}$$

For each finite place $v$ of $F$ and $w|v$, there is a place of $E$ above $w$.

We have $I(w|v) = D(w|v) \subseteq \text{Gal}(E/F)$.

Let $P_v(\sigma, X) = \det((1 - \sigma(\text{Fr}_w))_V^{-1})_{V(w|v)}$

Then $w$ is unramified $w$, if $w|v$, then $\exists \tau \in \text{Gal}(E/F)$ s.t. $\tau w = w_v$.

Then $D(w_v|v) = \tau D(w|v) \tau^{-1}$
$I(w_v|v) = \tau I(w|v) \tau^{-1}$

$\& \{ \text{Fr}_{w} \} = \{ \tau \circ \text{Fr}_w \circ \tau^{-1} \}$

We have $\sigma(\tau): V \to V$ & $V^{I(w|v)} \to V^{I(w_v|v)}$

**Def:** $L(\sigma, s) = \prod_{v \text{ finite}} P_v(\sigma, (N_v)^{-s})^\frac{1}{s}$, the **Artin L-function**

$P_v(\sigma, X) = (1 - X_1^v)(1 - (1 - X_1)|v|)$, the Euler product will converge if Re(s) > 0.

**Eg:** If $\sigma$ is the trivial rep,

$L(\sigma, s) = \zeta_F(s) = \prod_{v} (1 - (N_v)^{-s})^\frac{1}{s} = \sum_{\text{all f.g. un.} \sigma}$

of $F$.
**Proposition 1.**

1) If \( E \supseteq E \supseteq F \), \( \sigma \) is a rep of \( \text{Gal}(E/F) \), and \( \sigma \) is a rep of \( \text{Gal}(E'/F) \) given by

\[ \sigma: \text{Gal}(E'/F) \rightarrow \text{Gal}(E/F) \rightarrow \text{GL}(V) \]

then \( L(\sigma, s) = L(\sigma, s) \)

2) If \( \sigma_1, \sigma_2 \) are reps of \( \text{Gal}(E/F) \), then

\[ L(\sigma_1 \otimes \sigma_2, s) = L(\sigma_1, s)L(\sigma_2, s) \]

3) If \( E' \supseteq E \supseteq F \) and \( \sigma \) is a rep of \( \text{Gal}(E'/E) \), (don't need \( E/F \) Galois)

then \( L(\sigma, s) = L(\text{Ind}_{G(E/F)}^{G(E'/E)} \sigma, s) \).

**Proof:** Omitted. (i) and (ii) are easy — just look at Euler factors. (iii) is a little harder — need places of \( E \leftrightarrow \text{places of } E' \).

It was inspired by abelian L-series:

\[ L(\chi \psi, s) = \prod_{\text{primes } p} \left( 1 - \frac{\chi(p) \psi(p)}{p^s} \right) \]

Now write \( G = \text{Gal}(E/F) \). The regular rep of \( G \) decomposes as

\[ \rho_{\text{reg}} = \bigoplus_{\chi} \chi \otimes \sigma \]

as \( \sigma \) runs through all irreducible reps of \( G \), \( \chi \) runs through all characters.

Also, \( \rho_{\text{reg}} = \text{Ind}_{E \rightarrow F}^{G} (\text{trivial}) \).

\[ \Rightarrow L(\rho_{\text{reg}}, s) = \prod_{\chi} L(\chi \otimes \sigma, s) \]

& hence \( \frac{S_E(s)}{S_F(s)} = \prod_{\chi} \frac{L(\sigma, s)}{L(\chi \sigma, s)} \)

**Question:** Does \( S_E/S_F \) have an entire ext to \( C \)?

This would follow from

**Conjecture (Artin):** If \( \sigma \) is a non-trivial, irreducible Galois rep, then \( L(\sigma, s) \) has an entire ext to \( C \).
Firstly note that there is a meromorphic continuation.

Let $p = \mathfrak{X}$ be a $L$-unit char. of $\text{Gal}(E/F)$. Then $\mathfrak{X}$ factors thru $\text{Gal}(E_{\mathfrak{X}}/F)$ where $E_{\mathfrak{X}}/F$ is an abelian ext.

Then by global CFT we have $G(E_{\mathfrak{X}}/F) \cong C_{\mathfrak{X}}/N_{E_{\mathfrak{X}}/F}C_{E_{\mathfrak{X}}}$

So $\mathfrak{X}$ on $G(E_{\mathfrak{X}}/F) \to \mathfrak{X}$ on $C_{\mathfrak{X}}$, a finite GC.

We also have an $L$-f for $\mathfrak{X}$ (see below)

**Claim** $L'(\mathfrak{X}, s) = L(\mathfrak{X}, s) = \prod_{v \text{ finite}} \left( 1 - \mathfrak{X}(v)(Nv)^{-s} \right)^{-1}$

We will justify this claim. Say the conductor of $E_{\mathfrak{X}}/F$ is $F_{\mathfrak{X}}$, the minimal $K$ s.t. $E_{\mathfrak{X}} \subseteq F_{\mathfrak{X}}(K)$. Then $F(\mathfrak{X})/F_{\mathfrak{X}}$ is the field of the conductor.

The global conductor is the product of local conductors.

Note $F^\infty \left( \bigotimes_{v} U_{v}^{\alpha_v^n} \right) / F^x \cong N_{E_{\mathfrak{X}}/F}C_{E_{\mathfrak{X}}}$

Here $n_v$ is the power of $v$ in $F_{\mathfrak{X}}$

$\mathfrak{X}$ is faithful in $E_{\mathfrak{X}}$ and $\mathfrak{X}$ in $F_{\mathfrak{X}}$

Then the ramified primes of $E_{\mathfrak{X}}/F$ are the ramified primes of $\mathfrak{X}$.

For $v \nmid F_{\mathfrak{X}}$ we have $\mathfrak{X}(v) = \mathfrak{X}(F_{\mathfrak{X}})$, same Euler factor at $v$.

For $v \mid F_{\mathfrak{X}}$ we're ramified $\mathfrak{X}$. Then $\mathfrak{X}(I_v)$ is non-trivial so $V_I \neq 0$.

So indeed the two L-series are the same.
Hecke showed that if \( \sigma \) is any \( \mathrm{GC} \) on \( \mathbb{C} \), then \( L(\sigma,s) \) has a meromorphic continuation to \( \mathbb{C} \), which is in fact entire iff \( \sigma \neq \mathrm{Id} \).

(Here, of course, \( \|\omega\|_{\mathbb{C}} = \|\omega\|_{\mathbb{C}} \) i.e. all \( \omega \) to make \( \mathbb{F}^s = \ker \|\) \( \| \))

If \( \sigma \) is of the \( \sigma \)-adic type, then \( L \) has a simple pole at \( s=1 \).

So if \( \sigma \) is a 1-dim, non-trivial Galois character, or if \( \sigma \) is induced from a non-trivial char, then \( L(\sigma,s) \) has an entire continuation.

**Theorem 1.2.** (Bauer) If \( G \) is a finite group, \( \sigma \) is a 1-dim. virtual rep, then \( G \) subgps \( H \) of \( G \) & \( (1\text{-dim}) \) chars \( 
\chi_i \) on \( H_i \), & also \( n_i \in \mathbb{Z}, \) s.t.

\[
\sigma = \sum n_i \text{Ind}_{H_i}^{G} \chi_i
\]

(finite sum, of course)

So then \( L(\sigma,s) = \prod L(\chi_i,s)^{n_i} \)

& thus each \( L(\sigma,s) \) indeed has a mer. continuation to \( \mathbb{C} \), & in fact also satisfies a functional eqn.

Now say \( \rho \) is a \( \mathrm{GC} \) on \( \mathbb{C} \).

For \( v \mid \infty \) define \( G_v(s) = \begin{cases} \chi(2\pi)^{-s} \Gamma(s) & \text{if } s \text{ complex} \\ \chi^{-v} \Gamma(\frac{3}{2}) & \text{if } s \text{ real} \end{cases} \)

Then we can define the \( \text{completed L-series} \)

\[
\Lambda(\rho,s) = \prod_{v \neq \infty} \left( L G_v(s) \right) L(\rho,s)
\]

Then we have a \( \text{functional eqn} \)

\[
\Lambda(\rho,s) \Lambda(\rho,1-s) = \prod_{v} \langle \sigma_v,1-s \rangle
\]

where \( |\sigma_v| = 1 \) & \( \sigma_v = 1d_f |N_{F_v} \mathbb{F}_v \) \( (d \text{ is a } \text{disc. part}) \)

Mike will now spend a while making a fool of himself trying to reach the top board. Use the stick, Mike.
For each place \( v \) define \( n_v(\sigma) \) thus:

\[
\nu|_{\infty} \quad n_v(\sigma) = n = \text{dim}(\sigma)
\]

\[
\nu|_{\text{finite}} \quad n_v(\sigma) = \sum \frac{|G_v|}{|G_v^{\sigma}|} (\text{dim} \nu - \text{dim} \nu G_v^{\sigma}), \quad \text{that well-known integer.}
\]

Here \( V \) is the space that \( \sigma \) acts on, \( G_v^{\sigma} \) is the \( \ell \)-th ramification group \( \leq G_v \leq G \leq \text{Gal}(E/F) \). It's only a finite sum, so eventually \( |G_v^{\sigma}| = F_\ell \)

\[
\text{dim} V = \text{dim} V G_v^{\sigma}.
\]

If \( v \) is a real place, \( n_v(\sigma) = n_v^\mu(\sigma) + n_v^\nu(\sigma) \), the dimension of the \( \pm 1 \)-eigenspaces for the action of the generator \( \omega \). \( \text{We}: G_v \leq G \)

\[
\begin{align*}
\varphi & \in \mathbb{R} \quad \omega \\
\nu & \in \mathbb{R} \quad \nu
\end{align*}
\]

Then \( \Lambda(\sigma, s) = \prod_{v \text{complex}} T \left( G_v^{\mu(\sigma)} \right) \prod_{v \text{real}} T \left( G_v^{\nu(\sigma)} \right) \Lambda(\sigma, s) \)

(thus is probably adf)

We have a functional eqn:

\[
\Lambda(\sigma, s) = \left[ \Lambda(\sigma, 1 - s) \right] \left[ \prod_{v \text{finite}} N_{G_v^\sigma}^F \right] \left( s \right) \Lambda(\sigma, 1 - s)
\]

where \( F_\sigma = \prod_{v \text{finite}} \nu_v(\sigma) \)

Now let's write everything in our new modern technology.

Say \( \sigma \) is restriction of \( \sigma \) to \( G_v \leq G \) for finite places ( & infinite ones if you like)

\( D(w/\sigma) \)

We get a semisimple rep of \( W_F \) of Galois type

\( \rightarrow \) WD rep with \( N = 0. \)

To find its reps \( \tau_v \) of \( \text{WD}_F \), we can associate \( L(\tau_v, s) \) & \( \epsilon(\tau_v, s) \)

It's now 5pm. For \( \sigma \), the local Euler factors of \( \Lambda(\sigma, s) \) are \( L(\sigma_v, s) \)

\( \epsilon(\sigma_v, s) \) are \( \epsilon(\sigma_v, s) \) (2)
Now say \( n = 2 \) or \( 3 \) & \( F/\mathbb{Q}_p \). Recall Local Langlands (a thin as \( n = 2,3 \))

\[
\left\{ \text{Conjugacy classes of semisimple } n\text{-dim reps of } \text{WF} \right\} \leftrightarrow \left\{ \text{irred adms reps of } GL_n(F) \right\}
\]

\[ \rho \mapsto \pi(\rho) \]

Unramified reps of \(WF\) \( \mapsto \) unramified reps of \( Unramified \text{ exts?} \)

\[ \rho(\text{Fr}_\nu) \sim (\rho^\sigma) \]

\[ \rho \mapsto \text{ext^1 corr. to } \alpha, \beta \]

Say \( \chi : F^\times \to \mathbb{C}^\times \)

\[ \rho \otimes \chi \mapsto \pi(\rho) \otimes (\chi \text{ det}) \]

\[ W_{\pi,\rho} = \text{det } \rho \cdot \pi(\rho) \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) = W_{\pi,\rho}(\sigma) I \]

& \( \pi(\rho) = \overline{\pi(\rho)} \)

We can then define \( L \) and \( \varepsilon \)-factors for \( \pi(\rho) \) coming from \( \rho \).

We can do similar things for \( \mathbb{R} \) & \( \mathbb{C} \), replacing \( \text{WF} \) by \( WF \) and \( GL_n(F) \) by \( (\mathbb{C}, K) \text{-module} \).

If \( \text{dim } \sigma = 2 \) or \( 3 \), \( \sigma \mapsto \pi(\sigma) \), irred, adms, unram for almost all \( r \).

Define \( \tau(\sigma) = \otimes \pi(\sigma) \).

We've now got an irred adms (global) \( (\mathbb{C}, K) \times GL_n(\mathbb{R}_F^\times) \)-module.

Also define \( L(\tau(\sigma), s) = \prod L_v(\tau_v(\sigma_v), s_v) = \prod L_v(\sigma_v, s_v) : \Lambda(\sigma, s) \)

Hence \( L(\tau(\sigma), s) \) is entire \( \Rightarrow \Lambda(\sigma, s) \) is.

Our aim now is to show that \( \tau(\sigma) \) is actually an automorphic form.
We get $L_i, L_{i+1}$. We're ignoring $K$. So, if $K \nmid n$, $K$-ind. equality.

So the type of $M$ is controlled by type of $L_i$'s.

Interlacing up for irreducible reps $L_i$, general concept of $L_i$ as $L_i$ groups.

Last and realises [this is the end of another lecture.]

Euler product construction (yellow lecture notes 9/8).

Yesterday he talked a lot about 1-dim. reps & $L_i$'s. Today he'll do

§2 2-dim. irreducibles

Theorem 2.1: If $\sigma: G \rightarrow GL_n(\mathbb{C})$ (finite image: factors through $GL_n(\mathbb{E}/\mathbb{F})$ $\mathbb{E}/\mathbb{F}$ finite).

Then $\text{Im}(\sigma)$ is either i) $D_n$ of order $2n$, $n \geq 2$

or ii) $A_n$, $S_n$, $A_5$.

No time for p.f.

Neat little p.f. though

Corollary: In the dihedral case, $\sigma(G)$ has an abelian subgroup of index 2

I don't do p.f. of this (probably).

I quadratic ext $K$ of $F$ s.t. $\sigma(G)$ is abelian.

Then $\exists$ normal char $\chi$ of $G_{K}$ s.t. $\sigma = \text{Ind}_{G_{K}}^{G} \chi$.

Hence $L(\sigma, s) = L(\chi, s)$ is entire.

We want to attack $A_n$ & $S_n$. Here the methods fail for $A_n$.

$D_2 = \text{proj} \{ (0, 1), (0, 0) \}$

$A_n$, $S_n$, $A_r$ come 'geometrically' from embedding the rotation group of the tetrahedron, octahedron & icosahedron into $\text{PGL}_2(\mathbb{C})$, but he doesn't really understand how!!
The adjoint square

\[ \begin{array}{c}
\text{GL}_2(\mathbb{C}) \xrightarrow{\phi} \text{GL}_3(\mathbb{C}) \\
\text{PGL}_2(\mathbb{C}) \downarrow \quad \downarrow \text{ad action} \\
\text{PGL}_3(\mathbb{C})
\end{array} \]

where the action of \( \text{PGL}_2(\mathbb{C}) \) on its Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) at \( 0 \)

\[ \mathfrak{sl}_2(\mathbb{C}) \cong \{ \mathbf{M} \in \mathfrak{sl}_2(\mathbb{C}) | \mathbf{M} : 0 \} \] by conjugation

\( \chi(H) = xHx^{-1} \)

**Def** \( A(E) = 3 \)-dim rep \( \sigma : G_F \rightarrow \text{GL}_3(\mathbb{C}) \)

\( \sigma : G_F \rightarrow \text{GL}_3(\mathbb{C}) \) of tetrahedral type

Let \( E \) be the affine ext \( \text{f} \) s.t. \( \sigma(G_F) = V_4 \subset A_4 \)

**Lemma 2.2.** \( \sigma \) as above- \( A(E) \) is an irreducible rep \( \sigma \) \( \text{Ind}_{G_E}^{G} X \), for any non-trivial char \( X \) on \( G_E \)

**Prop.** Claim: \( A_4 \cong \sigma(G_F) \) is conjugate by \( \text{GL}_3(\mathbb{C}) \) to

\[ \text{proj} \left\{ \begin{array}{c}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\end{array} \right\} \]

Any finite \( G \leq \text{PGL}_2(\mathbb{C}) \) has a lift \( G' \) in \( \text{SL}_2(\mathbb{C}) \) s.t. \( G' \rightarrow G \)

In fact if \( G \) is simple then \( G' = \{ \pm \text{pullbacks to } \text{SL}_2(\mathbb{C}) \} \)

In fact \( G' \) is the left to \( \text{SL}_2(\mathbb{C}) \) as some theory argument shows

\( \chi(G) \& \chi(G') \) are common... 

\( \text{any lift } \chi(G') \& \chi(G) \) are common... 

Hence \( \chi(G) \) is the only lift of order 2

Elts of order 2 in \( \sigma(G) \) lift to elts of order 4. Conversely (as they can't lift to \( 1 \))

thing \( \chi(G) \) is \( X \) say. Choose a basis

\( \text{st. } X \in \{ \chi \} \)

Centraliser \( \text{sl}_2(\mathbb{C})X \) = diag matrices. \[ \{ Y \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \} \] then
\[ XY = \pm YX \quad \text{if} \quad YX = XY \quad \text{then} \quad Y \text{ is diagonal} \quad \& \quad Y = \pm e_0(\delta) \ast \ast \ast 1_{S_n}(\mathbb{C}) \]

\[ e_0(\delta) = -y(\delta, \delta) \quad \& \quad \text{after conjugating by diagonal elt get} \]

\[ y = (0, 1) \]

**Similarity for Z:** \[ Z \times Z^{-1} = \pm Y \]

So now we can work out \( \Pi^e \) etc. (see below)

From this explicit rep we can work out \( \Pi(e) \) easily & indeed \( \Pi(e) \) induced up from them \( K \)'s.

\[ \square \]

3. The Big Thin section

Mike has no idea how to prove any of the theorems in this section but they'll be used in §4 to prove cases of Artin.

\[ \text{2.3} \quad \Pi \text{ is an irreducible } (\Omega_w, K_v) \text{-rep } \times \text{G} \text{e}_0 (M^e) \text{-module} \]

\[ \Pi = \otimes \Pi_v \quad \text{By local Langlands then we get} \]

\[ \Pi_v = \tau_v (\sigma_v) \quad \text{for some } n \text{-diml rep of } W(D) \]

Then \( L_v (\Pi_v, s) = L (\sigma_v, s) \quad \varepsilon_v (\Pi_v, s) = \varepsilon_v (\sigma_v, s) \)

If \( v \) is a finite place, \( \sigma_v \rightarrow (\sigma_v, \mathbb{N}) \quad L_v (\sigma_v, s) = \det (1 - F_v (N) \mathbb{N})^{-s} \)

\[ \prod_v \]

\[ L (\Pi, s) = \prod_v L_v (\sigma_v, s) \quad \varepsilon (\Pi, s) = \prod_v \varepsilon_v (\Pi_v, s) \]

Not obviously e.g.

**Thm 3.1** \( (J = 1, n = 2, J - G, n = 3 - \text{if } n > 3 \text{ itself different as } \# J \text{-rep but you still get } \Pi \) on \( L \) if you try)

If \( \Pi \) is automorphic, then

(i) \( L (\Pi, s) \) has a meromorphic continuation extension to \( C \) with finitely many

(ii) \( L (\Pi, s) = \varepsilon (\Pi, s) L (1 - \Pi, s) \) if \( \Pi \) is the contragent

(iii) \( \Pi \) is cuspidal then \( L (\Pi, s) \) is actually entire. \( \square \)
There's also a wacky converse which to Mui seems much stronger:

If $\mathcal{V}_c L: G_F \to \mathcal{C}$, $\mathcal{L}(s, w, \omega)$ & $\mathcal{L}(s, w^2, \omega^2)$ are entire, bounded in vertical strips, & satisfy

$$\mathcal{L}(s, w, \omega) = \mathcal{L}(s, w^2, \omega^2) \mathcal{L}(1-s, w^2, \omega^2)$$

(\& possibly 1 or 2 other technical other conditions)

then $\pi$ is amb. cuspidal \& automorphic, I guess)

Let $\sigma: G_F \to GL_d(C)$ be red-dim. irreducible; then $\pi(\sigma) = \mathcal{O}^{d} \pi(\sigma)$

$$\sigma \circ \omega = \mathcal{O} \pi(\sigma, \omega, \omega) \& \pi(\sigma) \circ \omega = \mathcal{O} \pi(\sigma, \omega, \omega^2)$$

If $\sigma = \text{Ind}_{G_e} G_f X_{\sigma}, [E:F] = n$, then $\sigma$ is the corresponding local induced thing.

$w_{E/F} = w_{E/N_{E/F}} \quad \text{GC on } E$

$$\mathcal{L}(\pi(\sigma) \circ \omega, s) = \mathcal{L}(X, w_{E/F}, s); \quad \mathcal{L}(\pi(\sigma) \circ \omega^2, s) = \mathcal{L}(X^2, w_{E/F}^{-1}, s)$$

These are entire, bounded in vertical strips & satisfy functional eqn.

provided $X, w_{E/F} + \| w \|_F$ for some $w$ & some $\pi(\sigma) \in \text{H} = N_{E/F}$

$$\iff \pi \text{ is not of the form } \mathcal{L}(X, N_{E/F}, s), \text{ for some } X, \text{ GC on } F$$

$$\iff \pi \text{ is not the restriction of a chab } X \text{ on } G_F$$

If that were true, then $X$ would be a constituent of

$\text{Ind}_{G_e} G_f X_{\sigma} \circ \omega$ irreducible, 1-dim. $\&$, so indeed there are entire.

Jacquet, et al.

Thm 3.2. (Converse thm). If $\sigma$ is an irreducible dim. Galois rep. induced from a chab on a subgroup (monomial) then $\pi(\sigma)$ is cuspidal. \(\square\)

Base change for $GL_e$, coming up (another by thm)
Barb. change. If E/F is an ext. of no. fields

\[ \pi' = \otimes \pi\nu(\sigma\nu) \text{ -- cuspidal rep on } \text{GL}_1(\text{AF}) \]

\[ \pi' = \otimes \pi\nu(\sigma\nu) \text{ cuspidal on } \text{GL}_1(\text{AE}) \]

\[ \forall \nu, \pi' \text{ is a base change lift of } \pi \text{ iff } \text{for all } \nu, \pi\nu(\sigma\nu) \text{ is the restriction of } \pi\nu \text{ from } \text{WDE}_\nu \text{ to } \text{WDE}_\nu. \text{ This seems to be for all } \nu \text{ incl. local ones, (infinite ones?)} \]

If \( \pi \in G(E/F) \) then \( \pi \) acts on \( \text{GL}_1(E) \setminus \text{GL}_1(AE) \) & gives an action \( \pi' \) called \((\pi')^c\)

\[(\pi')^c = \otimes \pi\nu(\sigma\nu^c) ; \quad (\sigma\nu^c)^c : \text{WDE}_\nu \xrightarrow{\tau} \text{WDE}_\nu \quad \xrightarrow{\pi\nu(\sigma\nu^c)} \text{GL}_1(E) \]

In fact, \( \pi' : \pi(\sigma) \Rightarrow (\pi')^c = \pi(\sigma^c) \) where \( \sigma^c = \sigma^c(\text{conjugation by } \tau) \)

\( \pi' \) in Galov. -int \( \Rightarrow (\pi')^c = \pi' \forall \tau \in \text{Gal}(E/F) \).

**Theorem 3.3** (Base change for GL1 E/F cycles of prime degree)

(a) every cuspid. rep on \( \text{GL}_1(\text{AF}) \) has a base change lift to \( \text{GL}_1(\text{AE}) \)

(b) if a cuspidal rep on \( \text{GL}_1(\text{AE}) \) then it has a base change lift \( \Rightarrow \) it Galov. -int.

(c) If \( \pi \) & \( \pi' \) on \( \text{GL}_1(\text{AF}) \) have the same base change lift then \( \pi' = \pi \circ \omega \), \( \omega \) a char of \( G(E/F) \)

(d) If \( \pi' \) is a lift of \( \pi \), then \( \omega: \pi(\sigma) = \omega \circ \pi(\sigma) \)

\[ \pi((\sigma)) = \omega(\sigma) \cdot \pi((\sigma)) \]

"Can we give him 10 more minutes?" 11:20. No.6, shouts the crowd.
"5 more minutes?"

\[ \pi = \otimes \pi\nu(\sigma\nu) \text{ -- automorphic on } \text{GL}_1(\text{AF}) \]

**Def** \( \text{Ti} \) is a GL3 lift if \( \pi = \otimes \pi\nu(\sigma\nu) \) for almost all \( \nu \)

**Theorem 3.4** (a) Every cuspidal \( \pi \) has a GL3 lift to automorphic \( \pi' \)

(b) \( \pi' \) is cuspidal if \( \pi = \pi(\sigma) \) is a monomial rep in \( G_F \)
There's one more thing he'll need. It's why Tony mentioned.

**Criterion for equality of cuspidals**

\[ \pi = \pi', \pi' \text{ cuspidal reps on } \text{GL}_3(A_F) \]

\[ L(s, \pi \times \pi') \text{ is the Rankin product - this is for } \nu \text{ st. } \pi \text{ and } \pi' \text{ are unram.} \]

**Theorem 3.5** \( V \nu \text{ can define } L(s, \pi \times \pi') \text{ - coincides with } \pi \text{ in unramified case,} \)

\[ \text{st. } L(s, \pi \times \pi') = \prod V L(s, \pi \times \pi') \text{ converges in RH plane &} \]

(i) \( L(s, \pi \times \pi') \) has meromorphic continuation and functional eqn

(ii) \( L(s, \pi \times \pi') \) has poles at \( s=1 \) and \( s=\frac{1}{2} \)

(iii) \( V \nu \), \( L(s, \pi \times \pi') \) at \( s=1 \) and \( s=\frac{1}{2} \)

(iv) \( V \nu \), \( L(s, \pi \times \pi') \) in pole-free for \( \Re(s) \geq 1 \)

---

**Section 3.4: Artin's conjecture for tetrahedral reps**

\( \sigma: G_F \to \text{GL}_3(F) \text{ an irreducible rep, } \sigma(F_F) \cong A_4 \)

\( \pi_0(\sigma) = \pi, \pi_1(\sigma) \).

We want to show \( \pi(\sigma) \) is cuspidal.

**Step 1** Construction of \( \pi_{ps}(\sigma) \)

We have \( E/F, [E:F]=3, \sigma(G_E) \cong V_4 \cong A_4 \)

If \( \Sigma = \sigma|_{G_E} \), then we have that \( \Sigma \) is monomial.

By the converse thm, \( \pi(\Sigma) \) is cuspidal on \( \text{GL}_3(A_E) \).

\[ \Sigma^\sigma = \Sigma^\tau, \tau \in G(E/F) \Rightarrow \pi(\Sigma)^\sigma = \pi(\Sigma^\tau) = \pi(\Sigma) \]

By base change thm, \( \exists \pi_{ps} \) cuspidal rep on \( \text{GL}_3(A_E) \)

\[ s.t. \pi(\Sigma) \text{ is a base change of } \pi \]

\[ \pi = \pi^0(\sigma^0), \sigma^0 \text{ restrict to } \Sigma_w \text{ for } w/F, \text{ on restricting from } W_E \text{ to } W_{E_w} \]

There's some tricky reason why we don't use \( W_{E_w} \) when...
Replace \( \tau \) by \( \tau \omega \), \( \omega \) any char of \( G(E/F) \).

Want \( \omega \) s.t.
\[
(\omega_{\tau \omega}(\xi) = \det(\xi)) \quad \forall \xi.
\]

\[
(\omega_{\tau \omega})^2
\]

\[
(\omega_{\tau \omega}) \cdot N_{E/F} = (\omega_{\tau \omega}(1)) = \det \Sigma = \det \sigma \cdot N_{E/F}
\]

\[
(\omega_{\tau \omega}) \cdot \det \sigma \text{ agree on } N_{E/F}C_E \cdot C_F / N_{E/F}C_E \cong G(E/F) \cong C_3
\]

Can find \( \omega \) on \( G(E/F) \) s.t. \( \forall \sigma \), \( \omega \) is unique.

\[
\Pi_p(\sigma) = \Pi_{\tau \omega} \text{ for that } \omega
\]

\( \Phi_{\Pi_{\tau \omega}}(\sigma) \)

Need \( \omega' = \omega \) for almost all \( \nu \)

**Claim**
\[
A^v(\sigma'') = A^v(\sigma) \quad \text{for almost all } \nu
\]

**Step 1**
\[
\Rightarrow \sigma_v = \sigma_v'' \quad \text{for almost all } \nu
\]

If \( \nu \) splits in \( E/F \), \( \sigma_v = \Sigma = \sigma_v'\)

\[
\text{If } \nu \text{ remains prime in } E/F, \text{ we may as well restrict to when } \sigma_v, \sigma_v'' \text{ are unramified } \text{ & s.t. the claim is this:}
\]

\[
\sigma_v(Fr_v) \sim (0, b), \quad \sigma_v''(Fr_v) = (c, 0)
\]

As \( Fr_v^3 = Fr_v \in We_3 \), we have \( \sigma_v(Fr_v^3) = \sigma_v''(Fr_v^3) \)

\[
\begin{pmatrix}
(a, 0, b) \\
(0, b, 0)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(c, 0) \\
(0, 0)
\end{pmatrix}
\]

\[
(\nu=0) \quad c = 3^{1/3} \omega_v, \quad 3, 3', 3'' \text{ cube roots of } 1.
\]

\[
\prod \det(\sigma_v) = \prod \omega_{\tau \omega}(\xi) = \omega_{\tau \omega}(\xi) \omega_{\tau \omega}(\xi) = \det \sigma = \prod \det \sigma_v
\]

\[
\Rightarrow \det \sigma_v = \det \sigma_v \cdot \nu
\]

\[
\Rightarrow 3^{1/3} = 1 \text{ or } 3 = 3^2.
\]

That's about all we can make out here.
Recall $A^2: \text{GL}_3(\mathbb{C}) \to \text{GL}_3(\mathbb{C})$; ker $A^2$ scalars

$$A^3(\sigma_v) = A^2(\sigma_v) \circ (\sigma_v, 0, 0) \sim (\sigma_v, 0, 0) \sim (\lambda, 0, 0) \quad \text{for some } \lambda.$$

So I claim:

(i) $\exists \varphi = A^{2}\varphi, \quad \exists \psi = A^{2}\psi \quad \Rightarrow \lambda = 1 \Rightarrow \lambda = 1 \quad \Rightarrow \varphi = \varphi.$

(ii) $\exists \varphi = A^{2}\varphi, \quad \exists \psi = A^{2}\psi \quad \Rightarrow \lambda = 1 \Rightarrow \lambda = 1 \quad \Rightarrow \varphi = \varphi.$

\[ \lambda = 1, \quad \exists \lambda \]

\[ \sigma_v(F_{11}) \sim (\sigma_v, \rho, \sigma_v) \quad \text{prim. } 3 \text{th root of } 1 \]

\[ A^3(\sigma_v)(F_{11}) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} \quad \text{wrt basis } \left\{ (1, 0), (\sigma_v, 0), (\sigma_v^2, 0) \right\} \text{ basis of } L_{32}(\mathbb{C}) \]

\[ \text{order } 3 \quad \text{Ind} \left( A^3(\sigma_v) \right) \leq \text{Ind} (A^3(\sigma_v)) = A^2. \]

So it suffices to prove the claim.

**Proof of claim**

By lemma 2.2, $A^2(\sigma)$ is monomial.

\[ \Rightarrow \text{converse thm; } \tau^\ast = \bigoplus_{\mu} \text{Ind} (A^3(\sigma_v)) \text{ is cuspidal on } \text{GL}_3 \left( \mathbb{F}_{p} \right) \]

\[ \Pi_{G_{11}}(\tau) = \tau^{\ast}(\sigma); \quad \varphi: G_{11} \to \text{GL}_3 \left( \mathbb{F}_{p} \right) \Rightarrow \sigma^{\ast} = \sigma. \]

\[ \left[ \begin{smallmatrix} \sigma \end{smallmatrix} \right]_{G_{11}} = \left[ \begin{smallmatrix} \varphi \end{smallmatrix} \right]_{G_{11}} \Rightarrow \sigma^{\ast} = \sigma \otimes \omega, \quad \omega \in \text{char of } L_{32}(\mathbb{F}_{p}) \]

\[ \sigma^{\ast} = \text{Ind} (\text{Res}_{G_{11}} \sigma^{\ast}) = \bigoplus_{\omega \in \text{char of } L_{32}(\mathbb{F}_{p})} \sigma \otimes \omega \]

\[ \det \sigma^{\ast} = \det \sigma = \det \varphi \Rightarrow \omega^{1} \Rightarrow \omega^{1} \]

\[ \sigma \text{ is not monomial } \Rightarrow \exists \text{ GL}_3 \text{-lift } \Pi^{\ast} \text{ (cuspidal) } = \bigoplus_{\varphi \in \text{Ind}(\sigma^\ast)} \] for almost all $\nu$.

\[ \mathcal{A}_L(\sigma_v) = \mathcal{A}_L(\sigma_v). \] follows from $\Pi^{\ast} \Rightarrow L \left( \sigma_v, 1 \right) \otimes L \left( 1, \Pi^{\ast} \right)$
For almost all \( v \), \( L(s, \tau v, \nu(v) \tau v)) = L(s, A^*(\sigma_v) \circ A^*(\sigma_v)) \) (by def.)

\[ L(s, \tau v, \nu(v) \tau v)) = L(s, A^*(\sigma_v) \circ A^*(\sigma_v)) \]

(1)

\( v \) split: \( \sigma_v = \sigma_v - 2 \) local factors correspond

\( v \) remains prime, \( w/v \), \( A^*(\sigma_v) = \text{Ind}_{G_v}^{\text{Gr}_v}(\chi_v) \Rightarrow A^*(\sigma_v) = \text{Ind}_{G_v}^{\text{Gr}_v}(\chi_v) \)

\( A^*(\sigma_v) \circ A^*(\sigma_v) = A^*(\sigma_v) \circ \text{Ind}_{G_v}^{\text{Gr}_v}(\chi_v) \)

\[ = \text{Ind}_{G_v}^{\text{Gr}_v}(A^*(\sigma_v) \circ \chi_v) \]

So \( L(s, \tau v, \nu(v) \tau v)) = L(s, \tau v, \nu(v) \tau v)) \) for almost all \( v \), \( w/v \), \( S \) a finite set.

\[ L(s, \tau v, \nu(v) \tau v)) = \prod_{w/v} L(s, \tau v, \nu(v) \tau v)) \]

Thm 3.5 \( \Rightarrow L(s, \tau v, \nu(v) \tau v)) \) has a pole at 1.

\( \Rightarrow \) finite product over \( S \) is finite & non-zero @ 1

\( \Rightarrow \) RHS has a pole @ 1 \( \Rightarrow \) LHS has a pole @ 1

\( \Rightarrow (\text{Thm 3.5}) \tau v = \tau v \).

That's the end of the claim.

\( \tau_{p^2}(\sigma_v) = \Theta^* \tau v(\sigma_v) \), \( \tau v(\nu) = \Theta^* \tau v(\sigma_v) \)

\( \sigma_v = \sigma_v \) for almost all \( v \), including \( w/v \)

I finds set \( S \) of finite places where they could disagree

\[ \text{Lemma (J-L, Lemma 12.5) if } \forall v \in S \text{ then } \exists \text{ GC on } \frac{C_F}{w} \text{ s.t. } \text{ cond}(w) \text{ is highly divisible by all places in } S \text{ i.e. } \exists \]

& \( w/v \) unramified at \( v \) \( \Rightarrow \) (\text{Cond}(w) \text{ is arbitrary}) \( \square \)
Then \( \pi_v, \pi_0 \) & \( \pi_v(\sigma)\omega \) still agree locally \( \forall v \leq S \)

& both satisfy the eqns of the form

\[
L(-, s) = \varepsilon(\sigma) L(-, 1-s)
\]

: quotient of the 2 L-series for the twisted things.

If \( \omega \) is sufficiently ramified \( v \rightarrow \infty \) \( \forall v \leq S \), then the local Euler products of both twisted L-series are \( \pi_v(\sigma) \omega \) unchanged by twist \( \omega(\tau_v) \) on the left side & \( \omega(\eta_v) \) on the right-hand side.

\[ \Rightarrow \text{at } v_0, \quad L_v(\pi_v(\sigma)\omega) = L_v(\pi_v(\sigma)\omega) \]

\[ \Rightarrow L_v(\pi_v(\sigma)) = L_v(\pi_v(\sigma)) \]

I claim local \( \pi_v \)s are determined by \( L(\sigma) \)

\[ \Rightarrow \pi_v(\sigma) = \pi_v(\sigma) \Rightarrow \sigma = \sigma \]

So in fact \( \pi_v(\sigma) \cdot \pi(\sigma) \) is cuspidal in the 2-dim case.

So we've not only shown that the \( L \)-fs are entire, but that the rep is auto-cuspidal.
Laurenc wants to chat about lots of things. Maybe he'll talk about how to do (local rendition of yesterday's global stuff)

This morning however he will talk a lot about \( G \). In some sense \( G \) is easier, e.g. we have \( \varepsilon \)-\( \psi \) Fourier expansion, in \( G \)'s case we only have Whittaker models. \( J-L \) is important. A lot of the ideas behind the \( \psi \)'s we have met already.

Jean-Pierre did not follow John's lecture very well because he missed the vast majority of them. He may say stuff that John said already.

Say \( G = \text{GL}(2), \ F \text{global} \)

\[ \mathcal{A}(\text{GL}(2), \ F) = \text{the space of auto. forms}. \]

Auto reps are the subquotients of \( \mathcal{H} \) in this space.

Set \( \mathcal{H} \) \( \mathcal{H}_{\text{cusp}}(\text{GL}(2), \ F) = \{ f \text{ s.t. } \int f(ng) \, dn = 0 \ \forall g \}. \)

\[ \mathcal{H}_{\text{cusp}} = \oplus \text{ admissible rep with multiplicity 1}. \]

The complement of this space can be described by the space generated by the Eisenstein series.

Define \( f_N = \int f(n \bar{z}) \)

\[ \text{iff } \phi \in C^\infty(\mathbb{N}(A) \mathbb{P}(F) \backslash \text{GL}(N)) \text{ where } \mathbb{P} \text{ "paraboliques" } = (\mathcal{P}^\infty_0) \]

\[ \text{if we can form } \langle \phi, f_N \rangle = \int f_N \in C^\infty(\mathbb{N}(A) \mathbb{P}(F) \backslash \text{GL}(N)) \]

This converges.

The physicists are so important now that they can say has moved.

\[ \langle \phi, f_N \rangle = \int_{\mathbb{N}(A) \mathbb{P}(F) \backslash \text{GL}(N)} \phi(x) \, dx = \langle \varepsilon, \phi \rangle \text{ at } \]

\[ \text{when } E_{\psi}(x) = \sum_{\mathcal{P}(F) \mathcal{G}(F)} \psi(\mathcal{P} \bar{x}) \]

\[ \text{and } \langle E_{\psi}, f \rangle = \int_{\mathcal{G}(F) \mathcal{G}(A)} E_{\psi}(x) f(x) \, dx \]
\[ f \text{ is a cusp form } \iff f \perp \left\{ \psi \in C_c^\infty \left( N(A) \backslash G(A) \right) \right\} \]

Then

\[ L^2(G(F) \backslash G(A)) = L^2_{\text{cusp}} \oplus L^2_p \]

\[ L^2_p \text{ space generated by } \psi, \text{ I think.} \]

Now \[ \langle E_\psi, E_\xi \rangle_G = \langle \psi, (E_\xi)_N \rangle_p \]

\[ \psi \in \mathcal{P}(F) \backslash G(F) = \mathcal{P}(F) \backslash \mathcal{P}(N) \]

\[ (E_\xi)_N(x) = \int E_\xi(x \xi) = \int \sum_{w \in \mathcal{P}(N)} \psi(w x) \text{ d}x \]

\[ \mathcal{P}(F) \backslash G(F) = 1 \cup \mathcal{P}(N) \quad (w = (w_1)) \quad \text{(short decomp)} \]

\[ (E_\xi)_N(x) = \sum_{w \in \mathcal{P}(N)} \psi(x w) \text{ d}x \]

\[ (E_\xi)_N(x) = \psi(x) + (M\psi)(x) \]

where \[ (M\psi)(x) = \sum_{w \in \mathcal{P}(N)} \psi(x w) \text{ d}x \]

This kind of object may well be factorizable into a product of local objects, if you fancy studying it.

So

\[ \langle E_\psi, E_\xi \rangle_G = \langle \psi, \xi + (M\psi) \rangle_p \quad \text{( \( = \langle \psi, (1 + M)\psi \rangle \) for all \( \psi \), non-constant out there.)} \]

\[ L^2(G(F) \backslash G(A)) \quad L^2(N(A) \backslash G(A)) \]

\[ M \text{ spoils things a bit. Shame. We must take our spectral analysis further.} \]

"We are close to being done. Sense " though... \[ \mathcal{P}(A) / N(A) \mathcal{P}(F) \approx \mathcal{H}(A) / H(F) \]

Define \[ \psi(x, x) = \int_{\mathcal{H}(F) \backslash N(A)} \psi(x \xi) \text{ d}x \quad \text{for an } \psi \text{ quickly supported in some sense?} \]

Then \[ \psi(m, x) = \delta(m)^2 \mathcal{X}(m) \psi(x, x) \]

\[ \psi(n m, x) = \delta(m)^2 \mathcal{X}(m) \psi(x, x) \quad \text{It's a global version of} \]

\[ \psi(x, x) \]

\[ \psi(x, x) = \delta(m)^2 \mathcal{X}(m) \psi(x, x) \quad \text{pure sense?} \]
We have Fourier inversion too.

Write $E \psi$ for $E \gamma$.

$$E \psi = E \left( \int_{\mathbb{R}^d} \psi \cdot (x, \omega) \ d\mu(x) \right)$$

$$E \psi(x) = E \psi_0(x)$$

$$E \psi_0(x) = E(\gamma_0, \gamma, \chi) = \sum_{\rho \in \mathbb{R}^d \setminus \mathbb{R}_+} \gamma(\rho, \chi)$$

STOP this is not nec eqt.

Only apply this above formula "for good $\chi$".

Now $E_\chi(z) = \sum_{\rho \in \mathbb{R}^d \setminus \mathbb{R}_+} \mathcal{J}(\delta, g_\chi)^{1/2} \left( \frac{1}{d} \delta(z + \rho) \right)$

This stuff is eqt when $|\chi| = \delta^{1/2}$, $|\chi(m)| = \delta(m)^{1/2}$, see eqt.

Let's try doing $\sum \delta(\chi z) \delta(x) \delta(x - \rho) = \sum \delta(\chi m) \delta(x) \delta(x - \rho)$ (for $\delta(\rho m) = \delta(\rho)$ & $\delta$ is extended)

must study this bit.

This is the same as studying $\int_{\mathbb{R}^d} \delta(\chi m x) \ dx$

$G$ acts on $\mathbb{R}^d$ on the right.

$\mathbb{R}^d$ has some kind of $\| \cdot \|$.

$$\| \cdot \| = \| \cdot \|_1$$

$$e_2 = (0, 1), \ e_1 = e_2, \ \| e_2 \| = 1$$

$$\| \mathcal{J}(\omega) \| = 1$$

$F(\mathcal{J}(\omega)) = \| \omega \|_2$
\[ \| e_2^{(\theta_m^n)} \| = \| m \| \text{ so } \delta(m) = \frac{\text{det}(m)}{\| e_2 \|} \]

0. \[ \delta(\mathbf{w}_n) = \| e_2 \mathbf{w}_n \|^{-2} - \| e_2 \mathbf{n} \|^{-2} \quad \mathbf{n} = (0, 1) \]

\[ \delta(\mathbf{w}_n)^{\frac{1}{2}} \quad \mathbf{w}(\mathbf{u}) = \| (1, \mathbf{u}) \|^{-1} \quad \text{number theorists can do this explicitly} \]

\[ \int_{N(\mathbf{a})} \delta(\mathbf{w}_n)^{\frac{1}{2}} = \int_{\mathbf{a}} \delta(1, \mathbf{u})^{-\frac{1}{2}} \quad \text{& number theorists can do this explicitly} \]

At this point, we get \[ \int_{N(\mathbf{a})} \delta(\mathbf{w}_n)^{\frac{1}{2}} \]

It turns out to be \[ \frac{L(s, 1)}{L(\sigma s, 1)} \quad \text{(exercise)} \]

\[ L(s, 1) \]

For \( \Re(s) > 1 \) it's abscissa.

There's a pole \( @ s = 1 \).

We need this for \( J_2 \).

\[ E_1(x, y, \chi) \quad |x| = y^{1/2} \]

Pari series \( (1 - \chi) \cdot 1 - \chi^2 \) locally everywhere (maybe twisted by \( \chi \))

The Eisenstein series is nothing but an intertwining operator between induced rep of \( P \rightarrow G \) to auto form or stry.

\( \otimes \pi(1, \mathbf{v}, 1 - \chi^2) \)

we get hanks of local tau reps floating around

(5.4) \( E(x, y, \delta^s) \) in some sense reflects this.

Time is too short to explain this further.

\[ E(x, y, \chi) = \sum_{n=0}^{\infty} \sum_{n \not\equiv 0} \psi(\theta_n, \chi) = \psi(x, \chi) \cdot (N \chi (-, \chi))(x) \]

\[ \sum_{N(\mathbf{a})} \delta(\mathbf{w}_n)^{\frac{1}{2}} \quad \mathbf{w}(\mathbf{u}) = (1, \mathbf{u})^{-\frac{1}{2}} \]

\[ \int_{N(\mathbf{a})} \delta(\mathbf{w}_n)^{\frac{1}{2}} \quad \mathbf{w}(\mathbf{u}) = (1, \mathbf{u})^{-\frac{1}{2}} \]

we've made this computation already.
We get \( L(s, \chi_0) / L(s+1, \chi_0) \). So if, say, \( \chi = 1 \) & \( \psi \) is \( K \)-inf, we get equality

So the convergence of \( M \psi \) is controlled by the convergence of \( L \)-fcts

Intertwining op for par-supps... rep. L-group... general concept of L-funct on L-groups. Langlands realized this & defined L-fcts on L-groups in his lecture "Euler products" (Yellow lecture notes, 1966)

"He guesses he should stop here."

There are many things on the board. It appears that \( \chi_0 \) has summarized the previous lecture.

No I have no idea what order the board go in.

1. Functional eqn \( E(x, \gamma, \chi) = E(x, M\gamma, \chi) \) (Hold in \( \chi_0 \) or \( \chi_0^* \))

In prin \( E(x, \gamma, \chi_0 \delta^s) = E(x, M\gamma, \chi_0 \delta^{-s/2}) \)

& \( \psi(\chi x) = \psi(x) \), \( E(x, \gamma, \chi_0 \delta^s) = \frac{L(s, \chi_0)}{L(s+1, \chi_0)} \) \( E(x, \gamma, \chi_0 \delta^{-s}) \)

2. At noon-time you solved the exercise:

\[
\int \frac{1}{|L(m)|^{1-\delta}} \\, \text{d}m = \int \frac{1}{|L(x)|^{1-\delta}} \\, \text{d}x = \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha-\delta}{2})} \frac{1}{q^\delta} \quad \text{for} \quad 0 < \alpha < 2,
\]

\[
= 1 + \left(1 + \frac{1}{q^2}\right) \sum_{n=1}^{\infty} q^{-n} \approx \frac{1}{1-q^{-2}} = \frac{1 + q^{-2}(1 - \frac{1}{q^2})}{1 - q^{-2}}
\]

3. \( \frac{1 - q^{-2} + (q^{-2} - q^{-\infty})}{1 - q^{-2}} = \left(\frac{1 - q^{-2}}{1 - q^{-\infty}}\right)^{-1} \)

M intertwines \( X \) with \( X' \)

E intertwines "auto. forms" on \( N(A)(F) / G(A) \) with auto. forms on \( G(F) / G(A) \)
\[ E(x, \gamma, \chi) = \sum \chi(\delta x, \chi) \]

\[ E(x, \gamma, \chi)_{N} = \gamma(x, \chi) + (M\gamma)(x, \chi) \]

\[ \int \chi(mn, x) dm = \int \int \chi(m, mn, x) \overline{\delta(m)} \chi(m)^{-1} dm \, dn \]

\[ = \int \chi(mn, m, x) \overline{\delta(m)} \chi(m)^{-1} dm \, dn = (M\gamma)(x, \chi) \quad (\delta(m) = \delta(m)^{-1}) \]

5. **Particular case**: \( \chi(xz) = \gamma(x) \) & \( \gamma(zz) = \eta(x) \) & the centre

   & if \( x = \delta^{-1/2} x_0 \), then

   \[ (M\gamma)(x, \chi^2) = \frac{L(5, x_0)}{L(5, x)} \gamma(x, \chi^2) \]

   where \( \chi(x) = \chi(x_0) = x_0 \chi_0(x_0), |x_0|^5 = \chi_0(x_0) |x|^5 \)

6. \( L(5, x) = L(5, \dot{x}^2) \)

\[ \frac{L(5, x) L(-5, \dot{x}^2)}{L(5, \dot{x}) L(-5, x)} = 1 \quad M^2 = 1 \quad (?) \]

That's the end of what was on the board.

**Truncation operator** \( \Lambda_{\text{tr}} \)

(\text{Langlands, Arthur})

\[ \Lambda_{\text{tr}} \varphi = \varphi - E \hat{\gamma}(x \Delta \varphi) \]

where \( \varphi \in \mathcal{A}(GL_n, \mathbb{F}) \)

\[ (\Lambda_{\text{tr}} \varphi)(x) = \varphi(x) - \sum_{\delta \in \text{pargf}} \hat{\gamma}(\delta x) \varphi_\Delta(\delta x) \]

\[ \hat{\gamma}(x) = \begin{cases} 1 & s(m) > e_1 \\ 0 & \text{otherwise} \end{cases}, \quad x = \text{m} \text{mk} \]

\[ \Lambda_{\text{tr}} \varphi = \sum (-1)^q \lambda_q \varphi \quad \text{for} \quad q = \text{m} \text{mk} \]

Only 1 case in the adelic case.
Now \((\Lambda^T)^2 = \Lambda^T\), \(\Lambda^T = (\Lambda^T)^*\) so it is an orthogonal projector.

\[ \varphi, \Lambda \varphi = 0 \]

\(\varphi\) is a function, \(\Lambda\varphi\) is rapidly decreasing.

\(\Lambda^T\varphi\) is square integrable.

(Exercise: Compute the scalar product \(\int_{G(\mathbb{F})} \Lambda^T \varphi(x, \psi, \kappa) \Lambda^T \varphi(x', \psi, \kappa) \, dx \, d\psi \, d\kappa\).)

It only involves \(x, \psi, \kappa\).

If \(K_f(x, y)\) is the kernel, \(K_f(x, y) = \sum_{q \in G(\mathbb{F})} f(x^q y)\) is in \(C_c^\infty(G(\mathbb{F}))\).

Quotient not cpt. \(K_f\) not Hilbert Schmidt.

The truncated operator \(\Lambda^T K_f\) is Hilbert–Schmidt.

\[ \left| (K \Lambda^T \varphi)(x) \right| \leq C \| \varphi \|_{L^2(-)}, \quad \varphi \in L^2(-) \]

Therefore

\[ \sum_y f(x^y y) (\Lambda^T \varphi)(y) \, dy \]

\[ \lambda = n_z, a_k, b_k \]

\[ y = n_z, a_k, b_k \]

Keep \(\text{rep} s\) from \(k, s\).

\[ \sum_y f(x^y y) (\Lambda^T \varphi)(y) \, dy \]

\[ \gamma = (x^y \ldots) \]

\[ c(y) = 0 \]

A priori it diverges, a priori.

Use the fact that we're truncated.

Apply Poisson res, Fourier, we're slowly decreasing,

\[ \int f(a_z^k(a_z) \alpha_z) - \int f(a_z^k(a_z) \alpha_z) \, da_z \]

Make your majorization.
Define $\tilde{J^T}(f) = \int (\Lambda^k) (x, x) \, dx$ (T is some big real, Alan reckons)

$T$ large enough (w.r.t. ??? of $f$)
- this is a poly. w.r.t.

This is $\tilde{J}(g)$. This is what the trace formula is all about.

Eisenstein serie$\rightarrow$ (after many difficulties) trace formula. Spectral sequence.
It's the trace of nothing.

$\mathfrak{f} = \mathfrak{g} \mathfrak{v}$

If at 2 places $v_1, v_2$

\[ \begin{cases} O_\mathfrak{g}(\mathfrak{g} \mathfrak{v}) = 0 \\ \text{whenever } v_1 \neq v_2 \end{cases} \]

then the trace formula for $\mathbb{P}GL_2 = G$

\[ J^T(g) = \sum_{\text{elliptic}} \text{vol} \left( \frac{G_{\mathfrak{g}}(\mathfrak{f}) \setminus G_{\mathfrak{g}}(\mathfrak{m})}{G_{\mathfrak{g}}(\mathfrak{v})} \right) \times \left( \frac{\text{tr}(\mathfrak{f})}{\text{det}(\mathfrak{f})} \right) \]

Now we can quickly finish Jacquet-Langlands.

$\mathbb{D}^\prime(\mathfrak{f}) \mathcal{Z}(\mathfrak{g}) \backslash \mathbb{D}^\prime(\mathfrak{g}) \mathcal{Z}(\mathfrak{g})$

$\mathfrak{f} = \mathfrak{g} \mathfrak{v}$

$\mathbb{G}L(2, \mathfrak{f}) \mathcal{Z}(\mathfrak{g}) \backslash \mathbb{G}L(2, \mathfrak{g})$

$\mathfrak{f} = \mathfrak{g} \mathfrak{v}$

Compare 2 traces

$v \neq s \Rightarrow \mathbb{D}^\prime \text{ split} \& \text{ take } f' = f \mathfrak{v}$

\[ \begin{cases} v \text{ bad} \\ \mathbb{D}^\prime \xrightarrow{\mathfrak{f}} \mathbb{G}L(\mathfrak{f}) \end{cases} \]

buticare classes of $\mathbb{D}^\prime \notin \mathbb{G}L(\mathfrak{f})$

Richard Taylor has done exactly now what we need - matching stuff