RECENT PROGRESS ON STARK'S CONJECTURES
Explicit class field theory, special values of L-functions and beyond

London Number Theory Study Group, Spring 2020
Organised by Luis Garcia and Alice Pozzi

Abstract

We propose the plan for a study group on recent developments on the conjectures of Stark and Gross's p-adic analogues relating units of number fields to special values of L-functions. We will focus on the proofs of the Gross-Stark conjectures by Darmon-Dasgupta-Pollack and Dasgupta-Kakde-Ventullo, which leverage on the study of certain p-adic families of modular forms. We will then introduce the Eisenstein cocycle point of view, and explain the Charollois-Dasgupta approach to the construction of a p-adic zeta function for real quadratic field, and the relevance to the Gross-Stark conjectures.

Goal of the study group

The goal of explicit class field theory is providing explicit *analytic* formulas for generators of abelian extensions of number fields. The investigation of this question, known as Hilbert's Twelfth Problem, motivated great achievements; the more classical ones are the Kronecker-Weber Theorem for $\mathbb{Q}$ and the theory of complex multiplication for imaginary quadratic fields. However, no general strategy is available for arbitrary fields.
More recently, the conjectures of Stark and their p-adic analogues predict a link between certain *Gross-Stark units* of number fields and leading terms of (p-adic) L-functions. Certain instances of these conjectures have recently been proved; the goal of this study group is to give an overview of this recent developments.

The original conjectures give a relation between of units in abelian extensions of number fields and leading terms of L-functions, generalizing classical results on cyclotomic and elliptic units. Let us state a simple version. Let K/F be an abelian extension of number fields and S be a set of places of F containing all places that ramify in K as well as the archimedean places. For an integral ideal $\mathfrak{a} \subset \mathcal{O}_F$ not divisible by any prime that ramifies in K, denote by $\mathrm{Fr}(\mathfrak{a})$ the corresponding Frobenius element in $\mathrm{Gal}(K/F)$ under the Artin reciprocity map. For each $\sigma \in \mathrm{Gal}(K/F)$, define the partial zeta function

$$\zeta_{K/F,S}(\sigma, s) = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_F \\ (\mathfrak{a},S)=1,\ \mathrm{Fr}(\mathfrak{a})=\sigma}} \frac{1}{\mathrm{N}\mathfrak{a}^s}, \qquad \mathrm{Re}(s) > 1,$$

which admits meromorphic continuation to the whole s-plane with no pole at $s = 0$.

Conjecture [8]. Assume that $|S| \geq 3$ and that S contains exactly one place v that splits completely in K and fix a place w of K over v. Then $\zeta_{K/F,S}(\sigma,0) = 0$ and there exists $u \in K^\times$ such that

- $|u|_{w'} = 1$ if $w' \nmid v$
- Let e be the number of roots of unity of K. Then

$$\zeta'_{K/F,S}(\sigma,0) = -\frac{1}{e}\log|u^\sigma|_w \text{ for all } \sigma \in \mathrm{Gal}(K/F). \qquad (*)$$

- $K(u^{1/e})/F$ is an abelian extension.

In particular, the formula $(*)$ gives an analytic expression for $|u|_w$; thus, it is a step towards the explicit class field theory for F. Note however that the conjectures say nothing about the *argument* of the unit u!

Recent developments have been made towards proving instances of Stark's conjectures and refining their formulations. The strategies essentially fall into two categories, each influenced by a different classical approach to the rationality of the values of the partial zeta function for totally real fields.

1. Gross [6] proposed a p-adic refinement of the conjectures of Stark ; recently several cases have been proved by [3] and [5]. Their methods are automorphic, and relate L-values to constant terms of Eisenstein series. They exploit the connection between automorphic forms and Galois representations, and congruences between Eisenstein series and cuspforms, an idea that can be traced back to Ribet's proof of Herbrand's converse theorem.

2. The second direction of refinement concerns the Eisenstein cocycle, a $(n-1)$-cohomology class for $\mathrm{GL}_n(\mathbb{Q}_p)$. This method due to Sczech gives a cohomological interpretation to Shintani's approach to the rationality of L-values via the "geometry of numbers". In addition to encoding expressions for L-values, the Eisenstein cocycle has been used to provide conjectural formulas for the argument of Stark's units in some cases (see e.g. [1]). Thus, it suggests an approach to explicit field theory *beyond* special values of L-functions.

Our aim is to start with some classical cases of Stark's conjectures and move on to understand some of these recent developments.

Schedule of talks

The Study Group will meet Wednesdays, 2pm to 3.30pm. The talks will be at the UCL Mathematics building, Room 505, with the exception of 18 March where we will meet in Room LG04, 26 Bedford Way.

1. **Introduction** (15 January) Luis Garcia / Alice Pozzi
2. **Stark's proof for imaginary quadratic fields** (22 January) TBD

   In [8], Stark proves his conjecture for imaginary quadratic fields using the classical theory of complex multiplication. Review this theory as well as classical Eisenstein series and the Kronecker limit formula and discuss Stark's proof. Reference: Section 5 of [8].
3. **The conjectures of Gross** (29 January) TBD

   Present Gross' Tower of Fields Conjecture, and the implications in the $\mathbf{TR}_\infty$ and $\mathbf{TR}_p$ cases (reference: AWS notes, chapter 2). Discuss the rank one abelian case. Reference: [6].
4. **Ribet's converse to Herbrand's Theorem** (5 February) TBD

   Recent approaches to the Gross-Stark conjectures [3, 5] exploit the connection between automorphic forms and Galois representations to deduce information about special values of L-functions. This strategy is influenced by ideas appearing in Ribet's proof [7] of the converse to Herbrand's Theorem, which played a prominent role in the proof of the Iwasawa Main Conjecture by Mazur and Wiles.
5. **The Gross-Stark Conjecture I: rank 1 case** (12 February) TBD

   In [3], Darmon, Dasgupta and Pollack prove the Gross conjecture in the rank one case. Define the analytic and algebraic $\mathscr{L}$-invariants; explain the interpretation of the algebraic $\mathscr{L}$-invariant in terms of Galois cohomology.
6. **The Gross-Stark Conjecture II: rank 1 case, continued** (26 February) TBD

   Give an overview of the proof of [3]: construction of an infinitesimal eigenform, relation between its U_p-eigenvalue and p-adic L-functions, construction of a cocycle.
7. **The Gross-Stark Conjecture III: higher rank** (4 March) TBD

   Dasgupta, Kakde and Ventullo [5] generalize the ideas of [3] in higher rank. Construct a higher order eigenform by studying certain orbits for the Hecke action.
8. **Eisenstein cocycle I** (11 March) TBD

   Define the Eisenstein cocycle and describe its basic properties. Reference: Section 4 of Dasgupta and Greenberg's AWS notes [4].
9. **Eisenstein cocycle II** (18 March) TBD

   Charollois and Dasgupta [2] propose a construction of p-adic zeta function for totally real fields via a refinement of Sczech's cocycle. Introduce the "smoothed" integral version of Sczech's cocycle.
10. **Eisenstein cocycle III** (25 March) TBD

    Construct the p-adic zeta function as specialization of the refined Sczech's cocycle. Present Charollois and Dasgupta's proof of the result on the order of vanishing of the p-adic zeta function at $s = 0$, originally obtained by Mazur-Wiles.

References

[1] Pierre Charollois and Henri Darmon, *Arguments des unités de Stark et périodes de séries d'Eisenstein*, Algebra Number Theory **2** (2008), no. 6, 655–688. MR 2448667

[2] Pierre Charollois and Samit Dasgupta, *Integral Eisenstein cocycles on* $\mathbf{GL}_n$*, I: Sczech's cocycle and p-adic L-functions of totally real fields*, Camb. J. Math. **2** (2014), no. 1, 49–90. MR 3272012

[3] Samit Dasgupta, Henri Darmon, and Robert Pollack, *Hilbert modular forms and the Gross-Stark conjecture*, Ann. of Math. (2) **174** (2011), no. 1, 439–484. MR 2811604

[4] Samit Dasgupta and Matthew Greenberg, *The rank one abelian stark conjecture*, Lecture notes for the 2011 Arizona Winter School.

[5] Samit Dasgupta, Mahesh Kakde, and Kevin Ventullo, *On the Gross-Stark conjecture*, Ann. of Math. (2) **188** (2018), no. 3, 833–870. MR 3866887

[6] Benedict H. Gross, *p-adic L-series at* $s = 0$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 979–994 (1982). MR 656068

[7] Kenneth A. Ribet, *A modular construction of unramified p-extensions of* $Q(\mu_p)$, Invent. Math. **34** (1976), no. 3, 151–162. MR 419403

[8] Harold M. Stark, *L-functions at* $s = 1$*. IV. First derivatives at* $s = 0$, Adv. in Math. **35** (1980), no. 3, 197–235. MR 563924