The Langlands Programme -or- Instructional Course on Automorphic Forms

0: Introduction

Dr Richard Taylor

Mon 15th Feb '93
9am

Welcome to everybody. Richard's going to chat for ~ 10-15 mins,
then things'll get going at 10 with Martin.

This is supposed to be an introduction to automorphic forms for number theorists

An automorphic form is an analytic map on GL_n. Related to reps of $\mathrm{Gal}(\bar{\mathbb{Q}}/\overset{K}{\mathbb{Q}})$ &
also to arithmetic varieties.

Examples: $n=1$ - Grossencharacters
$n=2$ - Elliptic modular forms. Already very tricky

The aim of the course is to teach number theorists the flashy automorphic forms language

No attempt at an overview will be given. They decided to stick to GL_2 & do lots here.
L-functions, Eisenstein series, Theta series (Weil rep^n) will hardly be mentioned either.

In week 1 we'll tie up automorphic forms on GL_2 with elliptic modular forms, & show
why they're the same thing

In week 2 we'll take the trace formula (a technique from automorphic forms) & use it
to understand cyclic base change for GL_2.

He'll chat vaguely about base change now

If we have aut rep $\pi \Longleftrightarrow$ Gal rep ρ then we can do $S^2(\rho)$, $\Lambda^2(\rho)$, $\rho_1 \otimes \rho_2$ etc
If L/K then we can do $\rho|_{\mathrm{Gal}(\bar{K}/L)}$

If $\leftrightarrow$ really exists we should be able to do corresp things at automorphic rep^n level.
The easy things on RHS usually turn out to be quite tricky on LHS.

Eg $\rho|_{\mathrm{Gal}(\bar{K}/L)} \leftrightarrow \pi \mapsto \pi_L$, base change. If $\mathrm{Gal}(L/K) = \langle\sigma\rangle$, then π_L exists,
& Π arises $\Leftrightarrow {}^{\sigma}\Pi = \Pi$

Similarly, rep R of $\mathrm{Gal}(\bar{K}/L)$ arises in this way $\Leftrightarrow R^{\sigma} \cong R$ ie $R^{\sigma}(\tau) = R(\tilde{\sigma}\tau\tilde{\sigma}^{-1})$
(easy exercise)

In the case of GL_1 we can do this latter base change once we have CFT:
$\chi_L = \chi \circ N_{L/K}$. Tony Scholl will tell us more next week.

An underlying assumption is that we know more algebra than analysis.
Another underlying assumption is that we are not experts, even though Jean-Pierre Labesse at the back in fact is. So we'll spend lots of time doing stuff that experts would regard as standard & we'll have never seen before

The Courses: Week 1:

1) Martin Taylor - GL_1 = Class Field Theory & he'll show us that it's all as predicted
3) John Coates - auto forms on GL_2 $\Leftrightarrow$ modular forms & he'll explain the dictionary
2) Tony Scholl - Rep thy of $GL_2(\mathbb{R})$ & $GL_2(\mathbb{Q}_p)$ - necessary for 3)
4) Richard Taylor - Quaternion Algebras. He's doing this because it's easier than GL_2: less familiar but formally v. similar, no Eisenstein series, & analytically v. easier. He'll prove analogous results to John's course. He'll chat about the trace formula.

Week 2: Base change

quaternion algebras again
6) Tony Scholl again - use trace formula to establish base change results. He'll need some local analysis eg analysis on GL_n.
5), 7), 8) are orbital integrals & stuff - the local analysis required Local facts about orbital integrals
9) Michael Harrison - application of base change in split GL_2 case (nasty technicalities)
~~Lots of proofs~~, $\Rightarrow$ Langlands pf of cases of Artin conjecture.

Lots of proofs but limited objectives

1 Administrative thing: leccys @ 2:15 & 3:45 today & next Monday as there's a colloquium, & it would be political to allow people to go.

Tea & coffee & sarnies above. Lots of sarnies today & say if we want more in the future (Presumably you're not that bothered about this bit, John)

I Background & GL_1.

Martin Taylor

ecture 1
Mon 15th Feb '93
10am

He'd like to start with a few announcements

- Bursaries
- Register!!
- Food at lunchtime

He doesn't want to begin with Haar measure, but peer pressure is forcing him to.

§0 Haar measure

Principal object of study is G a locally cpct group.

Set $C_c(G) = \{ f : G \to \mathbb{R} \text{ cts} \mid \text{cpct support} \}$

A measure is a cts (def below) linear form $m : C_c(G) \to \mathbb{R}$.

For $K \subseteq G$ cpct $\exists C = C_K$ s.t. $m(f) \leq C_K \sup\limits_{x \in K} |f(x)|$ (this may be def of cts)

Write $m(f) = \int f = \int\limits_G f(x)\, d_m x$

John has nudged him into a defn of +ve measure:

Call m +ve if $f \geq 0 \Rightarrow m(f) \geq 0$

For $f \in C_c(G)$ & $s \in G$ define $f^\vee$, ${}_sf$, f_s by

$$f^\vee(x) = f(x^{-1}) \quad ({}_sf)(x) = f(s^{-1}x)$$
$$(f_s)(x) = f(xs^{-1})$$

Def: Measure m is called left invt if $m(f) = m({}_sf)$

Def: A non-zero, left invt, +ve measure on $C_c(G)$ is called a Haar measure.

Thm 0.1 $\exists$ Haar measure on G. Moreover, such a measure is unique up to a positive multiplicitive cst. $\square$

He's summoned Brian Birch to move a strange wooden thing which appears to be connected to the ground.

Modular f^n of G

$f \in C_c(G)$, $s \in G$. Define $f^{\mathrm{conj}(s)}(x) = f(sxs^{-1})$

$f \mapsto m(f^{\mathrm{conj}(s)})$ is also a measure: (Here m is Haar measure)

$m(({}_t f)^{\mathrm{conj}(s)}) = m(f(sxs^{-1}))$ (by left invariance) So this is also Haar measure.

so by the theorem, $m(f^{\mathrm{conj}(s)}) = \Delta_G(s)\, m(f)$

$\Delta_G : G \to \mathbb{R}_+$ is the modular function. It's a cts HM

It somehow measures how m fails to be right-invt.

If $m(f) \neq 0$, $\Delta_G(s) = \dfrac{\int_G f(sxs^{-1})\,dx}{\int f(x)\,dx}$

Note: if $s \in Z(G)$ then $\Delta_G(s) = 1$

Prop^n 0.2 The modular function Δ_G is identically 1 if a) G is cpct
b) G/Z is simple & non-abelian □

Def: If $\Delta_G \equiv 1$ we say G is unimodular

Prop^n 0.3 Let H be a closed normal subgp of G. Then $\Delta_G|_H = \Delta_H$

Pf By standard theory, G/H is locally cpct. So we have Haar measures on G, H, & G/H. Say $f \in C_c(G)$.

Let's define a gadget $n(f) = \int_{\bar{x} \in G/H} \left(\int_{h \in H} f(xh)\, d_m h \right) d_{\bar{m}} \bar{x}$

Here $d_m h$ is a Haar measure for H
$d_{\bar{m}} \bar{x}$ is a Haar measure for G/H

It's easy to check that n is a Haar measure for G.

Now say $s \in H$. Then $n(f_s) = \int_{\bar{x}} \int_h f(xhs^{-1})\, d_m h\, d_{\bar{m}} \bar{x}$

$= \int \Delta_H(s) \left(\int \quad \right)$

$= \Delta_H(s)\, n(f)$

$\therefore \Delta_G(s) = \Delta_H(s)$ □

He briefly wants to talk about going from G to G/H.

Say now $H \trianglelefteq G$, H cpct. We have $G \twoheadrightarrow G/H$ & this induces $C_c(G/H) \hookrightarrow C_c(G)$ & hence Haar measure on G induces one on G/H.

He'll now give us (hopefully lots of relevant) examples. Checking things are Haar measures is a worthwhile exercise, & not just laziness on behalf of the lecturer.

Examples

(1) G disc. top. Then $m(s) = 1 \ \forall s \in G$

(2) G profinite. If $H \subseteq G$, H open, then $m(H) = (G:H)^{-1}$

(3) $G = \mathbb{R}^+, \mathbb{C}^+$. $\mathbb{R}^+$ is Lebesgue measure

$\mathbb{C}^+$: volume form arising from $|dz \wedge d\bar{z}| = 2\,du\,dv$, $z = u+iv$

(4) (perhaps the most interesting yet)

$G = SO_2(\mathbb{R}) = \left\{ s \in \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a^2+b^2=1 \right\}$; $d_m s = \frac{da}{b} = \frac{db}{a}$ (I think this should be $\frac{-db}{a}$?)

(5) $K/\mathbb{Q}_p$ finite, $\mathcal{O}$ integral closure of $\mathbb{Z}_p$ in K, $\mathfrak{p}$ max^l ideal

$|\cdot| : K^\times \to \mathbb{R}_+$, $|x| = (\mathcal{O} : x\mathcal{O})^{-1}$; $m(\mathfrak{p}^n) = (N\mathfrak{p})^{-n}$ (this is measure on $(K,+)$ or $(\mathcal{O},+)$ I guess)

$$m(t\mathcal{O}) = |t|\, m(\mathcal{O})$$
$$d(tx) = |t|\, dx$$
$$\frac{d(tx)}{|tx|} = \frac{dx}{|x|}$$

So $G = K^\times$ has Haar measure $dx/|x|$

In fact this is case $n=1$ of $\underset{\underset{s=(s_{ij})}{\cup\!\!\!\mid}}{GL_n(K)}$, which has Haar measure $\frac{\prod ds_{ij}}{\det(s)^n}$

$GL_n(K)$ is unimodular. By prop^n 0.3) $\Rightarrow SL_n(K)$ is also unimodular.

If §0 were at the end of the course then he would have already defined $\mathbb{A}_K$, K a no-field

K a no-field

$G = \mathbb{A}_K$: choose Haar measures $\{m_v\}$ s.t. $\underset{v<\infty}{m_v(\mathcal{O}_v) = 1}$ a.e.$^{\text{everywhere}}_{\text{most}}$

Then $m = \prod m_v$ is Haar measure.

$G = J_K = \mathbb{A}_K^\times$: pick $m_v^\times$ s.t. $m_v^\times(\mathcal{O}_v^\times) = 1$ almost everywhere

& $m^\times = \prod m_v^\times$

From (5) et al you may assume that all we're interested in is unimodular gps, but this is false.

(7) Non unimodular $G=\left\{\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix}\in GL_2(\mathbb{R})\right\}$

Check: $a^{-2}\,da\,db$ is left-inv. Then $\Delta_G\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}=a^{-2}$

That ends the Haar measure bit.

Now begins the course proper

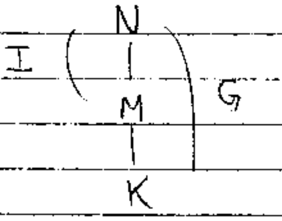
- little bit on local CFT
- substantial bit on global CFT
- lots more interesting stuff

§1 Local classfield theory

$K/\mathbb{Q}_p$ finite extension. $\mathcal{O}$ the integral closure of $\mathbb{Z}_p \supseteq \mathfrak{p}$ max^l ideal.

$|\cdot| : K^\times \to \mathbb{R}_+$. $N\mathfrak{p}=(\mathcal{O}:\mathfrak{p})$

If N/K is a field ext^n, Galois, & $G=G(N/K)$, define M = max^l n.r. ext^n of K in N, & $I=G(N/M)$ is the inertia gp, F = Frob automorphism



$G/I = H = \langle F\rangle$ Here $F(x)\equiv x^{N\mathfrak{p}} \bmod \mathfrak{p}_M$ $\forall x\in \mathcal{O}_M$.

Recall the local reciprocity map

$$\hat H^2(G,N^\times)\underset{\alpha\ (\text{split over } N)}{\cong} Br(N/K) \underset{\beta}{\cong} \tfrac{1}{|G|}\mathbb{Z} \bmod \mathbb{Z}$$

α is defined by Galois descent

By def^n $\hat H^{-2}(G,\mathbb{Z}) = H_1(G,\mathbb{Z}) = I_{aug}/I_{aug}^2 = G^{ab}$

$\hat H^0(G,N^\times) = K^\times/N_{N/K}(N^\times)$

The obvious generator $\frac{1}{|G|}$ mod $\mathbb{Z}$ of $\frac{1}{|G|}\mathbb{Z}$ mod $\mathbb{Z}$ can be pulled back to an elt of $\hat{H}^2(G,N^\times)$ which is called $c_{N/K}$, the <u>canonical class</u>

The cup product $\cup c_{N/K} : G^{ab} \to K^\times/N(N^\times)$ is an IM

We're really interested in its inverse, which we'll call $\theta = \theta_{N/K}$

Because $K^\times \twoheadrightarrow K^\times/N(N^\times)$ we'll call the composition $K^\times \twoheadrightarrow G^{ab}$ θ too.

<u>Facts</u> For simplicity assume G abelian. θ has the following properties

(1) $\theta(\mathcal{O}^\times) = I$

(2) $\theta(1+\mathfrak{p}) = \underline{P}$ = <u>wild inertia subgp</u> = p-Sylow subgp of I

(3) Let π be a uniformising parameter ie $\pi\mathcal{O} = \mathfrak{p}$. Then $\theta(\pi)|_M = F$, the Frobenius

Other people may need $\theta(\pi)|_M = F^{-1}$ but he'll stick with F.

<u>Functoriality properties</u>

If $N \supseteq L \supseteq K$, then

$$\begin{tikzcd} L^\times \arrow[r, "\theta_{N/L}"] \arrow[d, "N_{L/K}"'] & G(N/L)^{ab} \arrow[d, "\mathrm{inc}^{ab}"] \\ K^\times \arrow[r, "\theta_{N/K}"'] & G(N/K)^{ab} \end{tikzcd} \quad \& \quad \begin{tikzcd} K^\times \arrow[r, "\theta_{N/K}"] \arrow[d, hook] & G(N/K)^{ab} \arrow[d, "\mathrm{Ver}"] \\ L^\times \arrow[r, "\theta_{L/K}"] & G(L/K)^{ab} \end{tikzcd}$$

both commute.

& This only assumed N/K Galois. Suppose now L/K is Galois.

$$\begin{tikzcd} & G(N/K)^{ab} \arrow[d, "\mathrm{res}^{ab}"] \\ K^\times \arrow[ur, "\theta_{N/K}"] \arrow[r, "\theta_{L/K}"'] & G(L/K)^{ab} \end{tikzcd}$$

ie θ is natural w.r.t. restriction of AMs

So by taking limits we get $\theta_K : K^\times \hookrightarrow \mathrm{Gal}(K^{ab}/K) = \mathrm{Gal}(\bar{K}/K)/\overline{[\mathrm{Gal}(\bar{K}/K), \mathrm{Gal}(\bar{K}/K)]}$

which is cts (RHS has Krull topology) (= profinite topology)
& dense (as θ_K is surjective at finite levels)

<u>Archimedean case</u> Much easier. Only $\mathbb{C}/\mathbb{C}$, $\mathbb{C}/\mathbb{R}$, $\mathbb{R}/\mathbb{R}$

NB $\mathbb{R}_+$ = +ve reals $\mathbb{R}^+$ = $\mathbb{R}$, abelian gp, operator +

We have $\theta_{\mathbb{C}/\mathbb{R}} : \mathbb{R}^*/\underset{\mathbb{R}_+}{\overset{\shortparallel}{N(\mathbb{C}^*)}} \cong \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \rho \rangle$ when he'll write ρ for complex conjugation

ρ is called the Frobenius automorphism of $\mathbb{C}/\mathbb{R}$. The Frobenius automorphisms of $\mathbb{R}/\mathbb{R}$ & $\mathbb{C}/\mathbb{C}$ are of course trivial.

ecture 2
Mon 15th Feb '93
2:15 pm

Recall Tony's lecture is at 3:45

§2 Global Classfield Theory

K now a number field, v a place of K

If $v<\infty$, $|\cdot|_v : K_v^* \to \mathbb{R}_+$

If $v|\infty$, $K_v = \mathbb{R}$ or $\mathbb{C}$, & $|\cdot|_v$ = modulus. Set $n_v = [K_v : \mathbb{R}]$

<u>Product formula</u> for $x \in K^\times$, $\prod_{v<\infty} |x|_v \prod_{v|\infty} |x|_v^{n_v} = 1$

<u>Frobenius AM</u> If N/K infinite Galois ext^n, $G = G(N/K)$, v a place of K, w a place of N, w|v

Then $G_w = \{ g \in G \mid gw \sim w \}$

$$\begin{tikzcd} w \arrow[r, dash] \arrow[d, dash] & N \arrow[r, hook] \arrow[d, dash] & N_w \arrow[d, dash] \\ v \arrow[r, dash] & K \arrow[r, hook] & K_v \end{tikzcd}$$

Here $G(N_w/K_v) = G_w$

<u>Def</u>: If v is non-ramified in N/K, then $F = F(N_w/K_v)$ is an elt of G. We call this the Frobenius AM of w in N/K; it is characterised by the properties

(i) $v<\infty$ $F(x) \equiv x^{N\mathfrak{p}_v} \bmod \mathfrak{p}_w$ $\forall x \in \mathcal{O}_w$
(ii) $v|\infty$ $\langle F \rangle = G(N_w/K_v)$

Let's have a quick look at the functoriality properties of this new gadget.

The Frobenius AM satisfies

(1) $\sigma \cdot N \longrightarrow \sigma N$; $F(\sigma w, \sigma N / \sigma K) = \sigma \circ F(w, N/K) \cdot \sigma^{-1}$

(2) $v < \infty$
N/K Galois.

$$\begin{tikzcd} N \arrow[r, dash] \arrow[d, dash] & w \arrow[d, dash] \\ L \arrow[r, dash] \arrow[d, dash] & u \arrow[d, dash] \\ K \arrow[r, dash] & v \end{tikzcd}$$

$F(w, N/L) = F(w, N/K)^{f(u|v)}$

(3) If L/K is Galois in the above piccy, then $F(w, N/K)|_L = F(u, L/K)$

Fix v: $F(v) = \{F(w, N/K) \mid w|v\}$ This is a def^n. It is a conjugacy class in G.

<u>Passage to infinite ext's</u>

$K \subseteq N \subseteq K^c$ = alg. closure of K. Set Σ_K = set of places of K
$$\overset{\Sigma_K}{\|} \quad \Sigma_{K,\infty} \cup \Sigma_{K,f}$$

Rule Here K is a number field, & N/K may be infinite.

Put $\Sigma_N = \varprojlim \Sigma_L$ where $\underbrace{K \subseteq L}_{\text{finite}} \subseteq N$ & the limit is taken w.r.t. restriction of places.

So technically $w \in \Sigma_N$ is an infinite coherent vector of places.

$$w = (u_L), \quad u_L \in \Sigma_L$$

Define $G_w(N/K) = \varprojlim_L G_{u_L}(L/K)$

Say w is unramified if u_L is unramified $\forall L$.

If w is unramified then we have $F(w) = (F(u_L, L/K))$ (these are coherent by property (3) above)
$\{F(w), w|v\}$, form a conjugacy class in $G(N/K)$. Call this class $F(v)$.

So everything we did in the finite case goes through happily to the infinite case.

Let's now talk about

Density

Set $\Sigma = \Sigma_K$. N/K, finite again, Galois gp G.

For $S \subseteq \Sigma_f$ define $\delta_n(S) = \#\{s \in S \mid Ns \leq n\}$

We say that S has <u>density</u> $\delta \in [0,1]$ if $\lim_{n \to \infty} \frac{\delta_n(S)}{\delta_n(\Sigma_f)} = \delta$.

NB this is <u>natural density</u>. ~~Thus is~~ If S has natural density δ then it has Dirichlet density δ, but not conversely.

<u>Thm 2.1</u> (Chebotarev) Say N/K is an ext of number fields, with Galois gp G.

Let $X \subseteq G$ be stable under conjugation.

Let $P_X = \{ v \in \Sigma_{f,K} \mid F(v) \subseteq X \}$

Then P_X has density $|X| / |G|$. □

<u>Cor 2.2</u> Allow N/K to be infinite (inside K^c)
Suppose only a finite no. of places of K ramify in N.
Then the $\{F(w)\}$ ($w \in \Sigma_N$, non-ramified) is dense in $G(N/K)$. □

($N \supseteq L \supseteq K$ finite. Density follows as Frob of L/K cover Gal(L/K) by thm 2.1.

Ideles & Adeles

Given G_v a locally cpct group, $\supseteq H_v$ a cpct open subgp.

$S \subseteq \Sigma$ s.t. $|S| < \infty$. NB from now on S will always be finite.

Then $G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v$

Set $G = \bigcup_{\text{all } S} G_S$. It is a locally cpct gp, a restricted direct product.

G is topologised by: a fundamental system of nbds of 1 is given by those of the G_S for all S.

Note that he's at liberty not to define H_v for finitely many v because we can just enlarge S to include these v.

<u>Adeles</u> Take $G_v = K_v^+$, & for $v<\infty$ set $H_v = \mathcal{O}_v$.

The restricted product $\mathbb{A} = \mathbb{A}_K$ is the <u>adele ring</u>

Check: $\mathbb{A}$ is actually a topological ring.

<u>Ideles</u> $G_v = K_v^\times$, & for $v<\infty$ set $H_v = \mathcal{O}_v^\times$

The restricted product J_K is the ideles.

NB $\mathbb{A}_K^\times = J_K$

x is integral almost everywhere

We have a diagonal map $K^+ \hookrightarrow \mathbb{A}$, $x \mapsto \prod_v \sigma_v(x)$ where $\sigma_v : K \hookrightarrow K_v$

<u>Prop^n 2.3</u> K has discrete image in $\mathbb{A}$, and the quotient group $\mathbb{A}/K$ is compact for the quotient topology. $\square$

The same recipe gives us $K^\times \hookrightarrow J$, $x \mapsto \prod_v \sigma_v(x)$ (x is a unit almost everywhere

<u>Def^n</u> $C_K = J/K^\times$, the <u>idèle class group</u>

<u>Def^n</u> $\|\cdot\| : \underset{x}{\overset{\Psi}{J}} \to \mathbb{R}_+$

almost all 1 so the product makes sense

$$\|x\| = \prod_{v<\infty} |x_v|_v \times \prod_{v|\infty} |x_v|_v^{n_v}$$

By the product formula, we have $K^\times \subseteq \ker \|\cdot\|$, so we can view $\|\cdot\|$ as a map from C_K to $\mathbb{R}_+$

<u>Def^n</u> $J_K^0 = \ker(\|\cdot\| : J_K \to \mathbb{R}_+)$

<u>Prop^n 2.4</u> $K^\times$ is discrete in J, and $J^0/K^\times$ is cpct. $\square$

<u>Functoriality</u>

N/K number fields, v a place of K, w a place of N above v, so $K_v \hookrightarrow N_w$

(1) Inclusion: $J_K \hookrightarrow J_N$,

$\prod_v x_v \mapsto \prod_w y_w$ where $y_w = x_v$ for $v|w$

(2) Norm : $N_{N/K} : J_N \to J_K$

$\prod_w y_w \mapsto \prod_v x_v$, where $x_v = \prod_{w|v} N_{N_w/K_v}(y_w)$

This is cts.

Also, for $\mathbb{A}$, we get $\mathbb{A}_K \hookrightarrow \mathbb{A}_N$ & $Tr_{N/K}: \mathbb{A}_N \to \mathbb{A}_K$. Check the details.

To relate ideals to ideles we use the

<u>Content map</u> $I = I_K$, the gp of fractional $\mathcal{O}$-ideals

The content map
$$( ): J \to I$$
$$(\alpha) \mapsto \prod_{v<\infty} \mathfrak{p}_v^{v(\alpha_v)}$$

Here $v: K_v^\times \twoheadrightarrow \mathbb{Z}$ is the valuation associated to the place v.

By inspection, $\mathrm{Ker}\,( ) = \prod_{v<\infty} \mathcal{O}_v^\times \times \prod_{v|\infty} K_v^\times =: U_K$, <u>the unit ideles</u>

U_K is a basic open set in J. $(\cdot)$ is cts if I has the discrete topology.

He's now coming up to the concept of admissibility. This is distinct from the concept of an admissible repⁿ that Tony is talking about.

<u>Admissibility</u> part 1. Say $\Sigma_\infty \subseteq S \subseteq \Sigma$

I_S = group of fractional $\mathcal{O}$-ideals "prime to S"

<u>Defⁿ</u> $( )^S: J \to I_S$, defined by $(\alpha)^S = \prod_{v \notin S} \mathfrak{p}_v^{v(\alpha_v)}$

Related to this is the <u>group of S-ideles</u> $J^S = \{ j \in J \mid j_v = 1 \ \forall v \in S \}$

<u>Defⁿ</u> Let G be a topological abelian group. For a HM $\varphi: I_S \to G$, we call the pair (φ, S) <u>admissible</u> if for each open nhd $\mathcal{N}$ of 1 in G $\exists \varepsilon > 0$ s.t. $\varphi((a)^S) \in \mathcal{N}$ $\forall a \in K^\times$ with $|a-1|_v < \varepsilon$ $\forall v \in S$ ←(Script N, thanks to suggestion of B.C. Agboola)

Note that if G is discrete then we can take $\mathcal{N} = \{1\}$ & then get $(a)^S \in \ker \varphi$.

<u>Lecture 3</u>
<u>Tues 16th Feb '93</u>
<u>2:30pm</u>

He finished off yesterday with the defⁿ of admissibility in the non-Tony Scholl case.

Recall N/K an extⁿ of number fields, $\Sigma_\infty \subseteq S \subset \Sigma$, $|S| < \infty$

G a top. ab. gp; $\varphi: I_S \to G$; (φ, S) is admissible if $\forall$ nhds $\mathcal{N}$ of 1 in G $\exists \varepsilon > 0$ s.t. $\varphi((a)^S) \in \mathcal{N}$ $\forall a \in K^\times$ s.t. $\forall v \in S$ ~~sto~~ $, |a-1|_v < \varepsilon$.

Propⁿ 2.5 (Glorified weak approximation)

(i) Suppose G is complete, & (φ, S) is admissible. Then $\exists !$ HM $\psi : J \to G$ s.t.

(a) ψ cts
(b) $\psi(K^\times) = 1$
(c) $\psi(x) = \varphi((x)) \ \forall x \in J^S$

(ii) Suppose now $\exists$ open nhd 1 of G in which $\{1\}$ is the only subgp ("the no small subgp hypothesis"). Then, given cts HM. $\psi : J \to G$ s.t. $\psi(K^\times) = 1$, ψ comes from (φ, S) admissible as in (i)

Sketch proof (i) (φ, S) given. Given $x \in J$, choose $\{a_n \in K^\times\}_{n \geq 0}$ which converge to x^{-1} in K_v $\forall v \in S$.
Define $\psi(x) := \lim_{n \to \infty} \varphi((a_n x)^S)$

A key point is $\dfrac{\varphi((a_n x)^S)}{\varphi((a_m x)^S)} = \varphi((a_n a_m^{-1})^S) \to 1$

It's clear that $\psi(K^\times) = 1$.

(ii) Now given ψ. Then $U \cap J^S = U^S$ (this is a defⁿ of U^S; recall $U_K = U = \ker(\,)$)

We have U^T for $T \supseteq S$. These lie in arbitrarily small nhds of 1 in J_K and are groups. Hence by "no small subgps" we have $\psi(U^T) = 1$ for large T.
Then $\varphi : I_T \cong J^T / U^T \xrightarrow[\psi]{} G$. $\square$

The Artin map

Say N/K an ab. extⁿ of no. fields, & $G = G(N/K)$. & $\Sigma_\infty \subseteq S \subset \Sigma$.
Assume that S contains all ramified primes of N/K.

Define $F = F_{N/K} : I_S \to G$ by $F(v) = F(v, N/K)$.

F is onto. Here's a pf: If $H = \mathrm{Im}\, F$ then we get an extⁿ N^H/K & by Chebotarev we see $N^H = K$, so $H = G$.

This is one of the few bits of global classfield theory that he can prove. He'll just ~~sketch the rest, so~~ tell us about the rest of it in as attractive a manner as possible.

<u>Thm 2.6</u> (Main thm of CFT)

(a) F is admissible (i.e. (F,S) is admissible)
(b) By (2.5) F corresponds to $\theta = \theta_{N/K} : J/_{K^\times} \longrightarrow G$, with $\theta(x) = F((x))\ \forall x \in J^S$.
(c) $\operatorname{Ker}\theta = K^\times N_{N/K}(J_N)/_{K^\times}$

(d) Given any open subgp $X \subseteq C_K$ with finite index, $\exists$ ab. ext^n N/K with $\ker \theta_{N/K} = X$. ☐

A Galois-like arrangement here (it's killed by its dual)

If $N \supseteq L \supseteq K$ then

$$\begin{tikzcd} & G(N/K) \arrow[d, "res"] \\ C_K \arrow[ur, "\theta_{N/K}"] \arrow[r, "\theta_{L/K}"'] & G(L/K) \end{tikzcd}$$

Taking limits gives us $\theta_K : C_K \longrightarrow Gal(K^{ab}/K) = G_K^{ab}$.

Recall that in the local case θ was injective with dense image.
In the global case it's essentially the opposite.

<u>Thm 2.7</u> θ_K is surjective, & $\ker \theta_K$ is the connected component of the identity in C_K. ☐

More functoriality:

(1) If $\sigma \in G_{\mathbb{Q}}$ then

$$\begin{tikzcd} C_K \arrow[r, "\theta_K"] \arrow[d, "\sigma"'] & G_K^{ab} \arrow[d, "conj(\sigma)"] \\ C_{\sigma K} \arrow[r, "\theta_{\sigma K}"'] & G_{\sigma K}^{ab} \end{tikzcd}$$

(check for Frobenius & then patch up)

(2) If N/K is a not necessarily abelian ext^n, then

$$\begin{tikzcd} C_N \arrow[r, "\theta_N"] \arrow[d, "N_{N/K}"'] & G_N^{ab} \arrow[d, "inc^{ab}"] \\ C_K \arrow[r, "\theta_K"] & G_K^{ab} \end{tikzcd} \quad \& \quad \begin{tikzcd} C_K \arrow[r, "\theta_K"] \arrow[d, "N_{N/K}"] & G_K^{ab} \arrow[d, "ver"] \\ C_N \arrow[r] & G_N^{ab} \end{tikzcd}$$

We'll next talk about local/global compatibility.

<u>Compatibility</u> N/K abelian ext^n of number fields, v a place of K, w a place of N, w|v.

$$\text{Recall} \quad \theta_v : K_v^\times \longrightarrow G(N_w/K_v) \longleftrightarrow G$$

<u>Thm 2.8</u> For $a \in J$, $\theta_K(a) = \prod_v \theta_v(a_v)$. Note that $\theta_v(a_v) = 1$ for almost all v. ☐

<u>§ Grössencharaktere</u>

This is far too long, as are so many Germanic words, to an Anglo-Saxon like Martin, so he'll call them GCs.

Say K is a number field & $\mathfrak{f}$ an $\overset{\mathcal{O}}{\mathbf{0}}$-ideal.

$$\mathbf{U}(\mathfrak{f}) := \{ u \in U_K \mid u_v \equiv 1 \bmod \mathfrak{f}_v \ \forall v < \infty,\ u_v = 1 \ \forall v | \infty \}$$

<u>Def^n</u> A <u>GC</u> of K is a cts hom. $\chi . J/K^\times \to \mathbb{C}^\times$, or $\chi : J \to \mathbb{C}^\times$

Note $\mathbb{C}^\times$ has the 'no small subgps' property.

I Let $\mathcal{N}$ be a small nhd of $1 \in \mathbb{C}^\times$.
Then $\chi^{-1}(\mathcal{N})$ is open, so contains some $U(\mathfrak{f})$, & thus $\chi(U(\mathfrak{f})) = 1$ as $\chi(U(\mathfrak{f}))$ is a group.
Call the largest such $\mathfrak{f}$ the <u>conductor of χ</u>; write $\mathfrak{f}(\chi)$.

II Can apply prop^n 2.5 to obtain an admissible pair (φ, S). (It seems that we can arrange S s.t. it is $\Sigma_\infty \cup \{v | \mathfrak{f}$
$\varphi : I_S \to \mathbb{C}^\times$, s.t.

<u>(3.1)</u> $\varphi((x)) = \chi(x) \quad \forall x \in J^S$.

<u>Def^n</u> Set $K_\mathfrak{f} = \{ \alpha \in K^\times \mid \underbrace{\alpha \gg 0}_{\text{ie } \alpha \text{ is totally positive}} \ \& \ \underbrace{\alpha \equiv 1 \bmod^\times \mathfrak{f}}_{\text{i.e. } v(\alpha - 1) \geq v(\mathfrak{f}) \ \forall v | \mathfrak{f}} \}$

<u>Def^n</u> For $x \in J$, <u>x_S</u> $\in J$ is the S-part of x.

$$\text{Then} \quad (x_S)_v = \begin{cases} x_v & v \in S \\ 1 & v \notin S \end{cases}$$

If $\mathfrak{f} = \mathfrak{f}(\chi)$, $\alpha \in K_\mathfrak{f}$, then $\chi(\alpha . \alpha_S^{-1}) = \varphi((\alpha \alpha_S^{-1}))$ by (3.1) Here $S = \{v | \mathfrak{f}\} \cup \Sigma_\infty$

$= \varphi((\alpha)) \quad (\text{as } (\alpha_S^{-1}) = 1)$

But $\chi(\alpha . \alpha_S^{-1}) = \chi(\alpha_S^{-1}) = \prod_{v \in S} \chi_v(\alpha_v^{-1})$

<u>(3.2)</u> Hence $\varphi((\alpha)) = \prod_{v | \infty} \chi_v(\alpha_v^{-1})$. Now $\chi_v : K_v^\times \to \mathbb{C}^\times$ & in the case $v | \infty$ we have $K_v^\times = \mathbb{R}^\times$ or $\mathbb{C}^\times$.

Lemma 3.3 Viewing K_v, $v|\infty$, as a subfield of $\mathbb{C}$, then any $\chi_v : K_v^\times \to \mathbb{C}^\times$ may be written (non-uniquely) in the form

$$\chi_v(x) = x^{a_v} |x|^{t_v}, \quad a_v \in \mathbb{Z}, \ t_v \in \mathbb{C}$$

Idea of pf $\mathbb{R}^\times = \mathbb{R}_+ \times \{\pm 1\}$ & $\mathbb{C}^\times = S^1 \times \mathbb{R}_+$. □

Def: χ is of type A if $t_v \in \mathbb{Q}$ $\forall v|\infty$

& of type A_0 if $\left\{ \begin{array}{l} t_v \in \mathbb{Z} \text{ if } v \text{ real} \\ t_v \in 2\mathbb{Z} \text{ if } v \text{ complex} \end{array} \right\} \forall v|\infty$.

For $v|\infty$ we have $\sigma_v : K \hookrightarrow K_v$.

For $\alpha \in K_f$, (3.2) becomes

$$\varphi((\alpha))^{-1} = \prod_{v|\infty} \chi_v(\sigma_v(\alpha)) = \prod_{v|\infty} \sigma_v(\alpha)^{a_v} |\sigma_v(\alpha)|^{t_v}$$

So if χ has type A_0, we have $|\sigma_v(\alpha)|^2 = \sigma_v(\alpha)\, \rho\circ\sigma_v(\alpha)$, v complex
$|\sigma_v(\alpha)| = \sigma_v(\alpha)$ v real

(3.4) So we obtain $\varphi((\alpha)) = \prod_{\sigma\in\Gamma} \sigma(\alpha)^{n_\sigma}$, where $\Gamma = \underset{\mathbb{Q}\text{-alg}}{\mathrm{Hom}}(K, \mathbb{C})$ ie field embeddings

& n_σ are integers.

We still don't have a feel as to why type A & type A_0 are of importance. It's to do with algebraicity!

Algebraicity of values.

Set E = compositum of σK, & P_f = ~~prime~~ principal ideals with a K_f-generator

Prop 3.5 a) If χ is of type A, then for $\alpha \in I_s$, we have $\varphi(\alpha)$ is alg $/\mathbb{Q}$.
b) If χ is of type A_0, then $\mathbb{Q}(\varphi(I_s))$ is a number field.

Pf Similar ideas do a) & b). Here's b). From (3.4) we have $\varphi(P_f) \subseteq E$.
But $(I_f : P_f) < \infty$. □

Lecture 4
Wed 17th Feb '93
2:30 pm

It's grossencharacters part 2 today.

The story so far:

$$\begin{tikzcd} \chi : J/K^\times \longrightarrow \mathbb{C}^\times \arrow[d, leftrightarrow] & ;\ f = f(\chi). & S = \mathrm{supp}(f) \cup \Sigma_\infty \\ (\varphi, S) \text{ admissible.} \arrow[urr, bend right] & \varphi : I_S \to \mathbb{C}^\times & \end{tikzcd}$$

We had a_v & t_v for $v | \infty$. χ could be of type A or A_0.

If $\alpha \in K_f$ then $\varphi((\alpha)) = \prod_{\sigma \in \Gamma} \sigma(\alpha)^{n_\sigma}$. Set $T = \sum n_\sigma \sigma \in \mathbb{Z}\Gamma$
T is the type.

Now we'll do

Purity of values

Clearly T can't be any old elt of $\mathbb{Z}\Gamma$. e.g. $\varphi((\alpha))$ only depends on (α), not α, so we immediately see that T must annihilate $Y_f = \{ x \in O^\times \mid x \gg 0 \ \& \ x \equiv 1 \bmod f \}$

Note $(O^\times : Y_f) < \infty$

Prop^n 3.6 If T is a type then $\forall \sigma \in \Gamma$ $n_\sigma + n_{\rho \cdot \sigma} = w$, a constant, indpt of σ.

Proof kit is all the bits you need to assemble a proof.

Define $\lambda' : K^\times \longrightarrow \mathrm{Map}(\Gamma, \mathbb{R})$
$(\lambda'(x))(\sigma) = \log(|\sigma(x)|)$ Note $\lambda'(x)(\rho \cdot \sigma) = \lambda'(x)(\sigma)$.

Set $\lambda = \lambda'|_{O^\times}$

$\mathrm{Ker}\,\lambda = \mu_K$, & $\lambda(O^\times)$ is a lattice in the vector space

$$H = \{ \text{maps } f \mid f(\rho \cdot \sigma) = f(\sigma),\ \sum_\sigma f(\sigma) = 0 \}$$

Define $\langle , \rangle : \mathrm{Map}(\Gamma, \mathbb{R}) \times \mathbb{Z}\Gamma \to \mathbb{R}$ in the obvious way.

$$\{1\} = (O^\times / \underset{\llcorner \text{roots of unity}}{\mu})^T \iff \langle \lambda(O^\times), T \rangle = 0 \iff \langle H, T \rangle = 0$$
$$\iff \sum f(\sigma) n_\sigma = 0 \quad \forall f \in H$$

$u \in O^\times$
get $\prod \sigma(u)^{n_\sigma} \in \mu_K$

That's all we need. □

Defⁿ An alg no. α ($\in M$ say) is called pure if $|\alpha|_u$ is cst $\forall u | \infty$.

Propⁿ 3.7 The values of φ (of type A_0) (NB this is abuse of notation: χ_f is A_0 & $\chi \rightsquigarrow (\varphi, S)$) are pure, &

$$|\varphi(\mathfrak{a})|_u = N\mathfrak{a}^{w/2} \quad \forall (\mathfrak{a}, f) = 1.$$ Moreover, the number field $\mathbb{Q}(\varphi(I_S))$ is either $\mathbb{Q}$ or a CM field.

Pf $(I_S : P_f) < \infty$. So ~~for~~ it suffices to prove the result for $\mathfrak{a} = \alpha\mathcal{O}$, $\alpha \in K_f$

$$\varphi(\mathfrak{a})\varphi(\mathfrak{a})^{\rho} = \alpha^{T + \rho T}$$

$$= \alpha^{w \cdot \Sigma \sigma} = N(\alpha)^w = N\mathfrak{a}^w$$

(up to sign, but $\alpha \gg 0$)

$\square$

That's the end of grossencharacters. He wants to talk about ℓ-adic repⁿs and Weil-Deligne repⁿs, & that'll be about it.

§4 ℓ-adic repⁿs

K a number field or local field. $G_K = G(K^c/K)$, ℓ a prime no., E a number field, λ a place of E, $\lambda | \ell$.

Defⁿ A λ-adic repⁿ is a cts HM $\rho : G_K \to GL(V)$, V = f.d. E_λ-v.s.

We know (1) $\operatorname{Im}\rho$ is cpct & closed.

(2) Say ρ is abelian if $\operatorname{Im}\rho$ is abelian. Then ρ factors through G_K^{ab}

(3) ρ is semisimple or ss if V is ss as a G_K-module, i.e. all G_K-submodules have complements.

(4) If $E = \mathbb{Q}$ & $\lambda = \ell$ then ρ is called an ℓ-adic repⁿ

Examples (1) $T_\ell(\mu) = \varprojlim \mu_{\ell^n}$, a $\mathbb{Z}_\ell$-module.

$V_\ell(\mu) = T_\ell(\mu) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. This affords $\kappa_\ell : G_K \to \mathbb{Q}_\ell^\times$, an abelian repⁿ.

(2) Because we've used E we'll let $\mathcal{E}$, fracteur E, be an elliptic curve$/K$. Set $\mathcal{E}_{\ell^n} = \operatorname{Ker}(\ell^n : \mathcal{E} \to \mathcal{E})$

$$T_\ell(\mathcal{E}) = \varprojlim \mathcal{E}_{\ell^n}$$

$$V_\ell(\mathcal{E}) = T_\ell(\mathcal{E}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$ a 2-dimˡ repⁿ.

I think he said that if $\mathcal{E}$ had CM then the repⁿ is abelian.

(note ker ρ may not be open so it's a bit tricky)

Def^n^ A λ-adic rep^n^ $\rho: G_K \to GL(V)$ is called unramified at $v \in \Sigma_{K,f}$ if

$$\rho(I_u(K^c/K)) = 1 \quad \forall u \in \Sigma_{K^c} \text{ over } v$$

or just 1 u because they're all conjugate.

In this case, let w be a place of $(K^c)^{\ker \rho}$ above v & define the char poly

$$P_{v,\rho}(T) := \det(1 - \rho(F_w)T)$$

Frobenius at w. It's indpt of w|v.

John notes that traditionally this def^n^ is uses the geometric Frobenius, but Martin will stick with his arithmetic one.

Def^n^ Let K be a number field. Then ρ is called rational (over E) if $\exists$ a finite set $S \subseteq \Sigma_K$ s.t.

(1) ρ is unramified outside S

(2) $\forall v \notin S$ we have $P_{v,\rho} \in E[T]$.

(pertaining to examples above)

Exercises (1) $S = \{v \in \Sigma_K \mid v \| \infty\}$ - ρ rational / $\mathbb{Q}$

(2) $S = \{v \in \Sigma_K \mid v\|\infty, \text{bad primes}\}$ - ρ is rational / $\mathbb{Q}$. (Weil)

Compatability Suppose $\rho_\lambda, \rho'_{\lambda'}$ are λ-adic & λ'-adic reps respectively with $\lambda \neq \lambda'$.

We say ρ_λ & $\rho'_{\lambda'}$ are compatible if $\exists$ finite set $S \subseteq \Sigma$ containing all places where either rep^n^ ramifies & s.t. $P_{v,\rho_\lambda}(T) = P_{v,\rho'_{\lambda'}}(T)$ $\forall v \notin S$.

A system $\{\rho_\lambda\}_{\lambda \in \Lambda}$ is called compatible if its elts are pairwise compatible.

We call this system strictly compatible if $\exists$ a finite set S s.t.

(1) $P_{v,\rho_\lambda} \in E[T]$ $\forall v \in S \cup S_\lambda$ (a rationality condition)

(2) Each pair λ, λ' has $P_{v,\rho_\lambda} = P_{v,\rho_{\lambda'}}$ $\forall v \notin (S \cup S_\lambda \cup S_{\lambda'})$

Note that in the 'compatible' def^n^, S depended on λ, λ'. Errors may clock up in the compatible case. In the strictly compatible case $\exists$ finite set S which deals with the lot. A minimal such S is called the exceptional set for $\{\rho_\lambda\}$ but he won't be using this.

A new gadget coming up.

Locally algebraic rep^ns

Now $K/\mathbb{Q}_p$. Consider T, the torus restricting $\mathbb{G}_m/K$ to $\mathbb{Q}_p$

$$T = \mathrm{Res}_K^{\mathbb{Q}_p}(\mathbb{G}_m)$$

For $L/\mathbb{Q}_p$ ~~define~~ we have $T(L) = (K \otimes_{\mathbb{Q}_p} L)^\times$

Def^n $\rho : G_K^{ab} \to GL(V)$ is a p-adic rep^n, we call ρ locally algebraic if $\exists$ algebraic HM (= HM of alg gps)

$$r : T \to GL(V)$$

s.t. $\rho \circ \theta(x) = r(x^{-1})$

$\forall x \in$ some nhd of 1 in $K^\times$.

Here θ is the local reciprocity map.

$$\begin{tikzcd} K^\times \arrow[r, "\theta"] \arrow[dr, dashed, "? \, r^{-1}"'] & G_K^{ab} \arrow[d, "\rho"] \\ & GL(V) \end{tikzcd}$$

(Here $r^{-1}(x) = r(x^{-1})$)

Note that if r exists it's unique because the nhd of 1 is ~~a~~ Zariski-dense in $K^\times$ or something.

Ex(3) If $\mathcal{F}$ is a Lubin-Tate formal group associated to unif. par π of K

$$T(\mathcal{F}) = \varprojlim \mathrm{Ker}[\pi^n] \quad , \quad V(\mathcal{F}) = T(\mathcal{F}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$\uparrow$ This has $\mathbb{Q}_p$-dimension $(K:\mathbb{Q}_p)$.

$\forall u \in \mathcal{O}^\times, \quad \theta(u)v = [u^{-1}]v$

$r : K^\times \to GL(V)$ defined by K-module structure of V

& I guess the point is that ρ is locally algebraic in this case, although I'm quite lost here.

Prop^n 4.1 ρ locally algebraic $\Rightarrow \rho|_{I = I(K^{ab}/K)}$ is ss.

Sketch pf (1) Any open nhd of $K^\times$ is Zariski dense in $K^\times$

(2) Any rep^n of torus is ss

Identify $I = O^\times$ by θ.

Have $U \underset{\text{open}}{\subseteq} O^\times$, s.t. $\rho(x) = r(x^{-1})$ $\forall x \in U$.

Let W denote any $\rho(O^\times)$-submodule of V. We need to find a complement.

W is $r(U)$-stable
W is $r(K^\times)$ stable by (1)

So by (2) $W \underset{\pi}{\overset{\hookrightarrow}{\longleftarrow}} V$ where π respects the $r(K^\times)$-action & so it clearly respects the $r(U)$ action.

So π respects the $\rho(U)$ action, as $\rho(x) = r(x^{-1})$ $\forall x \in U$.

We're trying to respect the $\rho(I)$ action but this is easy now, we just use the standard averaging trick:

$$\text{set } \pi' = \frac{1}{(O^\times : U)} \sum_{s \in O^\times/U} \rho(s)\pi\rho(s)^{-1} \qquad \square$$

Lecture 5
Thu 18th Feb '93
2:30pm

In the last lecture we talked about locally algebraic rep's.

$\rho : G_K^{ab} \to GL(V)$ was locally algebraic if

$$\begin{tikzcd} T(\mathbb{Q}_p) = K^\times \supseteq U \arrow[r, "\theta"] \arrow[dr, "r^{-1}"'] & G_K^{ab} \arrow[d, "\rho"] \\ & GL(V) \end{tikzcd}$$

for U some small nhd of 1.

How do you recognise when a rep^n is locally algebraic? We'll come to that.

Anyway, we must talk a little about characters.

Observe $T(\overline{\mathbb{Q}}_p) = (K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p)^\times$. Now $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \cong \prod_{\sigma \in \Gamma} \overline{\mathbb{Q}}_p$ where $[\sigma](x \otimes y) = x^\sigma y$

(Here σ is acting on x, I guess)

Now any character $\chi = \prod_{\sigma} [\sigma]^{n_\sigma}$, $n_\sigma \in \mathbb{Z}$.

Say $\rho : G_K \to GL(V)$ is locally algebraic.

By (4.1), $\rho|_I$ is s.s. By abuse of notation $\rho|_I = \rho|_{\mathcal{O}^\times}$

Hence $\rho|_U$ is still semisimple & algebraic (i.e. $= r^{-1}$) & abelian.

So extending from $GL(V)$ to $GL(V \otimes E)$ for some suitable finite extn $E/\mathbb{Q}_p$ we can diagonalise $\rho|_U$ & hence $\rho|_U = \Sigma \chi_i$, χ_i 1-diml

Here the χ_i are algebraic, the χ_i "come from" the torus T.

<u>Propn 4.2</u> ρ is locally algebraic $\Leftrightarrow$ $\rho|_U = \Sigma \chi_i$, U a nhd of 1,
where $\chi_i(u) = \prod_{\sigma} \sigma(u)^{n_i(\sigma)}$ $\square$

He hopes it's a bit more comprehensible, now.

The main business of the day is now coming up:

<u>l-adic repns & GCs</u>

E, K number fields, ρ a λ-adic repn G_K on V.

CFT
$$\theta : \begin{tikzcd} C_K \arrow[r, two heads] \arrow[dr] & G_K^{ab} \arrow[d, "\rho"] \\ & GL(V) \end{tikzcd}, \quad \ker\theta = C_K^\circ$$

Also, given $C_K \to GL(V)$, it must factor thru G_K^{ab} for some trivial reason, I think he said.
It's because GL(V) is totally disconnected.

This is all fine if $v < \infty$.

If $v | \infty$, E_v, V/E_v a v.s. GL(V) is not totally disconnected.

So we have $\rho : C_K \to \mathbb{C}^\times$
$\chi : G_K^{ab} \to \mathbb{C}^\times$ & $\chi \rightsquigarrow \rho$ but not the other way.

That was the preamble. Here's the business

Let $\chi : J/K^\times \to \mathbb{C}^\times$ be a GC of type A_0.

A_0 $\mathfrak{f} = \mathfrak{f}(\chi)$.

Set $T = \mathrm{Supp}(\mathfrak{f}) \cup \Sigma_\infty$

We get (φ, T) associated to χ by 2.5 as usual.

$\varphi : I_T \to E^\times$. Say λ is a prime of E.

Say $\lambda | \ell$. Set $S = T \cup \{v | \ell\}$

Choose $h : E \to E_\lambda$

Then $\varphi_\lambda : I_S \xrightarrow{\varphi} E^\times \xrightarrow{h} E_\lambda^\times$

We want to pull φ_λ back to a map on C_K, or something.

$$\begin{tikzcd} C_K \arrow[r, phantom, "\underset{\text{dense}}{\supseteq}"] & J^S K^\times / K^\times \arrow[r] \arrow[rrr, bend right, "\chi'"'] & J^S \arrow[r, "()"'] & I_S \arrow[r, "\varphi_\lambda"'] & E_\lambda^\times \end{tikzcd}$$

<u>Propⁿ 4.3</u> χ' extends to a λ-adic repⁿ $\chi_\lambda : C_K \to E_\lambda^\times$ which is locally algebraic.

(Note that because E_λ is p-adic, we get $\chi_\lambda \leftrightarrow \rho_\lambda : G_K^{ab} \to E_\lambda^\times$)

<u>Sketch proof</u> (Sutherland p158 for topology bit!)

Given a small nhd Y of 1 in $E_\lambda^\times$ need to show $\exists X \underset{\text{open}}{\subseteq} J^S K^\times / K^\times$ with $\chi'(X) \subseteq Y$.

If $n \gg 0$, $\alpha \in K_{\mathfrak{f}^n \ell^n}$, then

$$\varphi_\lambda((\alpha)) = h(\varphi((\alpha))) \underset{(3.4)}{=} h(\alpha^T)$$ where here T is the type of χ.

But $h(\alpha^T)$ is λ-adically small so by making n large we get $\chi'(X) \subseteq Y$. This is the basic observation that makes everything work.

So $\underset{\text{open}}{X} \rightarrowtail P_{\mathfrak{f}^n \ell^n} \to Y$ small nhd, & now we have to show χ_λ is locally algebraic.

Now pick $u \in \mathcal{O}_{K_\ell}^{\times} \hookrightarrow J$. Here K_ℓ is the semilocalisation of K at ℓ.

Pursue the def^ns & find that $\chi_\lambda(u) = u^{h \circ T}$.

The def^n of locally algebraic, & (4.2), shows χ_λ to be locally algebraic.

Remarks (1) χ_λ vanishes on C_K°. So $\operatorname{Im} \chi_\lambda$ must be cpct, as C_K / C_K° is.

$$\text{Hence } \operatorname{Im} \chi_\lambda \subseteq \mathcal{O}_{E_\lambda}^{\times}.$$

(2) (exercise) (although like Richard Taylor he'll give us the key hint)

The system $\{\chi_\lambda\}_{\lambda \in \Sigma_f}$, varying λ, gives us a family of strictly compatible rep^ns

Key point: $v \notin S$, π_v a uniformising parameter for v.

Then $\chi_\lambda(\pi_v) = h \circ \varphi(p_v)$ indpt of λ.

There's a thing that Martin calls "functorality" although Richard appears to refer to it as "Base Change for GL_1". He was going to talk about it tomorrow, but as he's so ahead of time he'll start now.

Functoriality

Keeping previous notation, $\chi_\lambda : C_K \to E_\lambda^\times$ has kernel $\supseteq C_K^\circ$

ie $\chi_\lambda = \rho_\lambda \circ \theta_K$, $\rho_\lambda : G_K \to E_\lambda^\times$.

Prop^n 4.4 Suppose now N/K is a finite extension. Then $\rho_\lambda |_{G_N}$ comes from

$$\chi \circ N_{N/K} : J_N \to \mathbb{C}^\times$$

Pf

$$\begin{tikzcd} G_N^{ab} \arrow[r, "inc^{ab}"] & G_K^{ab} \end{tikzcd}$$

$$\overset{\in}{h} \longmapsto g = inc^{ab}(h)$$

$$J_N^{S'} N^\times \underset{\text{dense}}{\subseteq} J_N$$

$h = \theta_N(y)$. Take $y \in J_N^{S'}$, S' = places of N above S.

$$g = inc^{ab} h = inc^{ab}\, \theta_N(y) = \theta_K \circ N_{N/K}\, y$$

$$\therefore \rho_\lambda(g) = \rho_\lambda \circ \theta_K \circ N_{N/K}\, y = \chi_\lambda \circ N_{N/K}\, y$$

$$\rho_\lambda(g) = \rho_\lambda(inc^{ab} h) = \rho_\lambda|_{G_N}(h) = \rho_\lambda|_{G_N}(\theta_N(y)) \qquad \square$$

He'll stop now, as he ran over last time

Lecture 6
Fri 19th Feb '93
2:30 pm

Whilst this isn't his swan-song, as John was reminding him, we are now on the final lap.

§5 Weil-Deligne rep^n's

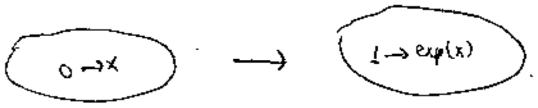
$K/\mathbb{Q}_p$ a local field.

Prop^n 5.1 Let ρ be an irred cts rep^n, $\rho : G_K \to GL_2(\mathbb{C})$. If $p>2$, then ρ is monomial.

Rk In fact we'll do considerably more - we'll get a handle on how which char induce ρ.

Note: $GL_n(\mathbb{C})$ does not have arbitrarily small subgps. This is because of a gadget that Tony mentioned: exp.

$$\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$$



small nhds

$\exp(nX) = (\exp X)^n$ & nX bursts out of its small nhd, so $\exp(nX)$ bursts out.

□ of note. Proof by pictures.

Pf of prop^n Replace G_K by $G_K / \ker\rho = G = \mathrm{Gal}(N/K)$

$$\rho : G \to GL_2(\mathbb{C}), \qquad \underbrace{1 \subseteq P}_{p\text{-gp}} \underbrace{\subseteq \overset{\text{inertia}}{I}}_{\text{cyclic}} \underbrace{\subseteq G}_{\text{cyclic}}$$

$\rho|_P$ = sum of 2 abelian chars, as $2 \nmid p$.

I is abelian-by-cyclic (this is what they seem to call it in Manchester anyway - sometimes it's cyclic-by-abelian!)

(*) { ∴ all irred reps of I are monomial, induced from ab. chars of some subgp $\supseteq P$. See eg Serre's bk on rep theory 8.2.

Case 1 $\rho|_I$ is reducible Then $\rho|_I = \chi_2 + \tilde{\chi}_2$

(G acts on chars of I, $\chi \to \chi^g$) Say $\Delta_2 = \mathrm{Stab}_G(\chi_2)$. Δ_2 / I is cyclic

Thus χ_2 extends to $\{\chi_2'\}$

By Frob. rec. ρ must occur in $\mathrm{Ind}_{\Delta_2}^{G} \chi_2'$ which is irred by Mackey's criterion.

Hence $\rho = \mathrm{Ind}_{\Delta_2}^{G} \chi_2'$.

Case 2 $\rho|_I$ is irred. Apply ⊛. Then $\rho|_I = \mathrm{Ind}_{I^2}^{I} \chi_3$.

Here $(I^\bullet : I^2) = 2$, $I^2 \geq P$ (I^2 isn't $I\times I$, its just notation)

Set $\Delta_3 = \mathrm{Stab}_G(\chi_3)$. Then he claims Δ_3 / I^2 is cyclic.

$$\Delta_3 I / I = \Delta_3 / \Delta_3 \cap I = \Delta_3 / I^2$$

Apply extⁿ. get ext's $\{\chi_3'\}$ of χ_3 to Δ_3

The endgame's the same: ρ occurs in some $\mathrm{Ind}_{\Delta_3}^{G} \chi_3'$ which is irred by Mackey.

□

The Weil gp

$$R : G_K \to G(K^{nr}/K) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}} \supset \mathbb{Z}$$

Ker $R = I$. (NB I will never be some ideal thing like I^5, as we're always local for this §)

Def: The Weil gp of K, $W_K = R^{-1}(\mathbb{Z})$, topologised by declaring I open

(I has a profinite topology)

$W_K \underset{\text{dense}}{\subset} G_K$. Pick $\Phi \in W_K$ s.t. $R(\Phi) = 1$

Then

$$\begin{tikzcd}
1 \arrow[r] & I(K^{ab}/K) \arrow[r] & G_K^{ab} \arrow[r, "R"] & \hat{\mathbb{Z}} \arrow[r] & 0 \\
1 \arrow[r] & \mathcal{O}^\times \arrow[r] \arrow[u, "\wr"] & K^\times \arrow[r, "v"] \arrow[u, "\theta"] & \mathbb{Z} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Easily check that $W_K^{ab} = \theta(K^\times)$.

Def: By a rep of W_K, we mean a cts HM $\rho : W_K \to GL_n(\mathbb{C})$

If $n=1$, we get $\chi : W_K \to \mathbb{C}^\times$. Call χ a character. (NB he usually calls it a quasicharacter but Tony says character)

Call χ unramified if $\chi(I) = 1$

Note that a cts HM $G_K \to \mathbb{C}^\times$ must have finite image, as $\mathbb{C}^\times$ has no small subgps. However, W_K chars are more exotic.

Example Fix $s \in \mathbb{C}$. Then ω_s is the unramified char.

$$\omega_s : W_K \to W_K^{ab} \xrightarrow{\theta^{-1}} K^\times \xrightarrow{|\cdot|^s} \mathbb{C}^\times$$

Set $\omega_{-1} = \|\cdot\|$. (This is a def^n)

~~Gatt s~~ Note this is already lots of ~~reps~~ chars of W_K. Lots more than there are ~~reps~~ chars of G_K.

Call a rep^n ρ of W_K of Galois type if it is the restriction of a rep^n of G_K.

ρ is of Galois type iff $\rho(W_K)$ is finite.

Prop^n 5.2 An irred cts rep^n $\rho : W_K \to GL_n(\mathbb{C})$ can be written in the form $\sigma \otimes \omega_s$ where σ is of Galois type.

Pf Use the fact that W_K ~~is an extension of~~ $\supseteq$ I (profinite)

Firstly, factor out $\ker\rho \cap I$

$$\begin{array}{ccccccc} 1 \to & I/_{\ker\rho\cap I} & \to & W/_{\ker\rho\cap I} & \to & \mathbb{Z} \to 0 \\ & & & \Phi^n & \longmapsto & n \end{array}$$

We have conj : $\mathbb{Z} \to \mathrm{Aut}(I/_{\ker\rho\cap I})$: has kernel $n\mathbb{Z}$

So $\rho(\Phi^n)$ central $=_{\text{Schur}} \mathrm{diag}(\lambda)$

Choose $s \in \mathbb{C}$ s.t. $\omega_s(\Phi^n) = \lambda$

Check that $\rho \otimes \omega_s^{-1}$ has finite image & is hence Galois. $\square$

So we've had Galois gps & Galois rep^{n}s ; Weil gps & Weil rep^{n}s.

Lastly, Weil-Deligne gps & their rep^{n}s.

Weil-Deligne gps

The bad news - they're not gps but gp schemes / defined /$\mathbb{Q}$. Fortunately after an initial foray we'll just be reduced to looking at the pts.

If G is a finite gp, the constant gp scheme $\underset{\sim}{G}$ is Spec(Map(G, $\mathbb{Q}$))

Have to be careful, as W_K isn't finite.

The constant group scheme $\underline{\underline{\underset{\sim}{W_K}}}$ over $\mathbb{Q}$ is $\mathrm{Spec}\,(\mathrm{Map}_{\mathrm{loc\,cst}}(W_K, \mathbb{Q}))$

Locally cst f has $f^{-1}(r)$ closed & open $\forall r \in \mathbb{Q}$.
I think this is a defn of loc. cst.

<u>Defn</u> The <u>Weil-Deligne gp</u> is the $\mathbb{Q}$-group scheme

$$WD_K = \mathbb{G}_a/_{\mathbb{Q}} \rtimes \underset{\sim}{W_K}, \text{ with action } \|\cdot\|.$$

$$WD_K(E) = \mathbb{G}_a(E) \rtimes W_K$$

of course $\|w_1\| \in q^{\mathbb{Z}}$ so this makes sense.

$$(a_1, w_1)(a_2, w_2) = (a_1 + \|w_1\| a_2, w_1 w_2)$$

<u>Defn</u> Let V be a f.d. E-v.s. Then a repn of WD_K/E is a HM of group schemes

$$\rho' : WD_K \times_{\mathbb{Q}} E \to GL_E(V)$$

(User-friendly translation:)

(a) (Restrict to W_K) $\rho'' : \underset{\sim}{W_K} \to GL_E(V)$

Ker ρ'' (as pts) is open in W_K.

We have fns x_{ij} on $GL_E(V)$. Then $(\rho''^{*})(x_{ij})$ is a fn on $\underset{\sim}{W_K}$

$((\rho''^{*})(x_{ij}))^{-1}(\delta_{ij})$ are all open, so the kernel is open.

(b) (much easier) (restrict to additive gp) $N \in \mathrm{End}_E(V)$ with

$$\circledast \quad \rho''(w) N \rho''(w)^{-1} = \|w\| N \quad \forall w \in W_K$$

Here's how you get N: ρ' gives an alg HM $\underset{\overset{\in}{e}}{\mathbb{G}_a(E)} \to GL_E(V)$

Such are of the form $\rho'(e) = \exp(eN)$. <u>$\exp(eN)$ is algebraic as N is nilpotent.</u>

This gives us N. We now use the composition law above. ~~to get~~

By $\circledast$, all eigenvalues of N are stable under mult by $q^{\mathbb{Z}}$
$\Rightarrow$ 0 is only eigenvalue
$\Rightarrow$ N nilpotent.

What's going on up here in these rather badly-taken notes is that given this complicated thing ρ' we pull out a repn ρ'' of W_K with open kernel & a nilpotent matrix N satisfying $\circledast$. So this gives us a better handle on ρ'.

In fact:

$$\rho' \longleftrightarrow (\overbrace{\underset{\substack{\uparrow\\ \text{open}\\ \text{kernel}\\ \text{rep}^n}}{\rho''}, \underset{\nwarrow\ \text{nilpt}}{N}}^{\text{satisfying } \circledast})$$

This is a bijection because given ρ'' & N we see can define

$$\rho'(a;w) = \exp(aN)\,\rho''(w)$$

Lecture 7
Sat 20th Feb '93
9:30am

He'll start this lecture with a brief recap.

$R: G_K \to \hat{\mathbb{Z}}$; $W_K = R^{-1}(\mathbb{Z})$; $WD_K = \mathbb{G}_a \rtimes W_K$.

For E char 0, a rep^n ρ': ~~WK~~ $WD_K \underset{\mathbb{Q}}{\times} E \to GL_E(V)$.

$\rho' \leftrightarrow (\rho'', N)$, $\rho'': W_K \to GL_E(V)$ open kernel
N nilpotent

$$\text{s.t.} \quad \circledast \ \rho''(w)\,N\,\rho''(w)^{-1} = \|w\|\,N$$

$$\text{where } \|\Phi\| = (O_K : \mathfrak{p}_K) \ \& \ (\ R(\Phi) = 1\ )$$

The reason we have ρ' & ρ'' is that there's a ρ coming.

<u>Def^n</u> Call ρ' <u>ss</u> iff ρ'' is ss

<u>Rk (1)</u> ρ'' is ss iff $\rho''(\Phi)$ is, because $\rho''(I)$ is finite, & so

$$(\operatorname{Im}\rho'' : \langle \rho''(\Phi) \rangle) < \infty :$$

So use the fact that in char 0 a rep^n is ss $\Leftrightarrow$ it is ss on subgps of finite index.

(2) Ker N is W_K-stable by $(*)$ - in fact it's WD_K-stable.

$\rho' = (\rho'', N)$ is irreducible $\Rightarrow$ $N=0$ (as $0 \neq \ker N$ is stable) & ρ'' is irred.
ie rep^n is lifted from Weil gp

Clearly $\Leftarrow$ is true too. So ρ' irred $\Leftrightarrow$ $N=0$ & ρ'' irred.

We've kicked the def^n around but haven't seen many examples yet.
We will get a handle on indecomposable reps of WD_K.

Example Sp(n), the special rep^n, is the rep^n (ρ'', N) of WD_K over $\mathbb{Q}$

$V = \mathbb{Q}e_0 \oplus \dots \oplus \mathbb{Q}e_{n-1}$

$\rho''(w)e_i = \omega_i(w)e_i$ (ss action)

(Recall $\omega_s : W_K \to W_K^{ab} \xrightarrow{\theta^{-1}} K^\times \xrightarrow{\|\cdot\|^s} \mathbb{C}^\times$

& $Ne_i = e_{i+1}$, $i < n-1$
$Ne_{n-1} = 0$

Sp(n) is semisimple & indecomposable.

Prop^n 5.3 Every ss indec rep^n of WD_K is of the form $\sigma' \otimes sp(n)$, where σ' is ~~indecomposable~~ irreducible. $\square$ (Deligne Antwerp for pf)

[Note: Here $\rho' \otimes \sigma' = (\rho'', N) \otimes (\sigma'', M) = (\rho'' \otimes \sigma'', N \otimes 1 + 1 \otimes M)$]

Conjecture (Langlands) There is a "natural" bijection between IM classes of dimension n ss WD_K reps & of $\underset{\text{irred ucible}}{}$ admissible reps of $GL_n(K)$
↑ Tony Scholl course def^n
not Tate def^n.

There's also L-series ~~an~~ & ε-factors which match up. ("natural" eg irred ↔ supercuspidal etc)

n = 1 is "just" classfield theory. ($n = \overset{1}{\cancel{0}} \Rightarrow N = 0 \Rightarrow$ rep of $W_K \overset{\theta}{\leftrightarrow}$ rep of $K^\times = GL_1(K)$

l-adic rep^{n}s of W_K

$K/\mathbb{Q}_p$; we also have $\overset{\text{residue field}}{k}$ and $q = |k|$

k_n is the unique ext^n of k of degree n (in k^c)
K_n is the non-ramified ext^n of K of degree n (in K^c)

(see eg $\underset{\text{assels}}{\text{C}}$&$\underset{\text{röhlich}}{\text{F}}$ p29) T_n is the max^l ab. totally tamely ramified ext^n of K_n (in K^c)

$$G(T_n/K_n) \cong k_n^\times \text{, a } G_K\text{-IM.}$$

$$I_K/P_K = \varprojlim G(T_n/K_n) \cong \varprojlim k_n^\times = \prod_{l \neq p} \mathbb{Z}_l$$

$$\left(G(T_{nm}/K_{nm}) \to G(T_n K_{nm}/K_n) \cong G(T_n/K_n) \right)$$

Def^n $t_l : I_K \to \mathbb{Z}_l^+$ is the tame character associated to l
i.e. $I_K \twoheadrightarrow I_K/P_K \xrightarrow{\sim} \prod_{l \neq p} \mathbb{Z}_l \underset{\text{project}}{\twoheadrightarrow} \mathbb{Z}_l$

It's not canonical because $\varprojlim k_n^\times \cong \prod_{l \neq p} \mathbb{Z}_l$ isn't.

All chars $I_K \to \mathbb{Z}_\ell^+$ are multiples of t_ℓ.

<u>Note</u> (1) $w \in W_K$, $t_\ell(w\sigma w^{-1}) = w(t_\ell(\sigma)) = \|w\| t_\ell(\sigma)$

$$(\Phi(\xi) = \xi^q = \xi^{\|\Phi\|})$$

The Haupsatz of the day:

<u>Thm 5.4</u> (Grothendieck) ; if $\ell \neq p$, $E/\mathbb{Q}_\ell$, V a f.d. $\mathbb{Q}_\ell$-v.space, & a rep^n ρ $W_K \to GL(V)$.

Then $\exists$ nilpotent $N \in End(V)$ s.t. $\rho(\sigma) = exp(t_\ell(\sigma) N)$ for σ some open nhd of 1 in I_K s.t. ⊛ $\rho(w) N \rho(w)^{-1} = \|w\| N$.

<u>Sketch pf</u> I cpct $\Rightarrow \rho(I)$ cpct in $GL(V)$
$\Rightarrow \rho(I)$ stabilizes some $\mathbb{Z}_\ell$-lattice in V

$$\begin{tikzcd} \rho \cdot I \arrow[r] & GL_n(\mathbb{Z}_\ell) & \\ U \arrow[u, phantom, sloped, "\subset"] \arrow[r] & 1+\ell^2 M_n(\mathbb{Z}_\ell) \arrow[u, phantom, sloped, "\subset"] \arrow[r, shift left, "\log"] & \ell^2 M_n(\mathbb{Z}_\ell) \arrow[l, shift left, "exp"] \end{tikzcd}$$

& replacing K by a finite ext^n

get $\rho : I \longrightarrow 1+\ell^2 M_n(\mathbb{Z}_\ell)$

$$\begin{tikzcd} I_K \arrow[r] \arrow[dr, "\rho"'] & I/P \arrow[r, "t_\ell"] & \mathbb{Z}_\ell \arrow[dl, dashed, "\bar{\rho}"] \\ & 1+\ell^2 M_n(\mathbb{Z}_\ell) & \end{tikzcd}$$

Say $\bar{\rho}(1) = exp\ N$

Then e.g. $\bar{\rho}(2) = \bar{\rho}(1)\bar{\rho}(1) = exp(N)^2 = exp(2N)$

$$\bar{\rho}(z) = exp(zN)$$

So we're shown $\rho(\sigma) = exp(t_\ell(\sigma) N)$ We've not shown ⊛ holds yet.

this is $\forall \sigma \in$ (slightly schrunken) I

Now note

$\rho(w\sigma w^{-1}) = exp(t_\ell(w\sigma w^{-1}) N) = exp(\|w\| t_\ell(\sigma) N)$

$\shortparallel$

$\rho(w)\rho(\sigma)\rho(w)^{-1} = \rho(w) exp(t_\ell(\sigma) N) \rho(w)^{-1}$.

Both these are $\equiv 1 \bmod \ell^2$, so we can take logs.

$\rho(\sigma) t_\ell(\sigma) N \rho(w)^{-1} = t_\ell(\sigma)\|w\| N$. Cancelling σ $t_\ell(\sigma)$ gives ⊛

Finally we have to show N is nilpotent.

But the eigenvalues of N are stable under multiplication by $q^{\mathbb{Z}}$, by the dodge we used yesterday, so we're done. □

<u>Cor 1</u> If now ρ is ss, then $\ker\rho$ is open (in W_K)

<u>Pf</u> ρ is ss = $\oplus$ ireds. So Wlog we can take ρ irred.

Certainly $\ker N$ is W_K-stable by $\circledast$ $\rho(w)N\rho(w)^{-1} = \|w\|N$

N is nilpotent $\therefore$ $0 \neq \ker N$ $\therefore$ $\ker N$ = everything & $N=0$.

$\rho(\sigma) = \exp(t_\ell(\sigma)N)$ $\forall \sigma \in I$ by thm.

So $\rho(\sigma) = 1$ ie kernel is open
$\forall \sigma \in I$ (beefed up!)

<u>Cor 2</u> If ρ is ss & irred then it can be written $\sigma \otimes \chi$ where χ is a char (ie what he calls a quasi-char), σ is irred, of Galois type.
In ptic, if $p \neq 2, \ell$, & ρ is 2-dim^l, then ρ is monomial

<u>Pf</u> Use Cor 1 & now just apply (5.1) & (5.2) □

The last of our results is

<u>Thm 5.5</u> (Deligne) ($\ell \neq p$)

$\Phi \in W_K$, $R(\Phi) = 1$.

Then for $\sigma \in I$, $n \in \mathbb{Z}$, the equality

(✱✱) $\rho(\Phi^n\sigma) = \rho''(\Phi^n\sigma)\exp(t_\ell(\sigma)N)$ sets up a bijection between

$$\{\ell\text{-adic } \rho \text{ reps of } W_K\} \overset{\alpha}{\underset{\beta}{\rightleftarrows}} \{\text{reps of } WD_K \text{ over } \mathbb{Q}_\ell\}$$

<u>Explanation of α, β</u>

α: Given ρ, apply (5.4) & this provides a nilpotent N
(✱✱) now defines ρ'' - check it's a rep^n.
Then by (5.4), $\ker\rho''$ is open

β: Given WD_K-rep^n (ρ'', N), then (✱✱) defines ρ □

II. GL_2 over a local field

Tony Scholl

Lecture 1
Mon 15th Feb 03
11:30am

Recall that the lecture this afternoon is at 2:15
John Coates has asked him to ask us to ask questions.

He'll be talking about reps of $GL_2(\mathbb{Q}_p)$, $GL_2(\mathbb{R})$ & what these have to do with classical modular forms.

Classical theory: F.d. v.s. $S_k(\Gamma)$ with operators (T_n) etc which have a rel^n with the Fourier coeffts of $f \in S_k(\Gamma)$

Adelic theory: ∞-dim^l rep of $GL_2(\mathbb{A}_\mathbb{Q}) = \prod_p' GL_2(\mathbb{Q}_p) \times GL_2(\mathbb{R})$

with certain distinguished f.d. subspaces + algebra of operators also called Hecke operators although Hecke would turn in his grave if he knew.

explicit models (Kirillov) of local rep's

Why? Well, adelic setting gives us i) ability to split stuff from global to local (especially helpful in case $K \neq \mathbb{Q}$)
ii) understand ramification of primes a bit better.

The plan: I) GL_2/p-adic eg principal series. This is more difficult than $GL_2(\mathbb{R})$ but much more number-theoretic & less analytical.
II) $GL_2(\mathbb{R})$, $GL_2(\mathbb{C})$

III) p-adic Kirillov model + Atkin-Lehner theory

IV) L & ε-factors, local Langlands correspondence for GL_2.

References: Jacquet-Langlands, Lecture notes vol 114(?)
Godement - "Notes on J-L theory"
Articles by Deligne, Casselman in Antwerp vol II (Modular f's of 1 variable II)
Corvallis proceedings: eg Cartier (generalities about p-adic gps
& $GL_2(\mathbb{R})$, $\mathbb{C}$ is tough to find a reference. Try $SL_2(\mathbb{R})$ by Lang.

So off we go

I GL_2 / p-adic field

§1 F a finite ext^n of $\mathbb{Q}_p$, $\mathcal{O}$, (π) max^l ideal

$$|x| = q^{-v(x)} \text{ if } q = \#(\mathcal{O}/(\pi))$$

$v(x)$ normalised : $v(\pi^n u) = n$ if $u \in \mathcal{O}^\times$

$= \mathrm{ord}(x)$ (as v is a useful letter elsewhere)

$G = GL_2(F)$, a topological group

$g, h \in G$ are congruent modulo some high power of π

$\Leftrightarrow$ $gh^{-1} \equiv I$ mod some high power of π.

$\therefore$ topology on G is generated by the open subgps

$$K_n = \{ g \in GL_2(\mathcal{O}) \mid g \equiv I \bmod \pi^n \}$$

& their translates.

K_n is compact, as it's profinite

Prop^n Any cpct subgp of G is conjugate to a subgp of $GL_2(\mathcal{O})$. In particular, $GL_2(\mathcal{O})$ is max^l cpct. (surely this isn't immediate: need $m(g^{-1}Hg) = m(H)$?)
(Aliter, look at normaliser of $GL_2(\mathcal{O})$) (i.e. G unimodular)

Pf K a cpct subgp of G. Then K leaves fixed some lattice $\Lambda = \mathcal{O}e_1 \oplus \mathcal{O}e_2 \subseteq F^2$.

(take $\Lambda' = \mathcal{O}^2$; then $K.\Lambda'$ is cpct, so generates a lattice, invt by K)

So for (e_1, e_2) standard basis, get $K \subseteq GL_2(\mathcal{O})$ $\Box$

Rk: Every open nhd of $e \in G$ contains a cpct open subgp

Always K = some cpct open subgp of G. It's often not important which, since $K \cap K'$ is of finite index in K, K'.

Haar measure Normalise the measure so that $\mathrm{meas}(K) = \dfrac{1}{(GL_2(\mathcal{O}) : K)}$ if $K \subseteq GL_2(\mathcal{O})$

Extend to cosets gK by invariance.

We can extend this to the functional $m : C_c^\infty(G) \to \mathbb{C}$, $\varphi \longmapsto \int_G \varphi \, dg$

$$C_c^\infty(G) = \left\{ \begin{array}{c} \text{locally cst} \\ \text{f}^n\text{s on G of} \\ \text{cpt support} \end{array} \right\} = \left\{ \begin{array}{c} \text{finite linear combinations} \\ \text{of char. f}^n\text{s of gKs} \end{array} \right\}$$

s.t. $\int_G \mathrm{char}_K \, dg = \mathrm{meas}(K)$

and $\int_G \varphi(x^{-1}g)\, dg = \int_G \varphi(g)\, d(xg) = \int_G \varphi(g)\, dg$

G is unimodular so $d(g) = d(gx)$

If $H \subseteq G$ is an algebraic subgp - will need

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} \neq 0 & 0 \\ 0 & * \end{pmatrix} \right\}, \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} = NA$$

We can normalise the Haar measure on H s.t. $\mathrm{meas}(H \cap GL_2(\mathcal{O})) = 1$

Measures: $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ is dx (additive Haar measure on F)

$A = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right\}$: $d^\times a_1 \overset{\text{(times)}}{\cdot} d^\times a_2$ where $d^\times a$ is multiplicative Haar measure on F^* s.t. $\mathrm{meas}(\mathcal{O}_F^*) = 1$

$$\left(\text{So } d^\times a = \frac{da}{a} \left(1 - 1/q\right)^{-1} \right)$$

$B = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \right\}$. $\left| \frac{a_1}{a_2} \right|^{-1} d^\times a_1 d^\times a_2 dx$ is the measure

NB $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & xb_1 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & (b_1/b_2)x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$ so we're not unimodular.

(NB: this all works for f^ns with values in any field of char. zero.)

Prop^n 2 (Iwasawa decomposition). Let $K = GL_2(\mathcal{O})$

Then $G = NAK = KAN = KB$

Moreover, $\int_G \varphi(g)\, dg = \int_{F \times F^{*2} \times K} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k \right) \left| \frac{a_1}{a_2} \right|^{-1} dx\, d^\times a_1 d^\times a_2 dk$

where dk = restriction of $d_G g$ to K

Proof $G/B \cong \mathbb{P}^1(F)$ because G acts transitively on $\mathbb{P}^1(F)$ by fractional linear transformations, & B = stabiliser of ∞. Also, $GL_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(F)$. (NB the put K)
RH measure is left invt under B, rt invt under K, so is multiple of Haar measure.
Finally if φ = char f^n of $GL_2(\mathcal{O})$ then LHS = RHS = 1 $\square$

Propn 3 (Cartan decomposition) $K = GL_2(\mathcal{O})$

$$G = KAK = \coprod_{m \geq n} K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^n \end{pmatrix} K$$

Pf amounts to showing that if Λ, Λ' are lattices in F^2, then $\exists$ bases $\{e_i\}$ $\{e_i'\}$ s.t. $e_1' = \pi^m e_1$, $e_2' = \pi^n e_2$ for some m, n $\boxdot$

Propn 4 (Bruhat decomposition) If $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $G = B \amalg BwB = B \amalg NwB = B \amalg BwN$ where $BwB = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot c \neq 0 \}$ (any field F)

Pf N acting on $G/B = \mathbb{P}^1(F)$ has 2 orbits, $\{\infty\}$ & F. $\boxdot$

BwB is the "big cell".

Defn The Hecke algebra $\mathcal{H}(G)$ is the space $C_c^\infty(G)$ of locally cst f's of compact support on G, with convolution product

$$(\varphi_1 * \varphi_2)(x) = \int_G \varphi_1(y) \varphi_2(y^{-1}x) \, dy$$

- an associative algebra (NB we have now fixed the Haar measure, to normalise $*$, st. the measure of the maxl cpct is 1) It has no unit (would be δ_e)

§2. Representations of G

Let $\pi: G \to GL(V)$ be a homomorphism, V a $\mathbb{C}$-vector space.

⌈ NB V will almost always be ∞-diml. The thing is, even though V is ∞-diml there are some easy cases when V is sort-of like a 1-diml thing. However, there are also some very nasty reps of G. ⌋

Defn $v \in V$ is smooth if its stabilizer is an open subgp of G

This is quite a strong condn.

Say (π, V) is a smooth repn if every $v \in V$ is smooth.

Propn 5 (i) (π, V) is smooth $\iff$ $V = \bigcup_K V^K$, K running over all cpct open subgps of G.

(Here V^K = K-invts of V)

(ii) $v \in V$ is a smooth vector $\Rightarrow$ span $\{\pi(k)v \mid k \in K\}$ is f.d. (for any K) This is "v is K-finite"

Pf (i) is obvious
(ii) Let K' be an open cpct subgp, fixing v. We can assume $K' \subseteq K$, so $K = \bigcup g_i K'$, a finite union
Then $\{\pi(k)v \mid k \in K\} = \{\pi(g_i)v\}$ is a finite set. $\square$

Propn 6 Every smooth irreducible repn of G of finite dimension is of the form

$$g \mapsto \chi(\det(g)) \in \mathbb{C}^*$$

where $\chi : F^* \to \mathbb{C}^*$ is a cts HM

Pf Let (π, V) be a finite-dim' smooth repn of G. The kernel $H = \ker(\pi)$ is an open normal subgp of G (as it's $\cap$ of stabilizers of elts of a basis for V).
So $H \supset \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ for $|x| \ll 1$. So $H \supseteq \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, as all $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$s, $x \neq 0$, are conjugate in G. Also $H \supseteq \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$

$$\text{So } H \supseteq \left\langle \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right) \right\rangle = SL_2(F) \quad \text{(exercise)}$$

So π factors thro' det, so Im π is abelian. Hence if V is irred, V is 1-dim' $\square$

Example of ∞-dim' smooth repn

Let $V = \{$locally cst fns $\varphi : G \to \mathbb{C}$, invt on the left by $B\}$

$= \{$loc cst fns on $\mathbb{P}^1(F)\}$

We have $\varphi(bg) = \varphi(g)$ $\forall b \in B$ $\forall g \in G$. G acts via

$(\pi(g)\varphi)(x) = \varphi(xg)$. It's a smooth repn.

It's one of a large family of reps that we'll look at in lecture 4.

The only invt subspace is $\{$cst fns$\}$ (we'll prove this later)

& so the quotient is an ∞-dim' irred smooth repn of G

ecture 2
Mon 15th Feb '93
3:45pm

Recall this morning we looked at (π, V) smooth rep of $G = GL_2(F)$.

This will give rise to a certain repⁿ of an algebra & this repⁿ will often be f.d.

Say $\varphi \in \mathcal{H}(G) = C_c^\infty(G)$ under $*$ = convolution

Define $\pi(\varphi) : V \to V$ by $\pi(\varphi)v = \int_G \varphi(g)\pi(g)v\, dg$

$$= \pi(g)v \cdot \int_K dg \text{ if } \varphi = \text{char}_{gK} \ \& \ v \in V^K$$

Note $\pi(\varphi_1 * \varphi_2) = \pi(\varphi_1)\pi(\varphi_2)$

so we have a HM $\mathcal{H}(G) \xrightarrow{\pi} \text{End}(V)$

Now fix K. Define $e_K = \text{char}_K / \text{meas}(K)$

Note $e_K^2 = e_K$

$$\pi(e_K)v \in V^K \text{ as } \pi(k)\pi(e_K)v = \int_K \pi(k)\frac{\pi(k')v}{\text{meas}(K)}\, dk'$$

$$= (\text{change of variables})\ \pi(e_K)v$$

So $\pi(e_K) : V \twoheadrightarrow V^K$ is a projector

Remark: If $\varphi \in \mathcal{H}(G)$, then $\exists K$ s.t. $\varphi(gk) = \varphi(g)$ for all $k \in K$, & hence $\varphi * e_K = \varphi$.

Given the action of $\mathcal{H}(G)$ on V, we can recover the action of G thus:

if $v \in V$ then $v \in V^K$ for some K, & $\pi(g)v = \pi(g)\pi(e_K)v$

$$= \pi(\underbrace{\text{char}_{gK}}_{\in \mathcal{H}(G)})v \,/\, \text{meas}(K)$$

Thm 1. The above construction gives a bijection

$$\left\{\begin{array}{c}\text{smooth reps}\\ \text{of } G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{reps } \mathcal{H}(G) \to \text{End}\, V \text{ which}\\ \text{are } \underline{\text{non-degenerate}}, \text{ ie s.t. } \forall v \in V\\ \exists \varphi \in \mathcal{H}(G) \text{ with } \varphi v = v\end{array}\right\} \quad \square$$

This in fact gives an equivalence of categories.

Now define $\mathcal{H}(G,K) = e_K \mathcal{H}(G) e_K$

$\mathcal{H}(G,K)$ is a subalgebra of $\mathcal{H}(G)$, with e_K as unit

If $\varphi \in \mathcal{H}(G,K)$, $v \in V$, then $\pi(\varphi)v \in V^K$, so we get a HM

$$\mathcal{H}(G,K) \to \mathrm{End}(V^K).$$

<u>Thm 2</u> Let (π_i, V_i), $i = 1,2$, be smooth rep's of G. Assume:-

(i) V_1 is generated as $\mathcal{H}(G)$-module by V_1^K

(ii) Every G-inv't subspace of V_2 contains a non-zero vector fixed by K.

Then $\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}_{\mathcal{H}(G,K)}(V_1^K, V_2^K)$ ☐

<u>Cor</u> If (π_i, V_i) are <u>irreducible</u>, & $V_1^K, V_2^K \neq \{0\}$, then V_i^K are irred $\mathcal{H}(G,K)$-modules, & $V_1^K \underset{\mathcal{H}(G,K)}{\cong} V_2^K \iff V_1 \underset{G}{\cong} V_2$.

<u>Pf of thm</u> Let $F : V_1 \to V_2$ be a G-HM (NB he'll never mention the field F so there's no notational problem). Then $F|_{V_1^K} : V_1^K \to V_2^K$ is an $\mathcal{H}(G,K)$-HM.

Suppose now that $F : V_1^K \to V_2^K$ is an $\mathcal{H}(G,K)$-module HM. We want to extend F ~~to~~ to a G-HM $\tilde{F} : V_1 \to V_2$.

Since $V_1 = \pi_1(\mathcal{H}(G)) V_1^K$, we try to define

$$\tilde{F}\Big(\sum_j \pi_1(\varphi_j) v_j\Big) = \sum_j \pi_2(\varphi_j) F(v_j), \quad v_j \in V_1^K, \ \varphi_j \in \mathcal{H}(G)$$ (NB he put $\mathcal{H}(G,K)$)

We need to check that this def^n is unambiguous, & then we're done because $\tilde{F}$ is clearly unique.

So STP $\sum_j \pi_1(\varphi_j) v_j = 0 \Rightarrow \sum_j \pi_2(\varphi_j) F(v_j) = 0$.

But LHS $= 0 \Rightarrow \sum_j \pi_1(e_K)\pi_1(g)\pi_1(\varphi_j e_K) v_j = 0 \ \forall g \in G$

$\Rightarrow \sum_j \pi_2(e_K)\pi_2(g)\pi_2(\varphi_j) F(v_j) = 0$

because $\pi_1(e_K)\pi_1(g)\pi_1(\varphi e_K) \in \mathcal{H}(G,K)$

$\Rightarrow$ G-module spanned by $\sum_j \pi_2(\varphi_j) F(v_j)$ has no K-inv'ts

$\Rightarrow \sum_j \pi_2(\varphi_j) F(v_j) = 0$ ☐

Recall that K is always a cpct open subgp of G.

Def: Let (π,V) be a rep^n of G. It is _admissible_ if

i) it is smooth (recall this is some sort of continuity cond^n)
ii) $\dim(V^K)<\infty$ $\forall K$. (This is a finiteness cond^n)

Note that if (π,V) is admissible irred & $V^K\neq\{0\}$ then we get a f.dim^l rep^n of $\mathcal{H}(G,K)$ on V^K, & this determines π up to isom.

Prop^n 7 (π,V) is admissible $\Leftrightarrow$ π is smooth & $\pi(\varphi)$ has finite rank for all $\varphi\in\mathcal{H}(G)$ $\Leftrightarrow$ for one (or any) K, $V=\bigoplus$ irred reps of K, f.d., cts, each occurring only a finite no. of times.

_Pf_ 1st equivalence $\dim V^K<\infty \Leftrightarrow \text{rank}\,\pi(e_K)<\infty$
Now taking $\varphi=e_K$ gives $(\Leftarrow)$, & taking K s.t. $e_K\varphi=\varphi$ gives $(\Rightarrow)$

2nd equivalence: V admissible, then for all normal, open $K'\subseteq K$, we have that $V^{K'}$ is a f.d. rep^n of K/K', so is completely reducible $\Rightarrow V=\bigoplus$ of irreds of K. Multiplicities finite, as if $K'=\ker(\rho: K\to GL_N(\mathbb{C}))$ then $\dim V^{K'}=N\times(\text{multiplicity of }\rho)<\infty$ $\square$

Now a def^n. NB the admiss irred reps of G can be classified, & this is what we'll be doing.

The _contragredient_ of a smooth rep^n (π,V):

Let $V^*=\mathrm{Hom}(V,\mathbb{C})$. We have $\langle,\rangle: V^*\times V\to\mathbb{C}$

Let $V^*(K)=\{v^*\in V^* \mid \langle v^*,\pi(e_K)v\rangle=\langle v^*,v\rangle\ \forall v\in V\}$

$=$ annihilator of $(\pi(e_K)-1)V \cong (V^K)^*$

Let $\tilde{V}=\bigcup_K V^*(K)$, & let G act by $\langle\pi(g)v^*,v\rangle=\langle v^*,\pi(g^{-1})v\rangle$ $\forall v\in V, v^*\in\tilde{V}, g\in G$.

It's easy to check that $\hat{V}$ is stable under this action, & hence defines a smooth (by def of $\hat{V}$) rep^n $(\tilde{\pi},\tilde{V})$, called the _contragredient_ of G.

If π is admissible, then $\tilde{V}^K=(V^K)^*$ is f.d., so $\tilde{\pi}$ is admissible & $\tilde{\tilde{\pi}}=\pi$.

(note $V=V^K\oplus(e_K-1)V$)

Schur's lemma

Let (π, V) be an irred. admiss rep^n of G, & let $\xi : V \to V^*$, linear, commute with $\pi(g)$ for all $g \in G$. Then ξ is a scalar.

Pf Take K; ξ commutes with e_K & with $\mathcal{H}(G,K)$; so ~~ξ_{V^K}~~ $\xi|_{V^K}$ is an endomorphism commuting with $\mathcal{H}(G,K)$. Take an eigenvalue λ; then $\ker(\xi - \lambda|_{V^K})$ is a subspace of V^K, invt under $\mathcal{H}(G,K)$. So by the Corollary to thm 2, $\xi = \lambda$ on V^K. True for $\xi = \lambda$ on V. ☐

Cor Let π be an irred admiss rep^n of G. Then $\exists$ HM $\omega = \omega_\pi : F^* \to \mathbb{C}^*$, cts, s.t. $\pi\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)v = \omega_\pi(a)\, v \quad \forall a \in F^*, \quad v \in V$. ω_π is called the central character of π. ☐

Remark if $\chi : F^\times \to \mathbb{C}^\times$ is a character (ie a cts HM) (ie $\chi = 1$ on an open subgp of $\mathcal{O}^*$) then we can define, given (π, V), a rep^n $(\pi \otimes \chi, V)$ via
$$(\pi \otimes \chi)(g) = \pi(g).\, \chi(\det g)$$

Thm Let π be irred admiss. Then $\tilde{\pi} \cong \pi \otimes \omega_\pi^{-1}$. NB this is deeper than you think. ☐

Lecture 3
Tues 16th Feb '93
4:00 pm

In this lecture he'll define a particular class of representations

§3 Unramified representations

Say (π, V) a rep. of G, assumed irreducible, admissible.

$\updownarrow$ no non-trivial G-invt subspace

$\updownarrow$ $v \in V \Rightarrow \mathrm{Stab}\, v$ open & $\dim V^K < \infty \;\; \forall K$

Last time we showed that if $K \subset G$ is open & cpct then we get a (f.d.) irreducible rep^n

$$\left\{ \begin{array}{l} \text{functions on } G \text{ with} \\ \text{compact support, left} \\ \text{\& right } K\text{-invt} \end{array} \right\} = \mathcal{H}(G,K) \longrightarrow \mathrm{End}(V^K)$$

& this "determines (π, V) up to isomorphism" (if we know it for ~~all~~ some K with various special properties)

For this §. we will set $K = GL_2(\mathcal{O})$, i.e. a maximal cpct.

Def^n (π, V) irred admiss is unramified if $V^{GL_2(\mathcal{O})} \neq \{0\}$.

We will completely determine all unramified πs.

Thm 3 (a classical thm) $\mathcal{H}(G,K)$ is commutative for this K.

Moreover, $\mathcal{H}(G,K) = \mathbb{C}[T_\pi, S_\pi, S_\pi^{-1}]$ • Here π is a uniformiser, not the repⁿ, & so we may well drop the π's later.

Here $T = T_\pi = \text{char}_{K\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)K}$ & $S = S_\pi = \text{char}_{K\left(\begin{smallmatrix}\pi & 0\\ 0 & \pi\end{smallmatrix}\right)}$

T_π & S_π will be basically the classical Hecke operators, modified by suitable powers of p.

Cor If (π, V) is unramified, then $\dim V^K = 1$, & its determined by

$$\pi(T) \in \mathbb{C}$$
$$\pi(S) \in \mathbb{C}^\times \quad \text{(up to isomorphism)} \quad \square$$

~~Beginning of pf of thm~~

$$G = \coprod_{\substack{m \geq n \\ m,n \in \mathbb{Z}}} K\begin{pmatrix}\pi^m & 0\\ 0 & \pi^n\end{pmatrix}K \qquad \text{(Cartan)}$$

& so $\mathcal{H}(G,K)$ has for a basis the functions $\Phi[\pi^m, \pi^n] = \text{char}_{K\left(\begin{smallmatrix}\pi^m & 0\\ 0 & \pi^n\end{smallmatrix}\right)K}$

We can now proceed classically by explicitly computing everything see e.g. chapter 3 of Shimura's book, where he does it for GL_n.

Alternatively, there's a more modern approach which generalises well:

Let $A = \left\{\begin{pmatrix} * & 0\\ 0 & *\end{pmatrix}\right\} \supseteq A^\circ = \left\{\begin{pmatrix} o_1 & 0\\ 0 & o_2\end{pmatrix} \,\middle|\, o_i \in \mathcal{O}^*\right\} = A \cap K$.

Form the Hecke algebra $\mathcal{H}(A, A^\circ) = \{\text{functions of finite support on } A/A^\circ \cong \mathbb{Z}^2\}$

as A is abelian

$$= \bigoplus_{r,s \in \mathbb{Z}} \mathbb{C}\, \lambda_{r,s}$$

with $\lambda_{r,s}$ the char fⁿ of $\begin{pmatrix}\pi^r \mathcal{O}^* & 0\\ 0 & \pi^s \mathcal{O}^*\end{pmatrix}$

It's easy to see that $\lambda_{r,s} * \lambda_{r',s'} = \lambda_{r+r', s+s'}$ (Here we have normalised the Haar measure s.t. meas $(A^\circ) = 1$)

& hence $\mathcal{H}(A, A^\circ) = \mathbb{C}[x, y, (xy)^{-1}]$ with $x = \lambda_{1,0}$, $y = \lambda_{0,1}$

Now define the Satake transform

$$\Sigma : \mathcal{H}(G,K) \longrightarrow \mathcal{H}(A,A^\circ)$$

by $(\Sigma\varphi)(a) = \delta(a)^{1/2}\int_F \varphi(a\begin{pmatrix}1&x\\0&1\end{pmatrix})\,dx$

where $\delta(a) = \left|\frac{a_1}{a_2}\right|$, this is the modular character for upper-triangular matrices, & it'll become clear why it appears, later.

$A^\circ \subseteq K \Rightarrow \Sigma\varphi$ is indeed invariant by A°, so we have a well-defined map.

Let $\mathfrak{S}_2$ = (German S)$_2$ be the symmetric group of degree 2, acting on A by letting the non-trivial element send $\begin{pmatrix}a&0\\0&b\end{pmatrix}$ to $\begin{pmatrix}b&0\\0&a\end{pmatrix}$.

($\mathfrak{S}_2$ is the Weyl group of GL_2). Then $\Sigma\varphi$ is invariant under $\mathfrak{S}_2$ (& hence the reason for $\delta(a)^{1/2}$)

Thm 4 Σ is an isomorphism of algebras $\mathcal{H}(G,K) \to \mathcal{H}(A,A^\circ)^{\mathfrak{S}_2}$ □

This is the Satake isomorphism & it generalises to a wide class of groups.

So we have shown $\Sigma : \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[x,y,(xy)^{-1}]^{\mathfrak{S}_2} = \mathbb{C}[x+y, xy, (xy)^{-1}]$.

We will show that $\Sigma : T \longmapsto q^{1/2}(x+y)$

& $S \longmapsto xy$, thus proving thm 3.

The proof: ① Algebra HM
② Calculate $\Sigma\Psi[\pi^m,\pi^n]$

① is tedious. If $\varphi,\varphi' \in \mathcal{H}(G,K)$, then $(\Sigma(\varphi*\varphi'))(a) = \delta(a)^{1/2}\int_N (\varphi*\varphi')(an)\,dn$

where $N = \begin{pmatrix}1&*\\0&1\end{pmatrix}$ is the unipotent matrices

$$= \delta(a)^{1/2}\int_{G\times N}\varphi(g)\varphi'(g^{-1}an)\,dg\,dn$$

But $G = BK$ by Iwasawa ∴

$$= \delta(a)^{1/2}\int_{B\times K\times N}\varphi(bk)\varphi'(k^{-1}b^{-1}an)\,db\,dk\,dn$$

$$= \delta(a)^{1/2}\int_{B\times N}\varphi(b)\varphi'(b^{-1}an)\,db\,dn$$

as φ,φ' are K-inv

So $(\Sigma(\varphi * \varphi'))(a) = \delta(a)^{1/2} \int_{B \times N} \varphi(b)\, \varphi'(b^{-1}an)\, db\, dn$

Now $((\Sigma\varphi) * (\Sigma\varphi'))(a) = \int_A (\Sigma\varphi)(a_1)\, (\Sigma\varphi')(a_1^{-1}a)\, da_1$

$$= \int_A \delta(a_1)^{1/2} \int_N \varphi(a_1 n_1)\, dn_1\ \delta(a_1^{-1}a)^{1/2} \int_N \varphi'(a_1^{-1}an_2)\, dn_2\, da_1$$

(Set $a_1 n_1 = b$)

$$= \delta(a)^{1/2} \int_{B \times N} \varphi(b)\, \varphi'(\underbrace{n_1 b^{-1} a n_2}_{= b^{-1} a n_1' n_2})\, dn_2\, db$$

Here $n_1' = a^{-1} b n_1 b^{-1} a$

~~There seems to be sthg wrong here.~~

$$= \delta(a)^{1/2} \int_{B \times N} \varphi(b)\, \varphi'(b^{-1}an)\, dn\, db \text{ , say, } n = n_1' a n_2$$

$$= \delta(a)^{1/2} \int_{B \times N} \varphi(b)\, \varphi'(b^{-1}an)\, dn\, db$$

② $\Psi[\pi^m, \pi^n] = \mathrm{char}_{K\begin{pmatrix}\pi^m & 0 \\ 0 & \pi^n\end{pmatrix}K}$, $m \geq n$

$$= \sum_{k=0}^{m-n} \sum_{b \bmod \pi^k} \mathrm{char}_{K\begin{pmatrix}\pi^{m-k} & b\pi^n \\ 0 & \pi^{n+k}\end{pmatrix}}$$

So $\Sigma\Psi[\pi^m, \pi^n]\begin{pmatrix}\pi^r & 0 \\ 0 & \pi^s\end{pmatrix} = 0$ unless for some k we have $m-k = r$, $n+k = s$

in which case we have

$$= q^{-(r-s)/2} \sum_{b \bmod \pi^k} \int_F \mathrm{char}_{K\begin{pmatrix}\pi^r & b\pi^{s-k} \\ 0 & \pi^s\end{pmatrix}} \begin{pmatrix}\pi^r & \pi^r x \\ 0 & \pi^s\end{pmatrix} dx$$

$= 0$ if $x \notin \pi^{s-k-r}$, 1 otherwise (for some b).

$= q^{-(r-s)/2}\ \mathrm{meas}(\pi^{s-k-r}\mathcal{O})$

$= q^{-(r-s)/2 - s+k+r} = q^{\frac{1}{2}(r-s+2k)} = q^{+(m-n)/2}$

$$\therefore \Sigma\Psi[\pi^m, \pi^n] = q^{+(m-n)/2} \sum_{k=0}^{m-n} \lambda_{m-k,n+k} = q^{(m-n)/2}(x^m y^n + x^{m-1}y^{n+1} + \dots + x^n y^m)$$

From this it follows that $\mathrm{Im}\,\Sigma = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]^{S_2}$ and that $\Sigma(T) = q^{1/2}(x+y)$

$\Sigma(S) = xy$

Remark ① $\mathcal{H}(G,K)$ is called the unramified or spherical Hecke algebra. ("Spherical" is in analogy with the real case where $K = SO_2(\mathbb{R})$ & f^ns which are bi-invt by K are called spherical f^ns.) Quite important for infinite tensor product.

If (π, V) is an unramified rep^n, then any non-zero $v \in V^K$ is called a spherical vector. The isom. class of such π is ~~deno~~ determined by $\pi(T) = q^{1/2}(\alpha+\beta)$ and $\pi(S) = \alpha\beta$, where α, β are "$\pi(x), \pi(y)$", or in other words by the conjugacy class of the semi-simple matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in GL_2(\mathbb{C})$. α & β are sometimes called the unramified parameters for π.

② The rep^n theory of $GL_2(\mathbb{C})$ (rational rep^ns) tells us that the ring of linear combinations of characters of ratl rep^ns of $GL_2(\mathbb{C})$ is

$$\mathbb{C}[x+y, xy, (xy)^{-1}], \quad \begin{array}{l} x+y \leftrightarrow \text{trace (char of std rep}^n) \\ xy \leftrightarrow \det \text{ (char of } \Lambda^2) \end{array}$$

$$\mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[GL_2(\mathbb{C})]$$

In fact, in general, $\mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[{}^LG]$ for a certain gp ${}^LG / \mathbb{C}$.

Example If $\pi(g) = \chi(\det g)$, $\chi : F^\times \to \mathbb{C}^\times$ is an unramified character

Then (watch out for 2 kinds of π.) $\quad \pi(S) = \pi\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} = \chi(\pi)^2$

$$\pi(T) = \pi\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b \bmod \pi} \pi\begin{pmatrix} 1 & b \\ 0 & \pi \end{pmatrix}$$

$$= (q+1)\chi(\pi)$$

$\therefore$ the parameter is $\begin{pmatrix} q^{-1/2}\chi(\pi) & 0 \\ 0 & q^{1/2}\chi(\pi) \end{pmatrix}$.

So every unramified rep^n with parameter α, β s.t. $\alpha\beta^{-1} \neq q^{\pm 1}$ must be ∞-dim^l.

Lecture 4
Wed 17th Feb '93
4:00pm

Most of this lecture will deal with 1 kind of rep^n of G, the principal series.

§4 The principal series of G

If $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in A$, then $\delta(a) = \left|\frac{a_1}{a_2}\right|$

The idea: $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ - a 1-dim^l rep^n of B defined by 2 chars (along the diagonal) can be induced up to a rep^n of G.

A def^n of a rep^n is coming up.

<u>Defⁿ</u> Let $\mu_1, \mu_2 : F^\times \to \mathbb{C}^\times$ be 2 characters (cts homs)

~~Letting G act by right multiplication translation g~~

Let $\mathcal{B}(\mu_1,\mu_2) = \left\{ \text{loc cst. } \Phi : G \to \mathbb{C} \;\middle|\; \Phi(ang) = \mu_1(a_1)\mu_2(a_2)\delta(a)^{1/2}\,\Phi(g) \right\}$

$\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = a \in A,\ n \in N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},\ g \in G \right)$

Letting G act by right translation gives a representation $\rho(\mu_1,\mu_2)$ of G on this space. This is <u>admissible</u>, since every Φ is determined by its restriction $\Phi|_K$ to $K = GL_2(\mathcal{O}_F)$, which is a locally cst f^n on K.

Note that $\rho(\mu_1,\mu_2)\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \mu_1\mu_2(a)$ is a scalar.

However, $\rho(\mu_1,\mu_2)$ may not be irreducible.

<u>Determine when $\rho(\mu_1,\mu_2)$ is irreducible</u>

Recall the Bruhat decomposition $G = B \amalg \underbrace{BwN}_{\substack{\text{dense}\\ \text{in } G}}$, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So $\Phi \in \mathcal{B}(\mu_1,\mu_2)$ is determined by $\varphi(x) = \Phi\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \Phi\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$

So by the idea $\Phi \mapsto \varphi$ we can replace $\mathcal{B}(\mu_1,\mu_2)$ by a space $V = V(\mu_1,\mu_2)$ of locally cst f^ns on F.

Write π for the repⁿ of G on V.

Then $\left(\pi\begin{pmatrix} a_1 & y \\ 0 & a_2 \end{pmatrix}\varphi\right)(x) = \mu_1(a_2)\mu_2(a_1)\delta(a)^{-1/2}\,\varphi\left(\frac{a_2x+y}{a_1}\right)$ (1)

since $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\begin{pmatrix} a_1 & y \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & \frac{a_2x+y}{a_1} \end{pmatrix}$

Also, $(\pi(w)\varphi)(x) = \mu_1(-1)\,\mu_2\mu_1^{-1}(x)\,|x|^{-1}\,\varphi(1/x)$ (2)

since $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1/x & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1/x \end{pmatrix}$

We get $\varphi(x) = c\,\mu_1^{-1}\mu_2(x)\,.\,|x|^{-1}$ for $|x| \gg 0$ (3)

as $\pi(w)\varphi$ is cst in a nhd of 0

So we can actually work out which functions we have here:

It's easy to see that $V \supseteq C_c^\infty(F) =_{\text{dfn}} \mathcal{S}(F)$, <u>the Schwartz space</u>

Hence $V = \mathcal{S}(F) \oplus \mathbb{C}\varphi_0$, with $\varphi_0(x) = \begin{cases} \mu_1^{-1}\mu_2(x) \cdot |x|^{-1} & \text{for } |x| \geq 1 \\ 0 & \text{for } |x| < 1 \end{cases}$

If $n = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in N$ then $\pi(n)\varphi(x) = \varphi(x+a)$ by (1)

Note $\mathcal{S}(F) \subseteq V$ is invariant under B by (1)

Let's find the B-invt subspaces of $\mathcal{S}(F)$

We need

<u>The Fourier transform</u> Pick a non-trivial additive character $\psi : F \to \mathbb{C}^\times$

$$\text{eg } \psi(x) = \exp(2\pi i . \mathrm{Tr}_{F/\mathbb{Q}_p}(x))$$

Then $\varphi \in \mathcal{S}(F) \rightsquigarrow \hat{\varphi} \in \mathcal{S}(F)$ given by $\hat{\varphi}(y) = \int_F \psi(xy)\varphi(x)\,dx$

and $\hat{\hat{\varphi}}(x) = c\varphi(-x)$, and $c = 1$ for a suitable choice of (ψ, dx).

Hence $\varphi(x) = \int_F \hat{\varphi}(y)\psi(-xy)\,dy$

$$\therefore \varphi(x+a) = \int_F \psi(-ay)\hat{\varphi}(y) . \psi(-xy)\,dy$$

$$\mathrm{Span}\{\psi(-ay)\hat{\varphi}(y) \mid a \in F\} \underset{\substack{\text{Fourier}\\ \text{Analysis}}}{=} \{\theta \in \mathcal{S}(F) \mid \mathrm{supp}\,\theta \subseteq \mathrm{supp}\,\hat{\varphi}\}$$

and so the N-invt subspaces of $\mathcal{S}(F)$ are in 1-1 correspondence with the open subsets $\Sigma \subseteq F$.

$$\Sigma \longleftrightarrow \{\varphi \in \mathcal{S}(F) \mid \mathrm{supp}\,\hat{\varphi} \subset \Sigma\} = U_\Sigma$$

$$\pi\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\varphi(x) = \mu_1(a)|a|^{1/2}\varphi(ax) = \mu_1(a)|a|^{1/2}\int_F \hat{\varphi}(y)\psi(-ayx)\,dy$$

$$= \mu_1(a)|a|^{-1/2}\int_F \hat{\varphi}(a^{-1}y)\psi(-xy)\,dy$$

So U_Σ is invt under $\pi\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \leftrightarrow a.\Sigma = \Sigma$

So the B-invt subspaces of $\mathcal{S}(F)$ are:

$$U_\phi = \{0\},\quad U_F = \mathcal{S}(F),\ \&\ U_{F^*} = \{\varphi \mid \overset{\hat{\varphi}(0)}{\overset{\shortparallel}{\int_F \varphi(x)\,dx}} = 0\}$$

$$= \mathcal{S}(F)^0$$

So any G-invt irreducible subspace $U \subseteq V$ either contains $\mathcal{S}(F)^\circ$ or is finite-dimensional & hence 1-dim^l.

<u>dim U = 1</u> : then $\pi(g)\big|_U = \chi(\det g)$, so if $U = \mathbb{C}\varphi$ say, we get

$\pi(N)\varphi = \varphi \quad \therefore \varphi = $ cst f^n, & the transformation formula (1) implies that $\mu_1\mu_2^{-1} = |\cdot|$. Conversely, if $\mu_1\mu_2^{-1} = |\cdot|$ then $V \supseteq \{\text{constants}\}$ as a G-invt subspace.

<u>Other case: $U \supseteq \mathcal{S}(F)^\circ$</u>

Reduce to first case by a duality argument : U has finite codimension

$$\langle\,,\,\rangle : V(\mu_1,\mu_2) \times V(\mu_1^{-1},\mu_2^{-1}) \longrightarrow \mathbb{C} \quad \text{defined by}$$

$$\langle\varphi,\varphi'\rangle = \int_F \varphi(x)\varphi'(x)\,dx$$

(NB from (3) $\Rightarrow$ $|\varphi\varphi'(x)| \ll |x|^{-2}$; it's a G-invt pairing by (1) & (2))

(NB if he hadn't put in that $\delta(a)^{1/2}$ factor, then we'd get some silly factors in here instead of μ_1^{-1}, μ_2^{-1}.)

So $U^\perp$ is a f.d. invt subspace of $V(\mu_1^{-1},\mu_2^{-1})$, & so $\mu_1^{-1}\mu_2 = |\cdot|$ if $U \neq V$ & we get $U = \{\varphi \in V \mid \int_F \varphi\,dx = 0\}$.

<u>Def^n / Thm 5</u> Let $\mu_1, \mu_2 : F^* \to \mathbb{C}^*$ be characters. Then $\rho(\mu_1,\mu_2)$ is indecomposable, &

(i) $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1} \Rightarrow \pi(\mu_1,\mu_2) =_{\text{defn}} \rho(\mu_1,\mu_2)$ is irreducible

(ii) $\mu_1\mu_2^{-1} = |\cdot| \Rightarrow \rho(\mu_1,\mu_2)$ has a unique 1-dim^l subrepresentation, denoted by $\pi(\mu_1,\mu_2)$, iso. to $(\mu_1|\cdot|^{-1/2})\circ\det$ & the quotient $\sigma(\mu_1,\mu_2)$ is irreducible

(iii) $\mu_1\mu_2^{-1} = |\cdot|^{-1} \Rightarrow \rho(\mu_1,\mu_2)$ has a ! 1-dim^l invt quotient $\pi(\mu_1,\mu_2)$ & the kernel $\sigma(\mu_1,\mu_2)$ is irreducible.

We have essentially done all the details. (just need to check these irreducibility claims Use duality) $\square$

<u>Def^n</u> The rep^{n}s $\sigma(\mu_1,\mu_2)$ are called <u>special</u> rep^{n}s

NB calling the $\pi(\mu_1,\mu_2)$ $\pi(\mu_1,\mu_2)$ even in the special case looks perverse but it will become clear once the Jacquet Langlands stuff start why this is the natural thing to do

<u>Thm 6</u> (i) $\pi(\mu_1,\mu_2) \cong \pi(\mu_1',\mu_2') \iff \{\mu_1,\mu_2\} = \{\mu_1',\mu_2'\}$

(ii) $\sigma(\mu_1,\mu_2) \cong \sigma(\mu_1',\mu_2') \iff \{\mu_1,\mu_2\} = \{\mu_1',\mu_2'\}$

(iii) no π is iso to any σ.

He may prove this later, once we've got some machinery.

NB these reps are quite easy but they're not all the collections of irred admiss reps of G, only a small subset. The rest are the <u>supercuspidal rep^n s</u>.
Somehow princ. series & special $\leftrightarrow$ reducible rep^n s of Galois gps
supercuspidal $\leftrightarrow$ irred reps.

Finally note that all 1-dim^l reps have shown up in the $\pi(\mu_1,\mu_2)$.

Unramified principal series : let $K = GL_2(\mathcal{O})$. Suppose $\mathcal{B}(\mu_1,\mu_2)^K \neq \{0\}$.

Because $G = BK$, every f^n in $\mathcal{B}(\mu_1,\mu_2)^K$ must be a multiple of

$$\Phi^{spher}(g) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \text{ if } g = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix} k,\ k \in K,$$

& because $\begin{pmatrix} \mathcal{O}^* & 0 \\ 0 & \mathcal{O}^* \end{pmatrix} \leq K$, Φ^{spher} exist $\iff \mu_1(\mathcal{O}^*) = 1 = \mu_2(\mathcal{O}^*)$

ie μ_i must be unramified characters.

(i) The case $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$; then $\pi(\mu_1,\mu_2)$ is an irred unramified rep^n

(ii) $\mu_1\mu_2^{-1} = |\cdot|$. Then $\pi(\mu_1,\mu_2) = (\mu_1 \cdot |\cdot|^{1/2}) \circ \det$ is unramified so

$$\sigma(\mu_1,\mu_2)^K = \{0\} \ \&\ \Phi^{spher} \in \pi(\mu_1,\mu_2).$$

(iii) similarly.

Let's work out T & S:

$$T_\pi \Phi^{spher}(I) = \Phi^{spher}\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} + \sum_{b \bmod \pi} \Phi^{spher}\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix}$$

$$= (\mu_2(\pi) q^{1/2} + q\,\mu_1(\pi)\, q^{-1/2})\, \Phi^{spher}(I)$$

$$= q^{1/2}(\mu_1(\pi) + \mu_2(\pi)) \quad \text{as } \Phi^{spher}(I) = 1.$$

$$S_\pi \Phi^{spher}(I) = \Phi^{spher}\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix} = \mu_1\mu_2(\pi)$$

So $\pi(\mu_1,\mu_2) \leftrightarrow$ conjugacy class of $\begin{pmatrix} \mu_1(\pi) & 0 \\ 0 & \mu_2(\pi) \end{pmatrix}$ in $GL_2(\mathbb{C})$

Hence $\{\pi(\mu_1,\mu_2) \mid \mu_i \text{ unramified}\}$ exhaust the set of (equiv. classes of) irred (unramified) admiss reps of G

Lecture 5
Thu 18th Feb '93
4:00pm

Today & tomorrow he's going to try & tell us about $GL_2(\mathbb{R})$, $GL_2(\mathbb{C})$. His knowledge of the situation here is much less than the p-adic case! He won't be speaking with as much authority. The analysis looks much more unfriendly to the average number theorist.

The first remark to be made is that a $GL_2(\mathbb{R})$ or $(\mathbb{C})$-module isn't really a rep^n of $GL_2(\mathbb{R}),(\mathbb{C})$!! It isn't a rep^n of any group at all, in fact.

II Representations of $GL_2(\mathbb{R}), GL_2(\mathbb{C})$

§5 $(\mathfrak{g}, K)$-modules

$G = GL_2(\mathbb{R}) \supseteq K = O(2) = \begin{pmatrix} \cos\theta & \pm\sin\theta \\ -\sin\theta & \pm\cos\theta \end{pmatrix}$, K the max^l cpct.

Note that if $v \in V$ is K-finite for some action of G on V, then

(ie span $\{\pi(k)v \mid k \in K\}$ f.d.

$\pi(g)v$ is gKg^{-1}-finite, & this in general has rather small intersection with K, as K isn't open & various other reasons.

So the main difference between $\mathbb{R}$ & p-adic case is that in general, G will not act on any space of K-finite vectors. This leads us to the idea of a $(\mathfrak{g}, K)$-module.

Here $\mathfrak{g}$ is the Lie algebra of G ; $\mathfrak{g} = M_2(\mathbb{R})$; $[X,Y] = XY - YX$

A representation of $\mathfrak{g}$ is a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$

st. $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$.

Recall $\exp : \mathfrak{g} \to G$, $X \mapsto e^X = \sum_{n \geq 0} X^n/n!$

Now let $\pi : G \to GL(V)$, V a complex Hilbert space st.

- $\pi(g)$ are bounded $\forall g$
- cts in the sense that $\forall v$ the map $g \mapsto \pi(g)v$ cts

Assume also that $\pi|_K$ is a rep^n by unitary transformations of V.

Then $V = \hat{\bigoplus}_{\rho} V(\rho)$, each $V(\rho)$ = sum of copies of some irred f.d. rep^n ρ of K. (since K is cpct)

Then $V \supseteq V^0 = \bigoplus_{\rho} V(\rho)$, an algebraic direct sum. V^0 is no longer a Hilbert space.

V^0 is the set of K-finite vectors in V

<u>Thm 7</u> Let (π, V) be as above, & say V is irreducible.

(i) If $v \in V^\circ$, then $g \mapsto \pi(g)v$ is C^∞, even real analytic, so if $X \in \mathfrak{g}$ we can define

$$(d\pi)(X)v = \frac{d}{dt}\left(\pi(e^{tX})v\right)\Big|_{t=0}$$

(ii) $(d\pi)(X)$ takes V° onto itself, & gives a repⁿ of $\mathfrak{g}$ on V°.

(So there's not an action of G on V° but there is an action of $\mathfrak{g}$.

(iii) Now assume (π, V) is unitary & irred. Then $\dim V(\rho) < \infty$.

$\uparrow$

i.e. topologically irred: $\nexists$ non-trivial closed inv't subspace.

Moreover, V° is jointly irred as a $\mathfrak{g}$ & K-module. Moreover, if V' is another irred. unitary rep, then $V \cong V'$ as unitary G-modules $\iff \exists$ isom. $V^\circ \xrightarrow{\sim} V'^\circ$ which commutes with the action of $\mathfrak{g}$ & K. $\square$ (Note that info about $\mathfrak{g}$ action is not good enough, as G isn't connected. That's a reason why K is here)

He won't indicate how to prove all this. It's a theorem of Harish-Chandra. It's true for any reductive real Lie gp G with max' cpct K, although it's possible to extract a self-contained pf for $SL_2(\mathbb{R})$ from Lang's book.

So given a repⁿ of G we extract a $(\mathfrak{g}, K)$-module. The $(\mathfrak{g}, K)$-module is much more algebraic, as e.g. $V(\rho)$ is f.d. $\forall \rho$. There is some sort of way of going back, but you need conditions on $(\mathfrak{g}, K)$ V°. He'll go into this later.

<u>Def:</u> An <u>(admissible) $(\mathfrak{g}, K)$-module</u> is a complex vector space V, together with

- $\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}(V)$, a Lie algebra HM
- $\pi_K : K \to GL(V)$, making V the direct sum of f.d.' cts rep's of K (occurring with finite multiplicity)

exists by $\uparrow$ f.d. bit

s.t. $\downarrow$

(i) $d\pi_K : \mathfrak{k} \to \mathrm{End}(V)$ & $\pi_{\mathfrak{g}}|_{\mathfrak{k}}$ are equal. ($\mathfrak{k}$ = Lie alg of K)

(ii) if $k \in K$, $X \in \mathfrak{g}$, then
$$\pi(k)\pi(X)\pi(k^{-1}) = \pi(\mathrm{ad}(k)X) = \pi(kXk^{-1})$$

<u>Notation</u> Write $V = \bigoplus_\rho V(\rho)$ where ρ runs over inequivalent f.dim' reps of K, each $V(\rho)$ = sum of copies of ρ.

$V(\rho)$ are called K-types. V admissible $\iff \dim V(\rho) < \infty \ \forall \rho$.

All this goes through for general G, K.

Now we'll specialise to $GL_2(\mathbb{R})$. Note then we understand the f.d. reps of K. (I think he said they're all 1 or 2-dim^l, or something). We want to classify the irred admiss reps in this case.

$G = GL_2(\mathbb{R})$. We complexify to start off with.

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, which acts on any $(\mathfrak{g}, K)$-module by linearity.

A basis (a $\mathbb{C}$-basis, that is) for $\mathfrak{g}_{\mathbb{C}}$ is $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (not "the identity" as the identity is 0 !!)

$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

& $X_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$

Then $e^{i\theta H} = r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$[J, \mathfrak{g}] = 0 \; ; \; [H, X_+] = 2X_+ \; ; \; [H, X_-] = -2X_- , \; [X_+, X_-] = H$$

Let $\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in K$. Now $\mathfrak{g}_{\mathbb{C}} \supseteq \mathfrak{g}'_{\mathbb{C}}$, the complexified Lie algebra of $SL_2(\mathbb{R})$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\{X \in M_2(\mathbb{C}) \mid \mathrm{tr}\, X = 0\}}$, spanned by $X_{\pm}$ & H.

(NB we'll do a lot for $SL_2(\mathbb{R})$ & then show that $GL_2(\mathbb{R})$ follows.)

An admissible $(\mathfrak{g}, K)$-module, in the $GL_2(\mathbb{R})$ case, can be described as

- A graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, $\dim V_n < \infty$.
- operators $\pi(X_{\pm})$, $\pi(H)$, $\pi(J)$ on V, satisfying the commutation laws, & s.t.
- $V_n = \ker(\pi(H) - n \mid V)$
- operator $\pi(\varepsilon)$, with square 1, satisfying certain relations

Now, by Schur's lemma, if (V, π) is an irred admiss $(\mathfrak{g}, K)$-module, then $\pi(J) = c \in \mathbb{C}$ (since it has an eigenvector on some V_n)

So it is sufficient to classify $(\mathfrak{g}', K)$-modules.

If we restrict further to $(\mathfrak{g}', K')$ where $K' = SO(2)$, then either

V remains irreducible, or

$V = W \oplus W^{\varepsilon}$, where W is a $(\mathfrak{g}', K')$-module, irreducible, & W^{ε} = conjugate of W by the automorphism

$g \mapsto \varepsilon g \varepsilon^{-1}$ of $G' = SL_2(\mathbb{R})$

So now we will classify the irred admiss $(\mathfrak{g}, K')$-modules (π, V)

- $\pi(X_+) : V_n \to V_{n+2}$, since if $v \in V_n$, then

$$\pi(H)\pi(X_+)v = \pi([H,X_+])v + \pi(X_+)\pi(H).v$$
$$= (2 \qquad + n)\pi(X_+)v$$

$\therefore\ \exists\, m \in \{0,1\}$ s.t. $V = \bigoplus_{n \equiv m \,(\mathrm{mod}\, 2)} V_n$

Now we'll define (some multiple of) the Casimir operator,

namely $\mathcal{D} = X_+X_- + X_-X_+ + \frac{1}{2}H^2$.

Formally $\mathcal{D} \in U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Alternatively think of it as composition of operators. It is nothing to do with multiplication of matrices.

$\mathcal{D}$ acts on any rep^n space of $\mathfrak{g}$, & $\mathcal{D}$ commutes with $\mathfrak{g}$. (In fact the centre $\mathfrak{z}(\mathfrak{g})$ of $U(\mathfrak{g})$ is $\mathbb{C}[\mathcal{D}]$.)

So by Schur's lemma, $\pi(\mathcal{D}) = d \in \mathbb{C}$.

$$\mathcal{D} = 2X_+X_- - H + \tfrac{1}{2}H^2 = 2X_-X_+ + H + \tfrac{1}{2}H^2$$

$$\therefore\ \pi(X_+)\pi(X_-)\big|_{V_n} = \tfrac{1}{2}(d + n - n^2/2)$$

$$\& \ \pi(X_-)\pi(X_+)\big|_{V_n} = \tfrac{1}{2}(d - n - n^2/2).$$

NB the ~~di~~trichotomy occurs depending on when either of these are ever 0.

Now let $r \in V_k \setminus \{0\}$, some k. Then the span of $\{\pi(X_+)^r v, \pi(X_-)^r v \mid r \geq 0\}$ is stable under $\mathfrak{g}$, and so equals V by irreducibility.

Hence $\dim V_n \leq 1\ \forall n$.

Moreover, since V is irreducible, if $V_k \neq 0$ then both of $\pi(X_\pm) : V_k \to V_{k\pm 2}$ are surjective (else $\bigoplus_{n \leq k} V_n$ or $\bigoplus_{n \geq k} V_n$ are invariant submodules).

So there are 3 cases left to consider, the same number as there are minutes left in the lecture.

Case(i) $\exists k$ s.t. $V_k \neq 0$, $\pi(X_-)(V_k)=0$. Then $V_n=0$ $\forall n<k$, so k is !, and

$$\pi(X_+)\pi(X_-)\Big|_{V_k}=0 \text{ , so } d=\frac{k^2}{2}-k$$

If $k \le 0$ then $\pi(X_-)\pi(X_+)\Big|_{V_{-k}} = \frac{1}{2}(d-(-k)-\frac{k^2}{2}) = 0$

so $V = \bigoplus_{\substack{-|k|\le n\le |k| \\ n\equiv k \bmod 2}} V_n$ is finite-dimensional.

If $k>0$ then $\pi(X_-)\pi(X_+)\Big|_{V_n} \neq 0$ for all $n \ge k$, so

$$V = \bigoplus_{\substack{n\equiv k(2) \\ n\ge k}} V_n \text{ is } \infty\text{-dim}^l$$

Let $v \in V_k$, $v\neq 0$. If we write $\varphi_n = \begin{cases} \frac{2^r(k-1)!}{(k+r-1)!}\pi(X_+)^r v & \text{if } n=k+2r \ (\ge k) \\ 0 & \text{otherwise} \end{cases}$

then $V = \bigoplus_{\substack{n\ge k \\ n\equiv k(2)}} \varphi_n \mathbb{C}$; $\pi(X_\pm)\varphi_n = \frac{1}{2}(k\pm n)\varphi_{n\pm 2}$

$$\pi(H)\varphi_n = n\varphi_n$$

discrete series $\mathcal{D}_k^+$, $d=\frac{1}{2}k(k-2)$

Lecture 6
~~Thu~~ Fri 19th Feb '93
4:00 pm

Recall we were looking at $(\mathfrak{g}, K') = (\mathfrak{sl}_2(\mathbb{R}), SO(2))$-modules (π, V), assumed irred & admiss.

$V = \bigoplus V_n$ under K'

Recall $\pi(X_+)\pi(X_-)\Big|_{V_n} = \frac{1}{2}(n+d)-\frac{1}{4}n^2$, d = eigenvalue of $\mathcal{D}$, Casimir operator.

$\dim V_n \le 1$, $\exists m\in\{0,1\}$ s.t. $V = \bigoplus_{n\equiv m(2)} V_n$

(i) $\exists k$ s.t. $V_k \neq 0$, $\pi(X_-)V_k=0$, if $k\le 0$ then $\pi(X_+)V_{-k}=0$ & V is f.dim^l

$k>0 \Rightarrow V = \bigoplus_{\substack{n\ge k \\ n\equiv k(2)}} V_n$ Called $\mathcal{D}_k^+$

Now on to case (ii).

<u>Case (ii)</u> $\exists k$ s.t. $V_{-k} \neq 0$ but $\pi(X_+)(V_{-k}) = 0$

2 cases again: $k \leq 0$ ($\therefore -k \geq 0$) & $V = \bigoplus_{-|k| \leq n \leq |k|} V_n$ is f.d. (same as (i))

or $k > 0$, in which case

$$V = \bigoplus_{\substack{n \leq -k \\ n \equiv k \bmod 2}} V_n$$

& we get an equiv. class of rep^ns $\mathcal{D}^-_k$

$\mathcal{D}^-_k$ is the conjugate of $\mathcal{D}^+_k$ by ε

<u>Case (iii)</u> $V_n \neq 0$ for all $n \equiv m \bmod 2$

So $V = \bigoplus_{\substack{n \in \mathbb{Z} \\ n \equiv m \bmod 2}} V_n$, $\dim V_n = 1$ $\forall n \equiv m\ (2)$.

Then $\pi(X_+)$, $\pi(X_-)$ injective $\Rightarrow d \neq \frac{1}{2}n^2 - n$ for any $n \equiv m \bmod 2$.

$\therefore$ we can write $d = \frac{s^2-1}{2}$, $s \in \mathbb{C}$, $s \not\equiv 1+m \pmod{2\mathbb{Z}}$

V is then determined uniquely by m & the action of $\pi(X_-)\pi(X_+)$ on V_m, i.e. by d & by s.

An explicit basis: let $v \in V_m \setminus \{0\}$; define

$$\varphi_n = 2^{(n-m)/2} \frac{\Gamma\left(\frac{s+m+1}{2}\right)}{\Gamma\left(\frac{s+n+1}{2}\right)} \pi(X_+)^{(n-m)/2} v$$

Then $\pi(X_\pm)\varphi_n = \frac{1}{2}(s+1 \pm n)\varphi_{n\pm 2}$

This rep^n is denoted $\mathcal{B}_s^{(-1)^m}$, the <u>principal series</u>

So we have proved

<u>Thm 8</u> Every irred admiss $(\mathfrak{g}, K')$-modules of ∞ dimension is isomorphic to $\mathcal{B}^\pm$ or a $\mathcal{D}^\pm_k$; the only equivalences are $\mathcal{B}^\pm_s \cong \mathcal{B}^\pm_{-s}$

$\mathcal{D}^-_k$ = ε-conjugate of $\mathcal{D}^+_k$

$\mathcal{B}^\pm_s$ is $\cong$ to its ε-conjugate. $\varphi_n \leftrightarrow \varphi_{-n}$. ☐

NB we seemed to use admissibility to show that the Casimir operator acts as a scalar. It's a thm of Harish-Chandra that irred $\Rightarrow$ admiss for real reductive gps, I think he said this.

We'll now go back to our original task.

Classification of (irred admiss, presumably) $(\mathfrak{g}, K)$-modules, $G = GL_2(\mathbb{R})$.

$\mu_1, \mu_2 : \mathbb{R}^\times \to \mathbb{C}^\times$

$\mu_i(t) = |t|^{s_i} (\mathrm{sgn}\, t)^{m_i}$, $s_i \in \mathbb{C}$, $m_i = 0$ or 1

Set $s = s_1 - s_2$, $m = |m_1 - m_2|$

$$\mathcal{B}(\mu_1, \mu_2) = \bigoplus_{n \equiv m \,(\mathrm{mod}\, 2)} \mathbb{C}\varphi_n.$$

with action of $(\mathfrak{g}, K)$.

$$\begin{cases} \pi(X_\pm)\varphi_n = \frac{1}{2}(s+1 \pm n)\varphi_{n\pm 2} \\ \pi(H)\varphi_n = n\varphi_n \\ \pi(r_\theta)\varphi_n = e^{in\theta}\varphi_n \\ \pi(\varepsilon)\varphi_n = (-1)^{m_1}\varphi_{-n} \\ \pi(J)\varphi_n = (s_1+s_2)\varphi_n \end{cases}.$$

$\mathcal{B}(\mu_1, \mu_2)$ can be identified with the space of right K-finite functions

$$\Phi : G \to \mathbb{C} \quad \text{s.t.} \quad \Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \Phi(g)$$

by $\varphi_n \leftrightarrow \Phi_n$ s.t. $\Phi_n(r_\theta) = e^{in\theta}$.

<u>Thm 9</u> (i) $\mathcal{B}(\mu_1, \mu_2)$ is an admissible $(\mathfrak{g}, K)$-module, & every irred admiss $(\mathfrak{g}, K)$-module is a submodule of some $\mathcal{B}(\mu_1, \mu_2)$.

(ii) $\mathcal{B}(\mu_1, \mu_2)$ is irreducible if $s \not\equiv 1+m \pmod{2\mathbb{Z}}$, i.e. it's irred. unless $\mu_1\mu_2^{-1} = \mathrm{sgn}.|\cdot|^s$, $s \in \mathbb{Z}$. Call this <u>$\pi(\mu_1, \mu_2)$</u>. I $\pi(\mu_1, \mu_2)$, irreducible, is the <u>principal series</u>

(iii) If $\mu_1\mu_2^{-1} = \mathrm{sgn}.|\cdot|^s$ and $s > 0$ is an integer then $\mathcal{B}(\mu_1, \mu_2)$ has a unique submodule of finite codimension, denoted <u>$\sigma(\mu_1, \mu_2)$</u>. It's equal to $\bigoplus_{|n| > s} \mathbb{C}\varphi_n$. Then let <u>$\pi(\mu_1, \mu_2)$</u> be the finite-dim^l quotient.

(iv) $\mu_1\mu_2^{-1} = \mathrm{sgn}\,|\cdot|^s$, $s < 0$ integer. Then $\mathcal{B}(\mu_1, \mu_2)$ has a unique (non-zero ?!) finite-dim^l submodule <u>$\pi(\mu_1, \mu_2)$</u> $= \bigoplus_{|n| \le -s-1} \mathbb{C}\varphi_n$, and the quotient <u>$\sigma(\mu_1, \mu_2)$</u> is irreducible.

The σ's are <u>discrete series</u>.

(v) Finally, a silly case. If $\mu_1\mu_2^{-1} = \mathrm{sgn}$, then $s=0$ so the f.d. submodule disappears, and we get
$$\mathcal{B}(\mu_1,\mu_2) = \underline{\pi(\mu_1,\mu_2)} \text{ is irreducible.}$$

This one is called limit of discrete series.

~~Thm 10 Let (π, V) be an admiss $(\mathfrak{g},K)$-module and $(,)$ a K-invt inner product, s.t. $\pi(X)v,v'$~~

The only isomorphisms.

$$\pi(\mu_1,\mu_2) \cong \pi(\mu_1',\mu_2') \iff \{\mu_1,\mu_2\} = \{\mu_1',\mu_2'\}$$

$$\sigma(\mu_1,\mu_2) \cong \sigma(\mu_1',\mu_2') \iff \{\mu_1',\mu_2'\} = \{\mu_1,\mu_2\} \text{ or } \{\mu_1\,\mathrm{sgn}, \mu_2\,\mathrm{sgn}\}$$

No π is equivalent to a μ.

Thm 10 Let (π, V) be an admissible $(\mathfrak{g},K)$-module, and $(,)$ a K-invt inner product s.t.
$$(\pi(X)v,v') = -(v,\pi(X)v') \text{ if } X \in \mathfrak{g}_{\mathbb{R}},\ v,v' \in V$$

Then there's a unique unitary rep^n of G on the completion $\hat{V}$ s.t. (π,V) is the $(\mathfrak{g},K)$-module of K-finite vectors in $\hat{V}$. □

This is true for any reductive (real?) gp & is due to Harish-Chandra.

So
$$\begin{pmatrix} \text{irred unitary} \\ \text{reps of } G, \text{ up} \\ \text{to unitary} \cong \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{irred admiss} \\ (\mathfrak{g},K)\text{-modules with} \\ \text{an invt inner product} \end{pmatrix}.$$

Thm 11 $(\mathfrak{g},K) = (\mathfrak{gl}_2(\mathbb{R}), O(2))$, the irred admiss $(\mathfrak{g},K)$-modules associated to unitary rep's of G are

- $\pi(\mu_1,\mu_2)$, μ_i unitary
- $\pi(\mu_1,\mu_2)$ where $m=0$, $s_1 = \sigma + i\tau = -\bar{s}_2$, $0<\sigma<1$
- $\sigma(\mu_1,\mu_2)$ with $|\mu_1\mu_2(t)| = 1$ □

Pf by using thm 10 & classification, & trying to attach invt inner products. Not too bad. He'll omit it.

Anyway, we've talked about $(\mathfrak{g},K)$-modules. There's an action of $\mathfrak{g}$ & one of K. It would be nice to find 1 object & 1 action instead. The Hecke algebra, of course. Analysis sort of disappears - it goes into construction.

§6 The Hecke algebra at infinity

We have $\mathfrak{g}$, & a Lie algebra structure $[\,,\,]$.

There is an associative algebra $U(\mathfrak{g})$, the universal enveloping algebra, with a unit, & a linear map $\mathfrak{g} \xhookrightarrow[j]{} U(\mathfrak{g})$, s.t. $j[X,Y] = jX.jY - jY.jX$, & s.t. any rep^n of $\mathfrak{g}$ extends !ly to a rep^n of $U(\mathfrak{g})$.

In fact, if $\mathfrak{g}$ has a basis $\{X_i\}$, $1 \le i \le d$, then ~~$U(\mathfrak{g}) =$~~

$$U(\mathfrak{g}) = (\text{Free associative algebra on } X_i \text{ s}) \Big/ \langle X_i X_j - X_j X_i - [X_i, X_j] \rangle$$

Thm 12 (Poincaré - Birkhoff - Witt)

A basis for $U(\mathfrak{g})$ is $\{ X_1^{a_1} X_2^{a_2} \dots X_d^{a_d} \}$. $\square$

$U(\mathfrak{g})$ has a centre $\mathfrak{z} \supseteq \mathbb{C}$. For $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$, $\mathfrak{z} = \mathbb{C}[J, \overset{\overset{\text{Casimir}}{\downarrow}}{D}]$.

In any irred f.d. reps of $\mathfrak{g}$, or any irred admiss $(\mathfrak{g},K)$-module, the elts of $\mathfrak{z}$ act via a character $\chi : \mathfrak{z} \to \mathbb{C}$.
χ is called the infinitesimal char of the module.

Corollary If $G = GL_2(\mathbb{R})$, then there is only a finite # of equiv classes of $(\mathfrak{g},K)$-modules with given infinitesimal character.

Pf (~~sketch~~) on $\mathcal{B}(\mu_1,\mu_2)$, $\pi(J) = s_1 + s_2$, $\pi(D) = \frac{s^2-1}{2}$, $s = s_1 - s_2$, so

there's only finitely many choices for s_1, s_2 (4 or 8 or sthg) $\square$

Thm 13 $\exists$ associative algebra $\mathcal{H} = \mathcal{H}(\mathfrak{g},K)$, without a unit, & a directed family of commuting idempotents $E \subseteq \mathcal{H}$ s.t.

$\mathcal{H} = \bigcup e\mathcal{H}e$, & there is an equivalence of categories

$(\mathfrak{g},K)$-modules $\longleftrightarrow$ non-degenerate reps ~~of~~ π of $\mathcal{H}$

a rep^n is nondegenerate if $\forall v\ \exists \xi \in \mathcal{H}$ s.t. $\pi(\xi)v = v$

admissibles $\longleftrightarrow$ $\pi(\xi)$ of finite rank $\forall \xi$.

<u>Pf of thm 13</u> (sketch) We have (π, V) a $(\mathfrak{g}, K)$-module.

(NB we need the thm to understand the decomposition of a global repⁿ into local rep's - see John's lecture next Monday)

Let A_K = algebra of left+right K-finite functions on K under convolution.

$\rho : K \to GL_N(\mathbb{C})$ irred repⁿ the coeffts $a_{ij}(k) \in A_K$
$k \mapsto (a_{ij}(k))$

and A_K is spanned by such matrix coeffts (all ρ)

$e_\rho = \frac{1}{\dim \rho} \operatorname{tr} \rho$ an idempotent. $A_K = \bigoplus_\rho M_{\dim(\rho)}(\mathbb{C})$, e_ρ a projector.

V as above, $\varphi \in A_K$

$$\pi(\varphi) v = \int_K \varphi(k) \pi(k) v \, dk$$

$\pi(e_\rho) : V \twoheadrightarrow V(\rho)$. So $\forall v \in V$ $\exists e = \underbrace{\text{finite sum of } e_\rho \text{ s}}_{\text{this set is } E}$, s.t. $\pi(e) v = v$

V is admissible $\Rightarrow \pi(\xi)$ has finite rank $\forall \xi \in \mathcal{H}$

$U(\mathfrak{g})$ also acts on V (since $\mathfrak{g}$ does)

The algebra $\mathcal{H}$ will be composed of products $X * \varphi$, $X \in U(\mathfrak{g})$, $\varphi \in A_K$

<u>Note</u> ① $X \in \mathfrak{k}$, $\varphi \in A_K \Rightarrow \pi(X)\pi(\varphi) = \pi(L_X \varphi)$

where $(L_X \varphi)(k) = \frac{d}{dt} \varphi(e^{-Xt} k) \Big|_{t=0}$

② $\pi(\varphi)\pi(X_i) = \sum \pi(X_j) \pi(m_{ij} \varphi)$ if $\mathrm{ad}(k) : X_i \to \sum m_{ij} X_j$

$X \in \mathfrak{g}$

Define $\mathcal{H} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} A_K$ (vector space). The $\otimes$ is over $U(\mathfrak{k})$ which acts on $U(\mathfrak{g})$ by right multⁿ & on A_K by L_X (to ensure ① holds)
$X * \varphi$

The product on $\mathcal{H}$ is $\varphi * X_i = \sum_j X_j * (m_{ij} \varphi)$.

Then $(\mathcal{H}, E)$ has the required properties. □

ecture 7
Sat 20th Feb '93
1:00 am

To conclude the archimedean theory, he'll spend 2 minutes on

§7 $GL_2(\mathbb{C})$

Let $\mu_1, \mu_2 : \mathbb{C}^* \to \mathbb{C}^*$ be cts HMs

$$\mathcal{B}(\mu_1,\mu_2) = \Big\{ \text{right } K\text{-finite fns } \Phi : GL_2(\mathbb{C}) = G \longrightarrow \mathbb{C} \text{ s.t. } \Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|_{\mathbb{C}}^{1/2}\Phi(g)\Big\}$$

Here $|z|_{\mathbb{C}} = z\bar{z}$, the local norm.

Here K = max^l cpct subgp $= U(2) \subset G$

$\mathcal{B}(\mu_1,\mu_2)$ is an admissible $(\mathfrak{g}, K)$-module

<u>Thm 14</u> (i) If $\mu_1\mu_2^{-1}$ is not of the form $z^p\bar{z}^q$ for $p,q\in\mathbb{Z}$, $\overset{\text{product}}{pq>0}$, then $\mathcal{B}(\mu_1,\mu_2)$ is irreducible; call it $\pi(\mu_1,\mu_2)$.

(ii) If $\mu_1\mu_2^{-1} = z^{\pm p}z^{\pm q}$, $pq > 0$, then $\mathcal{B}(\mu_1,\mu_2)$ has a ! (non-zero) f.d. subquotient $\pi(\mu_1,\mu_2)$ which is iso. to

$$\mathrm{Sym}^{p-1}\otimes\overline{\mathrm{Sym}^{q-1}}\otimes(\mu_1|\cdot|^{-1/2}\circ\det)$$

& the $\pi(\mu_1,\mu_2)$s exhaust all irred admiss $(\mathfrak{g},K)$-modules.

(iii) $\pi(\mu_1,\mu_2) \cong \pi(\mu_2,\mu_1)$ & there are no other equivalences. □

Hence rep's are classified by conjugacy class of semisimple HMS

$$\mathbb{C}^\times \to GL_2(\mathbb{C}), \quad \pi(\mu_1,\mu_2) \leftrightarrow \begin{pmatrix} \mu_1(z) & 0 \\ 0 & \mu_2(z) \end{pmatrix}$$

That is all & more than he wants to say about the archimedean case.

III §8 The Kirillov model (& Atkin-Lehner theory)

(although it's not really what Atkin-Lehner had in mind!

Notation as in I : $F/\mathbb{Q}_p$, (π, V) irred admiss rep^n of $G = GL_2(F)$.

Let $\psi : F \to \mathbb{C}^*$ be a non-trivial additive character.

<u>Thm 15</u> Assume π is ∞-dim^l. Then $\exists!$ space $\mathcal{K}(\pi)$ of functions on F^*, & a ! rep^n π' of G on $\mathcal{K}(\pi)$, which is equivalent to π, & s.t.

$$(\pi'\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi)(x) = \psi(bx)\,\xi(ax) \quad \forall a \in F^*, b \in F, \xi \in \mathcal{K}(\pi).$$

The support of any function in $\mathcal{K}(\pi)$ is contained in a cpct subset of F & the f^ns are locally cst. Moreover, $\mathcal{K}(\pi)$ contains $\mathcal{S}(F^*) = C_c^\infty(F^*)$, as a subspace of finite codimension.
loc cst ⟹ cts f^ns, cpct support

<u>Idea of pf</u>

(i) Construct a linear form $\lambda : V \to \mathbb{C}$ s.t. $\lambda(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}v) = \psi(x)\,\lambda(v)$

(ii) Deduce existence of a space $W(\pi)$ of f^ns on G, the <u>Whittaker model</u>, iso. to V, by

$$v \mapsto f \,, \quad f(g) = \lambda(\pi(g)v)$$

$$\left(\text{so } f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi(x) f(g)\right)$$

(iii) $\mathcal{K}(\pi)$ is obtained by restricting f^ns in $W(\pi)$ to $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ □

<u>Ex</u> $\pi = \pi(\mu_1, \mu_2)$ irred princ. series or $\sigma(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1} = |\cdot|^{-1}$

(so π is a subrep^n of $\mathcal{B}(\mu_1, \mu_2)$)

Then $\mathcal{B}(\mu_1, \mu_2) \xrightarrow{\sim} V(\mu_1, \mu_2)$, space of f^ns on F, satisfying

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\varphi(x) = \mu_2(a)\,|a|^{-1/2}\,\varphi\left(\frac{x+b}{a}\right)$$

(I think he said $\varphi(x) = f^n$ corresp to eval^n at $\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}$)

The Fourier transform, <u>formally</u>, $\Rightarrow \pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\hat{\varphi}(y) = \mu_2(a)\,|a|^{1/2}\,\psi(-by)\,\hat{\varphi}(ay)$.

The theory is, the funny f^ns in $\mathcal{B}$ with support non-cpct have $\hat{\varphi}$ diverging.

You have to be careful with details for $\hat{\varphi}$ then.

So, to get $\mathcal{K}(\pi)$, <u>define</u> $\xi = \mu_2(x)|x|^{1/2}\hat{\varphi}(x)$ for a mapping $V(\mu_1,\mu_2) \to$ (f^ns on F^*).

We need to define $\hat{\varphi}$ carefully when $\varphi \notin \mathcal{S}(F)$.

Anyway, in this way we can explicitly work out what $\mathcal{K}(\pi)$ is:

<u>Prop 8</u> $\mathcal{K}(\pi) = \{$ loc cst $\xi : F^* \to \mathbb{C}$ s.t. $\xi(x) = 0$ for $|x| \gg 0$, with behaviour as $|x| \to 0$ ~~open~~ given below. $\}$

(i) $\pi(\mu_1,\mu_2)$ irred princ. series ($\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$) & $\mu_1 \neq \mu_2$.

Then $\xi(x) = c_1\mu_1(x) + c_2\mu_2(x)$ for $|x| \ll 1$

(so $\mathcal{S}(F^*)$ has cod 2)

(ii) $\pi(\mu_1,\mu_2)$ irred PS, $\mu_1 = \mu_2$

val; I guess

$\xi(x) = c_1\mu_1(x) + c_2 v(x)\mu_2(x)$, $|x| \ll 1$

(so $\mathcal{S}(F^*)$ has cod 2 again)

(iii) $\sigma(\mu_1,\mu_2)$, $\mu_1\mu_2^{-1} = |\cdot|^{-1}$

$\xi(x) = c\mu_2(x)$ for $|x| \ll 1$

(so $\mathcal{S}(F^*)$ has cod 1)

(c, c_1, c_2 arbitrary)

□

<u>Because of uniqueness of $\mathcal{K}(\pi)$ we can use this to deduce thm 6</u>
<u>(eg $\pi \not\cong \sigma$ as cod $\mathcal{S}(F^*)$ is different)</u>

<u>Remark</u>

$\xi \in \mathcal{K}(\pi) \Rightarrow \xi(ax) = \xi(x)$ for $a \in$ open subgp of $\mathcal{O}^*$. Use this & the fact that $\mathcal{K}(\pi) \supseteq \mathcal{S}(F^*)$ to prove

Thm 16. See next page.

Thm 16 (local Atkin-Lehner thm) Define for $k \geq 0$

$$G_1(k) = \{\gamma \in GL_2(\mathcal{O}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \pi^k\}$$

(π, V) irred admis ∞-dim^l rep^n of G. Then $\exists$ unique $f \geq 0$ (the conductor of π) s.t.

$$V^{G_1(k)} = \{0\} \text{ if } k < f, \quad V^{G_1(f)} \text{ is 1-dim}^l.$$

We have $\mathrm{cond}(\omega_\pi) \leq f$.

Assume $\mathrm{cond}(\psi) = 0$ ($\Leftrightarrow \psi|_{\mathcal{O}} = 1$ & $\psi|_{\pi^{-1}\mathcal{O}} \neq 1$), & let $\xi \in \mathcal{K}(\pi)^{G_1(f)}$, $\xi \neq 0$. Then $\mathrm{supp}\,\xi \subseteq \mathcal{O}$, and $\xi(1) \neq 0$.

The fact that $\xi(1) \neq 0$ is the local analogue of the fact that $a_1 \neq 0$ for a newform $\underset{\substack{\uparrow \\ \text{i.e. a primitive form}}}{\sum a_n q^n}$

Pf Identify V with $\mathcal{K}(\pi)$, & write π for π'. Assume $\mathrm{cond}\,\psi = 0$.

Let $V_\infty = \{\xi \in V \mid \pi\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)\xi = \xi \quad \forall \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in GL_2(\mathcal{O})\}$

$$\xi \in V_\infty \Leftrightarrow \xi(x) = \psi(bx)\,\xi(ax) \quad \forall x \in F^*, a \in \mathcal{O}^*, b \in \mathcal{O}$$

$$\Leftrightarrow \xi \text{ is invt under } \mathcal{O}^* \text{ \& } \mathrm{supp}\,\xi \subseteq \mathcal{O}.$$

$V_\infty \neq \{0\}$ because, e.g., $\mathrm{char}_{\mathcal{O}^*} \in \mathcal{S}(F^*) \subseteq \mathcal{K}(\pi)$ is such a f^n.

If $\xi \in V_\infty$, then ξ is invt by $\left(\begin{smallmatrix} 1 & 0 \\ \pi^k\mathcal{O} & 1 \end{smallmatrix}\right)$ for some $k \gg 0$ [cog V is admissible!]

$$\therefore \xi \text{ is invt by } \underset{\uparrow}{\left\langle \left(\begin{smallmatrix} 1 & 0 \\ \pi^k\mathcal{O} & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{O}^* & \mathcal{O} \\ 0 & 1 \end{smallmatrix}\right) \right\rangle = G_1(k)}$$

$$\left[\text{hint for pf: } \begin{pmatrix} 1 & -b/(1+ab) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-a}{1+ab} & 1 \end{pmatrix} = \begin{pmatrix} 1/1+ab & 0 \\ 0 & 1+ab \end{pmatrix} \right]$$

In particular, ξ is invt by $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ if $a \equiv 1 \bmod \pi^k$

$$\Rightarrow k \geq \mathrm{cond}(\omega_\pi) \text{ if } \xi \neq 0.$$

So let $f = \min\{k \mid V^{G_1(k)} \neq \{0\}\} < \infty$. We have to prove $\dim V^{G_1(f)} = 1$.

Let $\xi \in V^{G_1(f)}$, & assume $\xi(1) = 0$, so $\mathrm{supp}\,\xi \subseteq \pi\mathcal{O}$.

Then $\xi' = \overset{\text{different } \pi\text{'s!!}}{\pi\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)}\xi$ is invt by $\left(\begin{smallmatrix} 1 & 0 \\ \pi^{f-1}\mathcal{O} & 1 \end{smallmatrix}\right)$ & $\mathrm{supp}\,\xi' \subseteq \mathcal{O}$.

$$\xi'(x) = \xi(\pi x)$$

So ξ' is invt by $\left\langle \begin{pmatrix}1&0\\\pi^{f-1}&1\end{pmatrix}, \begin{pmatrix}\mathcal{O}^*&\mathcal{O}\\0&1\end{pmatrix} \right\rangle$

Now $f \geq 1 \Rightarrow \xi'$ invt by $G_1(f-1) \Rightarrow \xi' = 0 \Rightarrow \xi = 0$ #

$f = 0 \Rightarrow \xi'$ invt under $\left\langle \begin{pmatrix}1&0\\\pi^{-1}\mathcal{O}&1\end{pmatrix}, \begin{pmatrix}\mathcal{O}^*&\mathcal{O}\\0&1\end{pmatrix} \right\rangle \supseteq SL_2(F)$

So $\xi' = 0$ as V is ∞-dim^l $\therefore \pi$ does not act via a character!

$\therefore \xi = 0$ again #. ∎

Hence $\xi(1) \neq 0$.

Finally, if $\xi, \xi' \in V^{G_1(f)}$ then some linear combination of ξ, ξ' vanishes at 1 $\Rightarrow$ (by preceeding bit) this linear combination is zero and ξ, ξ' are lin. dependent.

Hence $\dim V^{G_1(f)} = 1$. □

<u>Remark</u> The unique non-zero vector (up to scalar multiple) in $V^{G_1(f)}$ is called the <u>new vector</u> or <u>newvector</u>. In the Kirillov model it can be normalised by $\xi(1) = 1$.

<u>Example</u> Unramified $\pi(\mu_1, \mu_2)$. Then $f = 0$ and the newvector ξ is just the spherical vector $(G_1(0) = GL_2(\mathcal{O}))$

$$\text{Then } (T_\pi \xi)(x) = \sum_{b \bmod \pi} \psi(bx)\xi(\pi x) + \xi(x/\pi)\omega(\pi) \quad \left(\begin{pmatrix}\pi&b\\0&1\end{pmatrix} \ \& \ \underset{\begin{pmatrix}\pi&0\\0&\pi\end{pmatrix}\begin{pmatrix}\pi^{-1}&0\\0&1\end{pmatrix}}{\underset{\shortparallel}{\begin{pmatrix}1&0\\0&\pi\end{pmatrix}}} \right)$$

$$= q\,\xi(\pi x) + \xi(x/\pi)\,\omega(\pi) \text{ if } x \in \mathcal{O}$$

$$= \lambda \xi(x) \text{ for some } \lambda = q^{1/2}(\mu_1(\pi) + \mu_2(\pi))$$

λ is the eigenvalue of T_π

So if $A_m = q^m \xi(\pi^m)$, $\xi(1) = 1$

then $A_{m+1} = \lambda A_m - q\omega(\pi) A_{m-1}$

$$\Rightarrow \sum_{m=0}^{\infty} A_m q^{-ms} = \frac{1}{1 - \lambda q^{-s} + \omega(\pi) q^{1-2s}}$$

<u>ie A_m = eigenvalue of $\Psi[\pi^m, 1]$</u> $(= T_{\pi^m})$

Lecture 8
at 20th Feb '93
4:00pm

Recall this morning for ∞-dim^l (π, V) $\exists!$ $\xi \in \mathcal{K}(\pi)^{G_1(f)}$, $f = \text{cond}(\pi)$, $\xi(1) = 1$ (where $G_1(f) = \{ \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \bmod \pi f \}$)

ξ is invt under $\mathcal{O}^*$ & support $\subseteq \mathcal{O}$ $\therefore$ determined by $\{ \xi(\pi^n) \mid n \geq 0 \}$ $\updownarrow$ a_{p^n} in Fourier expansion.

e.g. $\pi(\mu_1, \mu_2) = \pi$ irred unram (i.e. μ_i unram & $\mu_1 \mu_2^{-1} \neq |\cdot|$)

$$\xi(\pi^n) = \begin{cases} 0 & n < 0 \\ q^{-n} A_n & n \geq 0 \end{cases}$$

Then $$\sum A_n q^{-ns} = \frac{1}{1 - \underset{\substack{\uparrow \\ \text{eigenvalue of } T_\pi}}{\lambda} q^{-s} + q^{1-2s}\omega(\pi)} = \frac{1}{(1-\mu_1(\pi)q^{1/2-s})(1-\mu_2(\pi)q^{1/2-s})}$$

$$A_n = \text{eigenvalue of } T_{\pi^n} = \Psi'[\pi^n, 1].$$

<u>Case when π has unramified central char, $f = 1$</u> ("$\Gamma_0(p)$")

Then ξ is invt by the <u>Iwahori subgp</u> $H = \{ \gamma \in GL_2(\mathcal{O}) \mid \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod \pi \}$ of G.

(H is usually denoted B but for us B is Borels)

$\eta = \left(\begin{smallmatrix} 0 & 1 \\ \pi & 0 \end{smallmatrix}\right)$ normalises H, so $\pi(\eta)\xi = c\xi$ for ξ a new vector.

$\eta^2 = \left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$ so $c^2 = \omega(\pi)$

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi^{-1} \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \eta$$

$$(\pi(w)\xi)(x) = \omega(\pi^{-1}) c\, \xi(\pi x) = c^{-1} \xi(\pi x)$$

$$GL_2(\mathcal{O}) = \bigcup_{a \bmod \pi} \underset{\substack{\| \\ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) w}}{\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}} H \cup H$$

If e = idempotent $e_{GL_2(\mathcal{O})}$; then $\pi(e)\xi = 0$ as it's $\in V^{GL_2(\mathcal{O})} = 0$.

$$\therefore 0 = \sum_{a \bmod \pi} \psi(ax) c^{-1} \xi(\pi x) + \xi(x)$$

$$\therefore \xi(\pi x) = -cq^{-1} \xi(x)$$
$$\therefore \xi(\pi^n) = (-cq^{-1})^n \quad \text{as } \xi(1) = 1$$

This is compatible with π of the form $\sigma(\mu_1, \mu_2)$ where μ_1, μ_2 are unramified. In fact, we'll see in a moment that this is the only possibility

This implies that the only (π, V) with $V^H \neq 0$ are subrep's of unramified principal series
(notice ξ not in $\mathcal{S}(F^*)$)

<u>Thm 17</u> If (π, V) irred admiss ∞-dim^l. TFAEquiv:

(i) π is not $\cong$ to a subrep^n of a $\mathcal{B}(\mu_1, \mu_2)$
(ii) $\mathcal{K}(\pi) = \mathcal{S}(F^*)$
(iii) $\forall v \in V$, $\forall n \gg 0$, $\int_{\pi^{-n}\mathcal{O}} \pi\begin{pmatrix}1&y\\0&1\end{pmatrix} v \, dy = 0$

(iv) matrix coeffts $<\pi(g)v, v'>$, $v \in V$, $v' \in \tilde{V}$, are cpct supported mod the centre of G.

<u>Def^n</u> π is <u>supercuspidal</u> if (iv) holds. (iv) is picked because it generalises.

<u>Idea of pf</u> (i) $\overset{\Leftarrow}{\cancel{\Leftrightarrow}}$ (ii) from how we've classified $\mathcal{K}(\pi)$ for π in $\mathcal{B}$.

Now $\mathcal{K}(\pi) \supsetneq \mathcal{S}(F^*)$
(↑ inv^t under B) $\Rightarrow$ quotient $\mathcal{K}(\pi)/\mathcal{S}(F^*)$ f.d. & we
get a 1-dim^l B-inv^t subgp as
B is soluble. quotient

So $\mathcal{K}(\pi)/\mathcal{S}(F^*) \twoheadrightarrow \mathbb{C}(\chi)$, χ char of B.
$\therefore \pi \to \mathrm{Ind}_B^G(\chi) = \mathcal{B}(\mu_1, \mu_2)$

So (i) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (iii) isn't so bad either $\xi \in \mathcal{K}(\pi)$; $\int_{\pi^{-n}\mathcal{O}} \pi\begin{pmatrix}1&y\\0&1\end{pmatrix} \xi(x) \, dy = \xi(x) \underbrace{\int_{\pi^{-n}\mathcal{O}} \psi(xy) \, dy}_{\neq 0 \Leftrightarrow x \in \pi^n\mathcal{O}}$

$\int$ vanishes $\Leftrightarrow$ ξ vanishes in nbd of 0. $\square$

He won't say anything more about supercuspidal rep^{n}s, which is a bit sad because they're the key.

NB if a global object contains a supercuspidal local object then the global object is cuspidal. That's why they're called supercuspidal.

Finally, something on L & ε factors.

§9 L, ε-factors & local Langlands

Noone has had the time to talk about Tate's thesis, & he'll have to assume it (GL₁ local L-f^ns or sthg).

(π, V) ∞ dim^c admiss irred. $\xi \in \mathcal{K}(\pi)$.

$$\text{Define } M(\xi, s) = \int_{F^*} \xi(x) |x|^{s-1/2} d^*x \in \mathbb{C}(q^{-s}) \text{ \& in fact } \in \mathbb{C}[q^{-s}] \text{ if } \xi \in \mathcal{S}(F^\times).$$

Now say $\xi = \xi^{new}$, new vector with $\xi(1) = 1$.

$$L(\pi, s) = M(\xi^{new}, s) = \sum_{n \geq 0} \xi^{new}(\pi^n) q^{n(1/2 - s)}$$

(*) NB here we're assuming cond $\psi = 0$. Then it's indpt of ψ.

e.g. unramified $\pi(\mu_1, \mu_2)$ ∞-dim^l:

$$L(\pi, s) = \frac{1}{(1 - \mu_1(\pi) q^{-s})(1 - \mu_2(\pi) q^{-s})}$$

Our local functional eqn:

$$\xi \in \mathcal{K}(\pi) \to \xi'(x) = \omega_\pi(x)^{-1} \pi(\underset{\binom{0\ 1}{1\ 0}}{\overset{\shortparallel}{w}}) \xi(x) \in \mathcal{K}(\tilde{\pi}), \quad \tilde{\pi} = \omega^{-1} \otimes \pi$$

Then $\dfrac{M(\xi, s)}{L(\pi, s)} \varepsilon(\pi, \psi, s) = \dfrac{M(\xi', 1-s)}{L(\tilde{\pi}, 1-s)}$ for certain $\varepsilon(\pi, \psi, s)$, indpt of ξ

$$\pi \longrightarrow L(\pi, s), \ \varepsilon(\pi, \psi, s)$$

NB
1) Can do this at all primes & multiply to get a global thing. Or something.
2) It all works at ∞ too. Need an understanding at ∞ to do local ⇝ global. However, local Langlands at ∞ is easy & he wants to talk about Local Langlands.

Now a rep^n ρ of WD_F on f.d. v.s. U. (ρ(Inertia) may be infinite)

$$\text{We get } \rho|_{W_F} : W_F \to GL(U) \text{ \& } \rho(N) \in \text{End}\, U$$

$\rho|_{W_F}$(Inertia) is finite

$$\text{s.t. } \rho(w) \rho(N) \rho(w^{-1}) = \|w\| \rho(N).$$

Now we get $L(\rho, s) = \det(1 - \rho(\Phi) q^{-s} \mid U^{N=0, I})^{-1}$

$$\text{If } \dim \rho = 1, \ \rho : F^\times \to \mathbb{C}^*, \ L(\rho, s) = \begin{cases} 1 & \rho \text{ ramified} \\ (1 - \rho(\pi) q^{-s})^{-1} & \end{cases}$$

~~(If we'd defined L-f^ns for σ we would write them down but we've not even defined them. Pf that εs are equal is harder than Ls)~~

Tates thesis gives us a defⁿ of $\varepsilon(\rho, \chi, s)$ for $\rho : F^* \to \mathbb{C}^*$

A deep thm of Deligne & Langlands (Langlands proved it first & noone has read his proof. Deligne subsequently proved it & lots of people have read Delignes proof because it has finite length.) implies that we can define ε for $\dim(\rho) > 1$ as well, agreeing with Tates ε & compatible with induction in degree 0.

<u>Thm 18</u> (Local Langlands conjecture for GL_2)

There exists a 1-1 correspondence between isom. classes

$$\begin{pmatrix} \text{2-dim}^{l} \text{ rep's of} \\ WD_F \text{, F-ss} \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{irred admis} \\ \text{reps of } G \end{pmatrix}$$

(↖ including 1-dimˡ ones)

$$\rho \longmapsto \pi(\rho)$$

s.t (i) If $\chi : F^* \to \mathbb{C}^*$ is a character, then $\pi(\rho \otimes \chi) \cong \pi(\rho) \otimes \chi \circ \det$
& also $\omega_{\pi(\rho)} = \det \rho$

(ii) $L(\pi(\rho), s) = L(\rho, s)$
$\varepsilon(---) = \varepsilon(---)$.

Examples: ① $\pi = \chi \circ \det$, 1-dim^l $\leftrightarrow$ $\rho = \chi|\cdot|^{1/2} \oplus \chi|\cdot|^{-1/2}$ (note ugly $|\cdot|^{1/2}$s. Theres serious problems with the normalisations)

② Reducible $\rho = \mu_1 \oplus \mu_2$ of $W_F^{ab} = F^*$.

$$\text{Then } \pi(\rho) = \pi(\mu_1, \mu_2).$$

$$\mu_1, \mu_2 \text{ unramified}, \quad L(\mu_1 \oplus \mu_2, s) = L(\mu_1, s) L(\mu_2, s)$$
$$= L(\pi(\mu_1, \mu_2), s)$$

②' ρ indecomposable but reducible

$$\rho = sp(2) \otimes \chi, \quad \rho\big|_{W_F} = \begin{pmatrix} \chi|\cdot| & 0 \\ 0 & \chi \end{pmatrix}, \ \rho(N) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\updownarrow$$
$$\text{special rep}^n \ \sigma(\mu_1, \mu_2) = \sigma(\chi|\cdot|, \chi)$$

Note thats why we have WD group: need to find a repⁿ which corresponds to the special case. $N \neq 0$ solves the problem

NB if we'd actually defined L-fs & ε-factos in this case then he might have said something about them. Pfs that εs are equal is harder than Ls, (as usual ?).

Finally

③ Irreducible ρ $\longleftrightarrow$ supercuspidal π (unsurprising as they're the only ones left!!)

Say $p \neq 2$. Then $\rho = \mathrm{Ind}_{E/F}(\theta)$, θ a char of E, E/F quadratic

The Weil repⁿ attaches to θ a repⁿ of G.

Hardest case: $p=2$. There are other irred ρ & other supercuspidals.
After partial results, the pf was completed by Kutzko, about 15 years ago.

He has ~ 5 minutes left, so he'll just mention the infinite case.

$\mathbb{R}, \mathbb{C}$; $W_{\mathbb{C}} = \mathbb{C}^*$, $W_{\mathbb{R}} = \langle \mathbb{C}^*, F \rangle$, $F^2 = -1$, $FzF^{-1} = \bar{z}$

W_F $\quad W_F^{ab} \cong F^*$.

We define L & ε-factors (they involve Γ)

Irreducible admissible $(\mathfrak{g}, K)$-modules are parameterised by semisimple 2-dimˡ rep's of W_F, $F = \mathbb{C}, \mathbb{R}$

$F = \mathbb{C}$ $\rho = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$; corresp. $(\mathfrak{g}, K)$-module is $\pi(\mu_1, \mu_2)$

$F = \mathbb{R}$ 1) ρ factors through $W_{\mathbb{R}}^{ab} = \mathbb{R}^*$ as $\mu_1 \oplus \mu_2$ $\rightarrow$ $\pi(\rho) = \pi(\mu_1, \mu_2)$

or 2) ρ irred. In this case, restriction to $\mathbb{C}$ is $z \mapsto \begin{pmatrix} (z/|z|)^s |z|^t & 0 \\ 0 & (\bar{z}/|z|)^s |z|^t \end{pmatrix}$

with $s \in \mathbb{Z}$, $t \in \mathbb{C}$

Then $\pi(\rho) = \sigma(\mu_1, \mu_2)$, $\mu_1 \mu_2^{-1}(x) = x^s \operatorname{sgn}(x)$, $t = s_1 + s_2$

Much easier!

III. GL_2 over a number field

John Coates

ecture 1
Tues 16th Feb '93
9:30am

There will be a survivors party 8:30pm a week Saturday @ 8:30pm @ John's house.
There is a sale of Birkhauser books outside, afterwards.

He wants today to talk about the classical theory of modular forms. He will be sticking to $\mathbb{Q}$ in this course, but a lot of the adelic approach goes through for a general no. field.

$\mathcal{H}$ = upper $\frac{1}{2}$ plane = $\{z = x+iy \mid y > 0\}$; $SL_2(\mathbb{Z})$ is a handy group
$q = e^{2\pi i z}$ is a handy notation.

Modular forms & stuff

Ex 1 $\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$. Note that if $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ then $\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta$.

Ex 2 $\varphi(z) = q \prod_{n=1}^{\infty} (1-q^n)^2 \prod_{n=1}^{\infty} (1-q^{11n})^2$

If $c \equiv 0 \ (11)$ then $\varphi\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 \varphi(z)$

If we write $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$, $\varphi(z) = \sum_{n=1}^{\infty} c(n) q^n$ then $\tau(n)$ & $c(n)$ have great arithmetical impotance.
Deligne attached reps of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ to Δ, reflecting properties of $\tau(n)$.
In ptic it showed $|\tau(p)| \leq 2p^{11/2}$

Also, $p - c(p)$ = # soln's of $y^2 + y \equiv x^3 - x^2 \bmod p$

If $E: y^2 + y = x^3 - x^2$ & $E_p = Ker\left(E(\overline{\mathbb{Q}}) \xrightarrow{p} E(\overline{\mathbb{Q}})\right)$

then we get
$$\begin{tikzcd} \mathbb{Q}(E_p) \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$
If $p \neq 5$ this is a nice non-ab ext^n.

Noone even knows which primes split etc. John hopes that the Langlands circle of ideas will solve this problem at some stage.

Notation $GL_2^+(\mathbb{R}) = \{\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ with $\det(\sigma) > 0\}$

It operates on $\mathcal{H}$ via $\sigma(z) = \dfrac{az+b}{cz+d}$. We will also define $j(\sigma, z) = cz+d$

Notation varies a lot in the books e.g. Shimura has a $(\det \sigma)^{k}$ factor in his j.

Note that for John's j we have $j(\sigma_1\sigma_2, z) = j(\sigma_1, \sigma_2 z) j(\sigma_2, z)$.

Now say $k \geq 1$ is an integer, & $f : \mathcal{H} \to \mathbb{C}$

Def: $(f|_k \sigma) : \mathcal{H} \to \mathbb{C}$ is defined by

$$(f|_k \sigma)(z) = f(\sigma z)\, j(\sigma, z)^{-k} (\det \sigma)^{k/2}$$

John thinks that this is the most usual def^n.

It works nice adelically: the centre of $GL_2^+(\mathbb{R})$ acts trivially.

Note that $f|_k(\sigma_1 \sigma_2) = (f|_k \sigma_1)|_k \sigma_2$. References: Shimura, Miyake books.

Now say $\Gamma \subseteq SL_2(\mathbb{Z})$ of finite index.

Define $V_k(\Gamma) = \{ f : \mathcal{H} \to \mathbb{C}$ such that (i) $f|_k \sigma = f \ \forall \sigma \in \Gamma$ (ii) f is holo. on $\mathcal{H} \}$

Cusps are $\mathbb{P}^1(\mathbb{Q})$ in this case.

Say k is even. Say $\alpha \in SL_2(\mathbb{Z})$.

Then $(f|_k \alpha)$ is invariant under $\begin{pmatrix} 1 & \text{mult of } N \\ 0 & 1 \end{pmatrix}$ some $N > 0$, $N = N(\alpha)$.

$$\therefore (f|_k \alpha)(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z/N}$$

This is the classical way of treating things.

f is *meromorphic* at the cusps if there are only a finite number of a_n with $n < 0$ which are non-zero.

f is *holomorphic* if $a_n = 0$ for $n < 0$.

The *cusp forms* are the holomorphic f with $a_0 = 0 \ \forall \alpha \in SL_2(\mathbb{Z})$

Notation $M_k(\Gamma) = \{ f \in V_k(\Gamma) ; f$ is holo @ cusps $\}$

$S_k(\Gamma) = \{ f \in M_k(\Gamma) : f$ vanishes @ cusps $\}$

eg $\Delta \in S_{12}(SL_2(\mathbb{Z}))$, $\varphi \in S_2(\Gamma_0(11))$

He'll really only be talking about cusp forms because they lie at the heart of the theory.

<u>Lemma</u> Assume $f \in V_k(\Gamma)$. Then $f(z)$ is a cusp form $\Leftrightarrow$ $|f(z)|(\operatorname{Im} z)^{k/2}$ is bounded on $\mathcal{H}$.
$\square$ (Proof in all the books.)

Now say $N \geq 1$

<u>Def:</u> $\Gamma_0(N) = \left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}$

<u>Def:</u> $\Gamma_1(N) = \left\{ \sigma \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N,\ d \equiv 1 \bmod N \right\}$

By using the Riemann-Roch theorem & stuff, & Riemann surfaces, we can deduce
<u>Fact</u> $M_k(\Gamma)$ & $S_k(\Gamma)$ are f.d./$\mathbb{C}$. In fact if $k \geq 2$ there's a nice formula for their dimension.

<u>Petersson inner product over $S_k(\Gamma)$</u>

$$(f,g) = \frac{1}{\operatorname{vol}(\Gamma \backslash \mathcal{H})} \int f(z)\overline{g(z)}\, y^k \,\frac{dx\,dy}{y^2}$$

He wants to talk about Hecke operators & diamond operators on $S_k(\Gamma)$

Note $0 \to \Gamma_1(N) \to \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$$

If $\alpha \in (\mathbb{Z}/N\mathbb{Z})^\times$ then $\exists\, \sigma_\alpha$ (this is a defn) $\in \Gamma_0(N)$ st $\sigma_\alpha \mapsto \alpha$.

If $f \in M_k(\Gamma_1(N))$ then define $\langle \alpha \rangle f = f|_k \sigma_\alpha$. Note - this is well-defined.

Hecke operators are difficult to explain classically. It's all tied up with double cosets, but it's a bit contorted. Adelically it's much easier. He'll just give the formula for Hecke operators.

Note that if $f \in M_k(\Gamma_1(N))$ then $\langle \alpha \rangle f \in M_k(\Gamma_1(N))$ & similarly for S_k.

Hence $M_k(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} M_k(\Gamma_1(N), \chi)$ & similarly S_k.

Anyway, back to Hecke operators. Here's a nice description, but it's not clear using this description why it fixes M_k or S_k.

Say $n \geq 1$

Defⁿ $$f|_k T_n = n^{k/2-1} \sum_{\substack{d|n \\ d>0 \\ ad=n \\ (a,N)=1}} \sum_{b=0}^{d-1} f|_k \sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$ Here $\Gamma = \Gamma_1(N)$

Fact $M_k(\Gamma)$ & $S_k(\Gamma)$ are f.d. / $\mathbb{C}$

$S_k(\Gamma)$ & $M_k(\Gamma)$ are stable under T_n

The T_n commute

Defⁿ $H_k(\Gamma) = \mathbb{Z}$-algebra in $\mathrm{End}(M_k(\Gamma))$ generated by all T_n, $n = 1, 2, \dots$

$h_k(\Gamma) = \mathbb{Z}$-algebra in $\mathrm{End}(S_k(\Gamma))$ generated by all T_n, $n = 1, 2, \dots$

Defⁿ If $(m,N)=1$, put $S_m = m^{k-2} \langle m \rangle$

Fact $S_m \in h_k(\Gamma)$ for all $(m,N)=1$

Fact T_n with $(n,N)=1$ are self-adjoint w.r.t. $(\,,\,)$. (so we can simultaneously diagonalise them)

Theorem Assume $f(z)$ is a non-zero elt of $S_k(\Gamma_1(N))$. Then TFAE:

(1) $f(z)$ is an eigenform for all the Hecke operators
(2) $\exists$ Dirichlet character χ mod N s.t. $f \in S_k(\Gamma_1(N), \chi)$
Moreover, if $f(z) = \sum_{n=1}^{\infty} c_n(f) q^n$ then $c_1(f) \neq 0$, & the following formal identity holds:

$$\text{Mellin Transform of } \underbrace{f = \sum_{n=1}^{\infty} \frac{c_n(f)}{n^s}}_{\text{def}^n \text{, if you like}} = c_1(f) \prod_p \left(1 - t_p(f) p^{-s} + \chi(p) p^{k-1-2s}\right)^{-1}$$

where $t_p(f) = c_p(f) / c_1(f)$

Moreover, when (1) & (2) hold, $T_n f = \dfrac{c_n(f)}{c_1(f)} f$

He finally in this lecture wants to talk about

Primitive forms in $S_k(\Gamma_1(N))$ (Atkin-Lehner)

NB this wasn't quite classical. It still has a nice adelic interpretation, though.

The crime that Atkin-Lehner committed was to call them newforms. It's a crime against the English language, especially to make it 1 word. John is plumping for primitive forms, which is what the French call them.

Define $S_k(\Gamma_1(N))^{old}$ or = oldforms If $M|N$ & d is a divisor of N/M

then define $[d] : S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$
$$f \mapsto f(dz)$$

& set $$S_k(\Gamma_1(N))^{old} = \sum_{\substack{M|N \,\&\\ M\neq N,\\ d|N/M}} S_k(\Gamma_1(M))\big|[d]$$

Now set $S_k(\Gamma_1(N))^{new}$ = the ~~newforms~~ new forms = the orthogonal complement of $S_k(\Gamma_1(N))^{old}$ under $(\,,)$.

Fact: $S_k(\Gamma_1(N))^{new}$ is stable under the whole of $h_k(\Gamma_1(N))$

Def^{n} A primitive form of level N is an elt of $S_k(\Gamma_1(N))^{new}$ which is an eigenform of $h_k(\Gamma_1(N))$ & s.t. $c_1(f)=1$.

If $f \in S_k(\Gamma_1(N))$ is an eigenform for $h_k(\Gamma_1(N))$

& $g \in S_k(\Gamma_1(M))$ is an eigenform for $h_k(\Gamma_1(N))$

& $f = \sum c_n(f)q^n$, $g = \sum c_n(g) q^n$,

then say $f \sim g$ if $\frac{c_p(f)}{c_1(f)} = \frac{c_p(g)}{c_1(g)}$ for all but a finite no. of p.

Thm (1) $\exists !$ primitive form in each equivalence class, say f. If $g \sim f$ then the level of f divides the level of g

(2) If M is any integer divisible by the level of f, then $\exists g$ of level M s.t. $g \sim f$. $\square$

ecture 2
Ved 17th Feb '93
9:30am

One of the big problems of the automorphic side of things is that there's no decent reference - either there's no proofs, or its too difficult for the beginner. John himself has been guided by a set of lecture notes of Richards, although he's normalised things differently.

Say $\mathbb{A}$ are the adèles of $\mathbb{Q}$. We'll stick with $\mathbb{Q}$ although one of the advantages of the adelic approach is that it goes through for any ~~field~~ number field.

Say $\mathbb{A}^\infty$ is the finite adeles (this crummy notation is due to Richard Taylor, so John takes no blame)

We have $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R}) = GL_2(\mathbb{A}^\infty).GL_2(\mathbb{R})$.

$$\begin{array}{c} GL_2(\mathbb{A}) \\ \cup\!\mid \\ GL_2(\mathbb{Q}) \end{array}$$ embedded diagonally.

If F is a field, set $B(F) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL_2(F) \right\}$ & $B'(F) = B(F) \cap SL_2(F)$.

Define $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Then $GL_2(\hat{\mathbb{Z}}) \subseteq GL_2(\mathbb{A}^\infty)$.

We need various little results & we'll put them together in a big lemma.

Lemma (i) $GL_2(\mathbb{Q}_p) = B(\mathbb{Q}_p)\, GL_2(\mathbb{Z}_p)$
$SL_2(\mathbb{Q}_p) = B'(\mathbb{Q}_p)\, SL_2(\mathbb{Z}_p)$
(ii) $\mathbb{A}^\times = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times \mathbb{R}^\times_{>0}$
(iii) $GL_2(\mathbb{A}^\infty) = B(\mathbb{Q})\, GL_2(\hat{\mathbb{Z}})$
$SL_2(\mathbb{A}^\infty) = B'(\mathbb{Q})\, SL_2(\hat{\mathbb{Z}})$
(iv) If $N \geq 1$ then $SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$, surjective.
(v) (Strong approx. for SL_2) $SL_2(\mathbb{Q})\, SL_2(\mathbb{R})$ is dense in $SL_2(\mathbb{A})$

Pf of (v) Every open subgp contains, for some N, the subgroup $V_N = \{ g \in GL_2(\hat{\mathbb{Z}}) \mid g \equiv I \bmod N \}$
So it suffices to show that $SL_2(\mathbb{Q}).V_N = SL_2(\mathbb{A}^\infty).V_N \quad \forall N$.

But $SL_2(\mathbb{Z})V_N = SL_2(\hat{\mathbb{Z}})$, as $SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\hat{\mathbb{Z}})/V_N = SL_2(\mathbb{Z}/N\mathbb{Z})$

so $SL_2(\mathbb{Q})\, V_N \supseteq B'(\mathbb{Q})\, SL_2(\hat{\mathbb{Z}}) = SL_2(\mathbb{A}^\infty)$. $\square$

(vi) If U is any open subgp of $GL_2(\mathbb{A}^\infty)$ s.t. $\det U = \hat{\mathbb{Z}}^\times$, then we have
$GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\, U\, GL_2^+(\mathbb{R})$

Pf of (vi) Follows quickly from (v). If $g \in GL_2(\mathbb{A})$, then $\det g = \alpha (\det u) \beta$, $\alpha \in \mathbb{Q}^\times, u \in U, \beta \in \mathbb{R}_{>0}$ by part (ii).

So $\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}^{-1} g u^{-1}\begin{pmatrix}\beta&0\\0&1\end{pmatrix} \in SL_2(\mathbb{A})$

& it's an elt of $\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}^{-1} g\, U\begin{pmatrix}\beta&0\\0&1\end{pmatrix} \cap SL_2(\mathbb{A})$ which is open & non-empty

$$\therefore \begin{pmatrix}\alpha&0\\0&1\end{pmatrix}^{-1} g u^{-1}\begin{pmatrix}\beta&0\\0&1\end{pmatrix} = \gamma v \delta, \quad \gamma\in SL_2(\mathbb{Q}),\ v\in U\cap SL_2(\mathbb{A}^\infty),\ \delta\in SL_2(\mathbb{R})$$

(by (v)).

$$\therefore g = \underbrace{\left(\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}\gamma\right)}_{\in GL_2(\mathbb{Q})}\underbrace{(vu)}_{\in U}\underbrace{\left(\delta\begin{pmatrix}\beta&0\\0&1\end{pmatrix}\right)}_{\in GL_2^+(\mathbb{R})} \qquad \square$$

(vii) If U is any open subgp of $GL_2(\mathbb{A}^\infty)$, & suppose $\mathbb{A}^\times = \coprod_{i=1}^{r} \mathbb{Q}^\times t_i \det(U)\mathbb{R}^\times_{>0}$. Then if we choose $g_i\in GL_2(\mathbb{A}^\infty)$ s.t. $\det g_i = t_i$, $1\le i\le r$, we have

$$GL_2(\mathbb{A}) = \coprod_{i=1}^{r} GL_2(\mathbb{Q})\, g_i\, U\, GL_2^+(\mathbb{R})$$

This is just a generalisation of (vi). $\square$

Def: $U_1(N) = \left\{ g\in GL_2(\hat{\mathbb{Z}}) \,\middle|\, g\equiv\begin{pmatrix}*&*\\0&1\end{pmatrix} \bmod N \right\} \subseteq GL_2(\mathbb{A}^\infty)$

Remarks: (i) $U_1(N)\cap GL_2^+(\mathbb{Q}) = \Gamma_1(N)$
(ii) $\det(U_1(N)) = \hat{\mathbb{Z}}^\times$

Conclusion: If $\xi\in GL_2(\mathbb{A}) = GL_2(\mathbb{Q})\,U_1(N)\,GL_2^+(\mathbb{R})$ then $\xi = \gamma u w$

$\gamma\in GL_2(\mathbb{Q})$
$u\in U_1(N)$
$w\in GL_2^+(\mathbb{R})$.

This decomposition is not unique.

Now say $f\in S_k(\Gamma_1(N))$, $k\ge1$. We will begin the translation.

Def: $\varphi_f : GL_2(\mathbb{A}) \to \mathbb{C}$

$$\varphi_f(\xi) = \varphi_f(\gamma u w) = f(wi)\,j(w,i)^{-k}(\det w)^{k/2}$$

$\uparrow$ Here $w\in GL_2^+(\mathbb{R})$ is acting on $i\in\mathcal{H}$, $i = e^{\pi i/2}$

NB (i) not clear that it's well-defined yet
(ii) It sort of doesn't matter what power of $(\det w)$ you put. It all boils down to personal taste. John likes $k/2$ best.

Let's deal with well-definedness. Say $\xi = \gamma_1 u_1 w_1 = \gamma_2 u_2 w_2$, $\gamma_i\in GL_2(\mathbb{Q})$

$u_i\in U_1(N)$
$w_i\in GL_2^+(\mathbb{R})$

Then $\delta = \gamma_2^{-1}\gamma_1 = \underset{GL_2(\mathbb{A}^\infty)}{\underset{\cap}{u_2u_1^{-1}}}\ \underset{GL_2^+(\mathbb{R})}{\underset{\cap}{w_2w_1^{-1}}}$

So in $GL_2(\mathbb{A}^\infty)$, $\delta = u_2u_1^{-1} \Rightarrow \delta\in U_1(N)$
& in $GL_2(\mathbb{R})$, $\delta = w_2w_1^{-1} \Rightarrow \det\delta>0$.

So $\delta \in GL_2^+(\mathbb{Q}) \cap U_1(N) = \Gamma_1(N)$ (slight abuse of notation - we're identifying δ with its finite part or its infinite part)

Hence $f|_k \delta = f$ as $f \in S_k(\Gamma_1(N))$.

ie $f(\delta(z))\, j(\delta, z)^{-k} = f(z)$

So if $z = w_1 i$, $\delta = w_2 w_1^{-1}$

$$\begin{aligned} & f(w_2 i)\, j(w_2, i)^{-k} (\det w_2)^{k/2} \\ = \ & f(\delta w_1 i)\, j(\delta w_1, i)^{-k} (\det \delta w_1)^{k/2} \\ = \ & f(w_1 i)\, j(w_1, i)^{-k} (\det w_1)^{k/2} \end{aligned}$$

Hence φ_f is indeed well-defined.

(Recall $GL_2^+(\mathbb{R})$ acts on $\mathcal{H}$, & also that $j(\sigma_1\sigma_2, z) = j(\sigma_1, \sigma_2 z) \times j(\sigma_2, z)$)

The stability subgp of i in $GL_2^+(\mathbb{R})$ is $\mathbb{R}^\times SO_2(\mathbb{R})$.
The stability subgp of i in $SL_2(\mathbb{R})$ is $SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \ u^2+v^2 = 1 \right\}$

Set $U_\infty = \mathbb{R}^\times SO_2(\mathbb{R})$

Then we get an identification
$$\begin{aligned} GL_2^+(\mathbb{R})/U_\infty &\longleftrightarrow \mathcal{H} \\ \beta U_\infty &\longmapsto \beta i \end{aligned}$$

<u>Properties of φ_f</u>

1) φ_f is left invt by $GL_2(\mathbb{Q})$ (as $\xi = \gamma u w$ & $\overset{f(wi)j(w,i)^{-k}(\det w)^{k/2}}{\varphi_f(\xi)}$ doesn't depend on γ)

2) φ_f is right invt by $U_1(N)$ (as $\varphi_f(\xi)$ doesn't depend on u either)

3) For all $\tau \in U_\infty$, we have $\varphi_f(\xi\tau) = \varphi_f(\xi)\, j(\tau, i)^{-k} (\det \tau)^{k/2}$ (easy by defn) just throwing in

(Recall $\xi = \gamma u w$, $\gamma \in GL_2(\mathbb{Q})$, $u \in U_1(N)$, $w \in GL_2^+(\mathbb{R})$)

4) Fix $g \in GL_2(\mathbb{A})$. Take any $z \in \mathcal{H}$, & pick any $w \in GL_2^+(\mathbb{R})$ s.t. $w(i) = z$.
Then the function
$$z \longmapsto \varphi_f(gw)\, j(w, i)^{k} (\det w)^{-k/2}$$ is well-defined by 3)
and in fact it's holomorphic as a function of z, for any $g \in GL_2(\mathbb{A}^\infty)$.

This is because if $g \in GL_2(\mathbb{A}^\infty)$ then $g = \underset{\substack{\uparrow \\ \mathbb{Q}\text{ bit}}}{\gamma} u \underset{\substack{\uparrow \\ \text{infinite bit}}}{\gamma^{-1}}$, so $gw = \gamma u (\gamma^{-1} w)$

Hence $\varphi_f(gw)\,j(w,i)^k(\det w)^{-k/2} = f(\gamma^{-1}wi)\,j(\gamma^{-1}w,i)^{-k}\det(\gamma^{-1}w)^{k/2}\,j(\overset{w,i}{\cancel{\quad}})^k(\det w)^{-k/2}$

$$= f(\gamma^{-1}\underbrace{wi}_{=z})\,j(\gamma^{-1},wi)^{-k}\det(\gamma^{-1})^{k/2}$$

$$= (f|_k\gamma^{-1})(z) \text{ which is holomorphic on } \mathcal{H}.$$

5) Fix $g\in GL_2(\mathbb{A}^\infty)$. Then the functions on $GL_2^+(\mathbb{R})$ given by $w\mapsto\varphi_f(gw)$ is bounded on $GL_2^+(\mathbb{R})$.

Pf We just showed $\varphi_f(gw)\,j(w,i)^k(\det w)^{-k/2} = (f|_k\gamma^{-1})(z)$, $z=wi$.

$$\text{Hence } |\varphi_f(gw)| = |(f|_k\gamma^{-1})(z)|\cdot\left|\frac{\det w}{|j(w,i)^2|}\right|^{k/2}$$

$$\text{If } z=wi \text{ then } \operatorname{Im} z = \frac{\det w}{|j(w,i)|^2}.$$

$$\text{Hence } \varphi_f(gw) = |(f|_k\gamma^{-1})(z)|\,|\operatorname{Im} z|^{k/2}$$

Now $f|_k\gamma^{-1}$ is a cusp form for $\gamma\Gamma_1(N)\gamma^{-1}\cap SL_2(\mathbb{Z})$

& so by a famous propertys of cusp form, $\varphi_f(gw)$ is indeed bounded.

Next time we'll show why properties 1).. 5) in some sense characterise φ_f, in that any φ with these properties is φ_f for some f.

ecture 3
Thu 18th Feb '93
9:30am

Recall $N\geq 1$, $f\in S_k(\Gamma_1(N))$, $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty)\times GL_2(\mathbb{R})$

$$U_1(N) = \left\{ g\in GL_2(\hat{\mathbb{Z}}) \,\middle|\, g\equiv\begin{pmatrix} * & * \\ 0 & 1\end{pmatrix} \bmod N\right\}$$

$$f\longmapsto\varphi_f : GL_2(\mathbb{A})\to\mathbb{C}$$

1) φ_f invt on left by $GL_2(\mathbb{Q})$
2) φ_f invt on right by $U_1(N)$
3)
3) $\varphi_f(\xi\tau) = \varphi_f(\xi)\,j(\tau,i)^{-k}(\det\tau)^{k/2}$ $\forall\,\tau\in U_\infty = \mathbb{R}^\times SO_2(\mathbb{R})$

4) Fix $g\in GL_2(\mathbb{A}^\infty)$. Then the function $\mathcal{H}\to\mathbb{C}$ given by

$$z = wi\longmapsto\varphi(gw)\,j(w,i)^k(\det w)^{k/2}, \quad w\in GL_2^+(\mathbb{R})$$

is holomorphic.

5) Fix $g\in GL_2(\mathbb{A}^\infty)$. Then the function on $GL_2^+(\mathbb{R})$ given by $w\mapsto\varphi(gw)$ is bounded.

<u>Lemma</u> Given $\varphi\ GL_2(\mathbb{A}) \to \mathbb{C}$ satisfying 1)-5), $\exists! f \in S_k(\Gamma_1(N))$ s.t. $\varphi = \varphi_f$.

<u>Pf</u> Given φ, define $f(z) = \varphi(w) j(w,i)^k (\det w)^{-k/2}$, $z = wi$, $w \in GL_2^+(\mathbb{R})$

Holomorphic map $\mathcal{H} \to \mathbb{C}$ by 4)
Well-defined by 3)

$\forall \alpha \in \Gamma_1(N)$ we ~~have~~ want $f|_k\alpha = f$.

But note $\alpha^{-1} \in \Gamma_1(N) \Rightarrow (\underset{\underset{f}{\uparrow}}{\alpha^{-1}}, \underset{\underset{\infty}{\uparrow}}{1}) \in U_1(N)$

Note $\varphi((\alpha^{-1},1)w) = \varphi(w(\alpha^{-1},1)) = \varphi(w)$ by 2)

$\shortparallel$

$\varphi(\alpha^{-1}(1,\alpha w)) = \varphi((1,\alpha w))$ by 1)

$= f(\alpha w i) j(\alpha w, i)^{-k} (\det \alpha w)^{+k/2}$

$= f(\alpha z) j(\alpha w, i)^{-k} (\det \alpha w)^{+k/2}$

Hence $f(z) j(w,i)^{-k} (\det w)^{+k/2} = \varphi(w) = f(\alpha z) j(\alpha w, i)^{-k} (\det \alpha w)^{+k/2}$

$\therefore f(z) = f(\alpha z) j(\alpha, z)^{-k} (\det \alpha)^{k/2}$

$= (f|_k \alpha)(z)$

Finally we want to show that f is cuspidal.

Note that $|f(z)| = |\varphi(w)| \left(\frac{|j(w,i)|^2}{\det w} \right)^{k/2}$

$z = wi \Rightarrow \operatorname{Im} z = \frac{\det w}{|j(w,i)|^2}$

$\therefore |f(z)| = |\varphi(w)| (\operatorname{Im} z)^{-k/2}$

$\Rightarrow f(z)$ is a cusp form by 5) □

<u>Defn</u> Let $\underset{\underset{\text{Gothic } S}{\uparrow}}{\mathfrak{S}_k}$ be the vector space of all functions $\varphi : GL_2(\mathbb{A}) \to \mathbb{C}$ satisfying 1) to 5), except that we weaken 2) to 2)':

2)': φ is right-invariant under some open subgp $U = U(\varphi)$ of $GL_2(\mathbb{A}^\infty)$

So we've encapsulated $S_k(\Gamma_1(N))$ for all N, & other stuff too!

Note There is not an action of $GL_2(\mathbb{A})$ on $\mathcal{S}_k$ by right multiplication, e.g.

if $g \in GL_2(\mathbb{A})$, try defining $\varphi_g(\xi) = \varphi(\xi g)$

Then $\varphi_g(\xi\tau) = \varphi(\xi\tau g)$ & to check property 3) we have to commute τ & g.

In general we can't commute them, so 3) does not hold.

If $g \in GL_2(\mathbb{A}^\infty)$, we can commute them.

Hence $\mathcal{S}_k$ is a $GL_2(\mathbb{A}^\infty)$-module under right translation.

Lemma (it's really a remark - he should avoid name-inflation!)

$$\begin{aligned} S_k(\Gamma_1(N)) &\cong \mathcal{S}_k^{U_1(N)} \\ f &\mapsto \varphi_f \end{aligned}$$

So there's some sort of admissibility condition here, like in Tony's lectures.

Say M is a v.s. $/\mathbb{C}$ with an action of $GL_2(\mathbb{A}^\infty)$

Def^n We say that M is admissible if

(1) M^U is f.d. $/\mathbb{C}$ for every open subgp U of $GL_2(\mathbb{A}^\infty)$
(2) The stabilizer of any $z \in M$ is open in $GL_2(\mathbb{A}^\infty)$

Lemma $\mathcal{S}_k$ is an admissible $GL_2(\mathbb{A}^\infty)$-module.

Pf Condition (2) is obvious by def^n: stab $\varphi \supseteq U(\varphi)$

(1) Say U is any open subgp of $GL_2(\mathbb{A}^\infty)$

Then $\exists g_1, \dots, g_r \in GL_2(\mathbb{A}^\infty)$, s.t. $GL_2(\mathbb{A}) = \coprod_{j=1}^{r} GL_2(\mathbb{Q}) g_j U GL_2^+(\mathbb{R})$

(by (vii) on page III.7)

For $1 \le j \le r$, define $\Gamma_j = g_j U g_j^{-1} \cap GL_2^+(\mathbb{Q})$

Define $\theta : \mathcal{S}_k^U \longrightarrow \prod_{j=1}^{r} S_k(\Gamma_j)$

$\theta(\varphi) = (f_1, \dots, f_r)$, where, for $1 \le j \le r$,

$f_j(z) = \varphi(g_j w) j(w,i)^k (\det w)^{-k/2}$; $z = wi$, $w \in GL_2^+(\mathbb{R})$, $f_j : \mathcal{H} \to \mathbb{C}$

Check $f_j \in S_k(\Gamma_j)$.

We need to check θ is injective; then we'll be home.

Say $\theta(\varphi)=0$. We need to show $\varphi=0$.

Say $\xi\in GL_2(\mathbb{A})$. We need $\varphi(\xi)=0$.

But $\xi=\gamma g_j u w$ for some j, $\gamma\in GL_2(\mathbb{Q})$, $u\in U$, $w\in GL_2^+(\mathbb{R})$

Then $\varphi(\xi)=\varphi(g_j u w)=\varphi(g_j w)=0$ as $f_i=0$. $\square$

In fact we can also show that θ is surjective. It's just a generalisation of the pf that $S_k(\Gamma_1(N))\cong \mathfrak{S}_k^{U_1(N)}$.

John now wants to talk about the interpretation of the classical diamond & Hecke actions in this new setting.

<u>Action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathfrak{S}_k^{U_1(N)}$</u>

Define $U_0(N)=\left\{g\in GL_2(\hat{\mathbb{Z}})\ \middle|\ g\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \bmod N\right\}$

$$\text{Then } 0\to U_1(N)\to U_0(N)\to \underset{\substack{\text{\rotatebox{90}{$\cong$}}\\ (\hat{\mathbb{Z}}/N\hat{\mathbb{Z}})^\times}}{(\mathbb{Z}/N\mathbb{Z})^\times}\to 0$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto d \bmod N$$

$$\tilde{\sigma}_d \longmapsto d \bmod N$$

<u>Lemma</u> If $f\in S_k(\Gamma_1(N))$, then $\varphi_{f|_k\sigma_d}=\tilde{\sigma}_d^{-1}\varphi_f$. $\square$ Easy lemma.

Hence, as in the classical case, $\mathfrak{S}_k^{U_1(N)}=\bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times}\mathfrak{S}_k^{U_1(N),\chi}$

Now let's look at the action of the centre, $\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix} : a\in(\mathbb{A}^\infty)^\times\right\}$

Lemma If $\chi:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}^\times$ then $\tilde{\chi}:\mathbb{A}^\times\to\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}\xrightarrow{\sim}\hat{\mathbb{Z}}^\times\twoheadrightarrow(\mathbb{Z}/N\mathbb{Z})^\times\xrightarrow{\chi}\mathbb{C}^\times$ is the associated Grossencharacter.

<u>Lemma</u> If $\varphi\in\mathfrak{S}_k^{U_1(N),\chi}$, then $\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\varphi=\tilde{\chi}(z)\varphi\ \ \forall z\in(\mathbb{A}^\infty)^\times$

Note that this is the advantage of the $(\det)^{k/2}$ factor that John has gone for - there's no extra (z)s floating around.

Proof Say $z=\alpha\eta$, $\alpha\in\mathbb{Q}^\times$, $\alpha>0$ WLOG, $\eta\in\hat{\mathbb{Z}}^\times$.

Then $\eta=\alpha^{-1}z\in\hat{\mathbb{Z}}^\times$ $\therefore$ $\exists\, d\in\mathbb{Z}$ s.t. $d\equiv\alpha^{-1}z \bmod N$.

$$\therefore \begin{pmatrix}z&0\\0&z\end{pmatrix}=\begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix}\tilde{\sigma}_d\, v, \quad v\in U_1(N)$$

$$\text{Then } \left(\left(\begin{smallmatrix}z&0\\0&z\end{smallmatrix}\right)\varphi\right)(\xi)=\varphi\left(\xi\left(\begin{smallmatrix}z&0\\0&z\end{smallmatrix}\right)\right)=\varphi\Big(\xi\underset{\substack{\text{finite}\\\text{embedding}}}{\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)}\tilde{\sigma}_d\, v\Big)$$

$$=\varphi\left(\xi\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\tilde{\sigma}_d\right)=(\tilde{\sigma}_d\varphi)\Big(\xi\underset{\substack{\uparrow\\\text{finite}\\\text{embedding}}}{\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)}\Big)$$

$=$ [illegible]

$$=(\tilde{\sigma}_d\varphi)\Big(\underset{\text{diag}}{\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)}\underset{\text{inf}}{\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)^{-1}}\xi\Big)$$

$$=(\tilde{\sigma}_d\varphi)\Big(\xi\underset{\substack{\uparrow\\\text{infinite bit}}}{\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)^{-1}}\Big)$$

$$=(\tilde{\sigma}_d\varphi)(\xi) \text{ by } 3)$$

$$=\chi(d)^*\varphi(\xi) \text{ as } \varphi\in S_k^{U_1(N),\chi} \text{ (\& def of this)}$$

$$=\tilde{\chi}(z)^*\varphi(\xi) \quad \square$$

Hecke operators on the S^k

Say U_1, U_2 are cpct open subgps of $GL_2(\mathbb{A}^\infty)$

Then $[U_1gU_2]: S_k^{U_2}\longrightarrow S_k^{U_1}$

defined thus: $U_1gU_2=\coprod_{j=1}^{r} g_jU_2$

Then $[U_1gU_2](\varphi)=\sum_{j=1}^{r} g_j\varphi$.

ecture 4
Fri. 19th Feb '93
9:30am

"Is David Reid here?" 11 people want sandwiches tomorrow. He'll ask again at the end.

I think he said he'll polish off 2 thms today.

Recall Hecke operators. Recall $\mathcal{S}_k = \{ \varphi : GL_2(\mathbb{A}) \to \mathbb{C} \mid 1), 2)', 3), 4), 5) \}$

Hecke operators : $U_1, U_2 \subseteq GL_2(\mathbb{A}^\infty)$, $g \in GL_2(\mathbb{A}^\infty)$

$$\text{Then } [U_1 g U_2](\varphi) = \sum_{j=1}^{\wedge} g_j \varphi, \text{ where } U_1 g U_2 = \coprod_{j=1}^{\wedge} g_j U_2$$

$$[U_1 g U_2] : \mathcal{S}_k^{U_2} \to \mathcal{S}_k^{U_1}.$$

Now say $U_1 = U_2 = U_1(N)$, $N \geq 1$

If p is a prime, define $\pi_p \in \mathbb{A}^{\infty\times}$ by $(\pi_p)_q = 1$ $q \neq p$
$(\pi_p)_p = p$

<u>Def</u> $\tilde{S}_p = [U_1(N) \begin{pmatrix} \pi_p & 0 \\ 0 & \pi_p \end{pmatrix} U_1(N)]$.

Of course $\begin{pmatrix} \pi_p & 0 \\ 0 & \pi_p \end{pmatrix} \in$ centre of $GL_2(\mathbb{A}^\infty)$

$$\therefore U_1(N) \begin{pmatrix} \pi_p & 0 \\ 0 & \pi_p \end{pmatrix} U_1(N) = \begin{pmatrix} \pi_p & 0 \\ 0 & \pi_p \end{pmatrix} U_1(N)$$

Lemma If $\varphi \in \mathcal{S}_p^{U_1(N)}$ then $\tilde{S}_p(\varphi) = \tilde{\sigma}_p^{-1} \cdot \varphi$ for $(p, N) = 1$.

Pf Recall $U_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times \to 0$ &
$\tilde{\sigma}_p \longmapsto p \bmod n$

So we have $\pi_p = p(1/p, 1/p, \ldots, 1/p, \underset{\substack{\uparrow \\ \text{place } v=p}}{1}, 1/p, \ldots, 1/p, \ldots)$

Pick $d \in \mathbb{Z}$ s.t. $dp \equiv 1 \bmod N$. Then $\tilde{S}_p \varphi = \tilde{\sigma}_p^{-1} \varphi$ (easy check) □

Now say p is any prime again.

Define $\tilde{T}_p = [U_1(N) \begin{pmatrix} \pi_p & 0 \\ 0 & 1 \end{pmatrix} U_1(N)]$. Recall $S_k(\Gamma_1(N)) \xrightarrow{\sim} \mathcal{S}_k^{U_1(N)}$
$f \longmapsto \varphi_f$

We can compare the action of $\tilde{T}_p$ with that of T_p.

Here's where John's action normalisation looks strange. Of course, it's a no-win situation - if this looked right then something else would look wrong.

Propⁿ $\forall f \in S_k(\Gamma_1(N))$, we have $(p^{k/2-1}\tilde{T}_p)(\varphi_f) = \varphi_{f|_k T_p}$

Example

$$f = \Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n\ ,\ k=12.\ \text{Then } \tilde{T}_p(\varphi_\Delta) = \frac{\tau(p)}{p^5}\varphi_\Delta$$

Pf of propⁿ Firstly, for $j \in \mathbb{Z}$, define $\alpha_j \in \mathbb{A}^\infty$ by $(\alpha_j)_v = 0\ \forall v \neq p$
$(\alpha_j)_p = j$

Write $B = U_1(N)\begin{pmatrix}\pi_p & 0\\ 0 & 1\end{pmatrix}U_1(N)$

Fact

(i) If $(p,N)=1$ then $B = \coprod_{j=0}^{p-1}\begin{pmatrix}\pi_p & \alpha_j\\ 0 & 1\end{pmatrix}U_1(N) \amalg \begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}U_1(N)$

(ii) If $p|N$ then $B = \coprod_{j=0}^{p-1}\begin{pmatrix}\pi_p & \alpha_j\\ 0 & 1\end{pmatrix}U_1(N)$

Convince yourselves. Away from p things look easy. At p its the classical decomposition of double cosets for the usual T_p, essentially. John claims that Tony said something about this (?)

Let's do $(p,N)=1$. Write $\eta_0,\dots,\eta_p$ for $\begin{pmatrix}\pi_p & \alpha_j\\ 0 & 1\end{pmatrix}$ $0\le j\le p-1$ & $\begin{pmatrix}1 & 0\\ 0 & \pi_p\end{pmatrix}$.

Then $(\tilde{T}_p\varphi_f)(\xi) = \sum_{j=0}^{p}(\eta_j\varphi_f)(\xi) = \sum_{j=0}^{p}\varphi(\xi\eta_j)$

Now $\xi \in GL_2(\mathbb{A}) = GL_2(\mathbb{Q})U_1(N)GL_2^+(\mathbb{R})$

Say $\xi = \gamma u w$. Then $\xi\eta_j = \gamma u \eta_j w$

We want to understand $u\eta_j$. But $u\eta_j \in B$

$\therefore u\eta_j = \eta_{\sigma(j)}u'$, $u' = u'(j,u) \in U_1(N)$
σ a permutation of $\{0,\dots,p\}$

$\therefore \xi\eta_j = \gamma\eta' u' w$, $\eta' = \eta_{\sigma(j)}$

~~Hence $(\tilde{T}_p\varphi_f)(\xi) = \sum_{j=0}^{p}\varphi(\gamma\eta_j u' w) = \sum_{j=0}^{p}\varphi(\eta_j w)$. Understand $\eta' = \eta_j$~~

Case 1 $\eta' = \begin{pmatrix}\pi_p & \alpha_h\\ 0 & 1\end{pmatrix}$, $h = 0,1,\dots,p-1$

Then $\xi\eta_j = \gamma\eta' u' w$

$$= \gamma\underset{\substack{\uparrow\\ \text{diagonal}}}{\begin{pmatrix}p & h\\ 0 & 1\end{pmatrix}}\Big(\underset{\substack{\uparrow\\ \text{finite}}}{zu'},\ \underset{\substack{\uparrow\\ \text{infinite}}}{\begin{pmatrix}p^{-1} & -hp^{-1}\\ 0 & 1\end{pmatrix}w}\Big)$$

$$\xi\eta_j = \gamma\eta_j' u' w = \gamma\begin{pmatrix} p & h \\ 0 & 1 \end{pmatrix}\Big(zu', \underbrace{\begin{pmatrix} p^{-1} & -hp^{-1} \\ 0 & 1 \end{pmatrix} w}_{w'}\Big)$$

where $(z)_q = \begin{pmatrix} p^{-1} & -hp^{-1} \\ 0 & 1 \end{pmatrix}, q \neq p$ & $(z)_p = 1$. Note the fact that $z \in U_1(N)$

$$\therefore\ \varphi_f(\xi\eta_j) = f(w'i)\, j(w',i)^{-k} (\det w')^{k/2}$$

$$= \varphi_{f|_k \begin{pmatrix} 1 & -h \\ 0 & p \end{pmatrix}}(\xi)$$

<u>Case 2</u> $\eta' = \begin{pmatrix} 1 & 0 \\ 0 & \pi_p \end{pmatrix}$, $(p,N) = 1$

Then $\xi\eta_j = \gamma\eta_j' u' w = \gamma\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\sigma_p^{-1}\Big(zu', \overbrace{\sigma_p \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} w}^{w'}\Big)$

where $(z)_q = \sigma_p \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$ if $q \neq p$ & $(z)_p = \sigma_p$

Note $z \in U_1(N)$ again

$$\therefore\ \varphi_f(\xi\eta_j) = f(w'i)\, j(w',i)^{-k} (\det w')^{k/2}$$

$$= \varphi_{f|_k \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}(\xi)$$

Hence $(\tilde{T}_p(\varphi_f))(\xi) = \sum_{j=0}^{p} \varphi(\xi\eta_j) = \sum_{\substack{\cancel{j=0} \\ h=0}}^{p-1} \varphi_{f|_k \begin{pmatrix} 1 & -h \\ 0 & p \end{pmatrix}} + \varphi_{f|_k \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$

Next note $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & +h \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & h-p \\ 0 & p \end{pmatrix}$

so this is $\sum_{j=0}^{p-1} \varphi_{f|_k \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}} + \varphi_{f|_k \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}$

$= p^{1-k/2} \varphi_{f|_k T_p}$ $\square$ ($p|N$ slightly easier)

Now we'll understand S_k a bit more as a rep^n of $GL_2(\mathbb{A}^\infty)$ (It'll turn out to be a direct sum of irreducible admis rep's)

There is an inner product on S_k. It doesn't make S_k into a Hilbert space or anything.

Say $\varphi_1, \varphi_2 \in S_k$. They behave well under right translation by $U_\infty = \mathbb{R}^\times SO_2(\mathbb{R})$

Hence $\varphi_1 \overline{\varphi_2}$ is invariant on the right by U_∞ (easy check)

<u>Def:</u> $$(\varphi_1, \varphi_2) = \int_{\mathbb{R}^* GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})} \varphi_1(\xi)\overline{\varphi_2(\xi)}\, d\xi$$

There's a fair chance that this could converge - c.f. Richard's Quaternion Algebra analysis. It does indeed converge.

Now say $f_1, f_2 \in S_k(\Gamma_1(N))$. Define $\varphi_i = \varphi_{f_i}$

Then if $\xi = \gamma u w$, usual notation,

~~$f_i(\xi) =$~~ $\varphi_i(\xi) = f_i(wi)$... etc Set $z = wi$.

$$\therefore \varphi_1(\xi)\,\overline{\varphi_2}(\xi) = f_1(z)\,\overline{f_2(z)} \left(\frac{\det w}{|j(w,i)|^2}\right)^k$$

$$= f_1(z)\,\overline{f_2(z)}\,(\operatorname{Im} z)^k$$

It's not too difficult to check that

$$(\varphi_1, \varphi_2) = \text{cst} \times \int_{\Gamma_1(N)\backslash H} f_1(z)\,\overline{f_2(z)}\, y^k \frac{dx\,dy}{y^2} \quad , \; z = x + iy$$

<u>Properties of adelic (,)</u> (NB φ_1, φ_2 are now general elts of S_k now)

1) (,) is $GL_2(\mathbb{A}^\infty)$-invt, i.e. $(g\varphi_1, g\varphi_2) = (\varphi_1, \varphi_2)$ $\forall g \in GL_2(\mathbb{A}^\infty)$

2) (,) restricted to $S_k^U \times S_k^U$, for U any cpct open subgp, is non-degenerate.

He's running out of time. He wanted to give a little algebraic argument, which would have yielded

<u>Thm</u> $S_k = \oplus W_i$, W_i admissible irreducible $GL_2(\mathbb{A}^\infty)$-subspaces, which are orthogonal under (,).

He'll talk more about this next time.

ecture 5
Mon 22nd Feb '93
9:30 am

Recall we're talking about $\mathcal{S}_k = \{\varphi : GL_2(\mathbb{A}) \to \mathbb{C} \mid 1) \text{ to } 5) \text{ hold}\}$, $k \geq 1$

$GL_2(\mathbb{A}^\infty)$ acts via right translation.

$$(\varphi_1, \varphi_2) = \int_{\mathbb{R}^* GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})} \varphi_1 \bar{\varphi}_2 \, d\xi.$$

It's $GL_2(\mathbb{A}^\infty)$-inv't. It doesn't make $\mathcal{S}_k$ complete.

However, $(\,,)$ restricts to a non-degenerate inner product on $\mathcal{S}_k^U \times \mathcal{S}_k^U$.

We will use $(\,,)$ to prove a theorem, coming up, which will convince us that $\mathcal{S}_k$ is <u>the</u> $GL_2(\mathbb{A}^\infty)$-module to be looking at.

<u>Thm</u> $\mathcal{S}_k = \bigoplus W_i$, where W_i is an irred admiss. $GL_2(\mathbb{A}^\infty)$-module.

<u>Notation</u> If $W \subseteq V \subseteq \mathcal{S}_k$, set $\underline{W^\perp(V)} = \{v \in V \mid (v, w) = 0 \ \forall w \in W\}$

<u>Lemma</u> If $W \subseteq V$ are $GL_2(\mathbb{A}^\infty)$-inv't subspaces of $\mathcal{S}_k$, we have

$$V = W \oplus W^\perp(V).$$

<u>Pf of lemma</u> Take $\varphi \in V$. We want $\varphi = \overset{\psi}{w} + \rho$, $\overset{\psi}{w} \in W$, $(\rho, W) = 0$.

Now $\varphi \in V^U$ for some cpct open subgp U of $GL_2(\mathbb{A}^\infty)$, & V^U is f.d.

Then $(\,,)$ is non-degenerate on V^U $\therefore$ $V^U = W^U \oplus W^{U\perp}(V^U)$

Hence $\varphi = \psi + \rho$, $\psi \in W^U \subseteq W$, $\rho \in W^{U\perp}(V^U)$ ie $(\rho, W^U) = 0$

He claims $(\rho, W) = 0$. Take $w \in W$; we must show $(\rho, w) = 0$

Now $w \in W^{U_1}$ & WLOG $U_1 \subseteq U$, U_1 normal in U, U_1 open

(surely he means $W^{U_1} = W^U \oplus Z^\perp$)

WLOG U/U_1 is a finite group. Hence $V^{U_1} = V^U \oplus Z$ Write $w = w_1 + w_2$

Now $\rho \in V^U$ $\therefore (\rho, w_2) = 0$ $\therefore (\rho, w) = (\rho, w_1) = 0$ as $w_1 \in W^U$ $\square$ of lemma.

Now hopefully a Zorns lemma-type argument will finish it off.

Pf of thm

Now pick a max^l family $\{V_i\}$ s.t. (i) V_i is an irred $GL_2(\mathbb{A}^\infty)$-submodule of S_k & (ii) $\sum V_i = \oplus V_i$ (Zorn)

Define $V = \oplus V_i$, a subspace of S_k

By our lemma, $S_k = V \oplus V^\perp$. Set $X = V^\perp$. Must show $X = 0$

Suppose for a contradiction that $X \neq 0$

Then there's a compact open U s.t. $X^U \neq 0$

With this U fixed, pick $A \subseteq X$ s.t. $A^U \neq 0$ & A^U is minimal w.r.t. not being 0 (X^U f.d)

Consider all $GL_2(\mathbb{A}^\infty)$-invt subspaces B of X s.t. $B^U = A^U$

Pick minimal such B (Zorn) Claim: B is irreducible.

For if $B_1 \overset{\subseteq}{\neq} B$ is a $GL_2(\mathbb{A}^\infty)$-invt subspace, then $A^U = B^U = B_1^U \oplus (B_1^\perp(B))^U$

$\therefore$ Minimality of $A^U \Rightarrow B_1^U = A^U$ or $(B_1^\perp(B))^U = A^U$

~~Maximality of $B \Rightarrow B_1^U$~~

If $B_1^U = A^U$ then by minimality of B we see $B_1 = B$

If $(B_1^\perp(B))^U = A^U \Rightarrow B_1^\perp(B) = B$ by minimality of B

$\Rightarrow B_1 \subseteq B_1^\perp(B) \Rightarrow B_1 = 0.$ □

Factorization

Say $\varphi : GL_1(\mathbb{A}) \to \mathbb{C}^\times$. Then $\varphi = \prod_v \varphi_v$, $\varphi_v : GL_1(\mathbb{Q}_v) \to \mathbb{C}^\times$

We want to do the same for $GL_2(\mathbb{A}^\infty)$-modules - eg $W_i = \otimes W_{i,p}$.

We will study $GL_2(\mathbb{A}^\infty)$, & $GL_2(\mathbb{Q}_p)$ (cf Tony)

Both of these are locally profinite groups

Say G is a locally profinite gp. We can define $\mathcal{H}(G)$ to be the locally cst compactly supported f^ns on G. $\mathcal{H}(G)$ becomes an algebra under $*$ once we've fixed a Haar measure.

$$\mathcal{H}(G) = \bigcup_K \mathcal{H}(G,K) \quad , \ \& \ \mathcal{H}(G,K) \text{ has unit } e_K = (\text{char f of } K) / \text{vol}(K)$$

<u>Fact 1</u> If V is a smooth G-module, we can endow V with a structure of $\mathcal{H}(G)$-module s.t. $V = \mathcal{H}(G)V$ (eh??)

$$\pi : G \to \mathrm{Aut}(V) \rightsquigarrow \pi : \mathcal{H}(G) \to \mathrm{End}(V)$$

$$\pi(f)v = \int_G f(g)\pi(g)v\,dg$$

(in fact ∃ equivalence between smooth G-mods & non-degenerate $\mathcal{H}(G)$-mods, or something)

<u>Fact 2</u>

<u>Irreducibility criterion</u>: A smooth G-module V is irreducible ⇔ V^K is an irreducible $\mathcal{H}(G,K)$-module for all K.

(Tony mentioned this)

We'll also need:

If $G = G_1 \times G_2$, G_1 & G_2 locally profinite,

& W_1 is an irred admiss G_1-module, W_2 an irred admiss G_2-module, $W = W_1 \otimes W_2$ is an irred admiss G-module.

<u>Thm</u> Let W be an irred admiss $G = G_1 \times G_2$-module, then ∃ irred admiss G_1-module W_1 & an irred admiss G_2-module W_2 s.t. W is G-isomorphic to $W_1 \otimes W_2$. Moreover W_1 & W_2 are ! up to isom (he said sthg about isotypic cpts)

<u>Classical result</u> (Bourbaki, Algebra, Chap VIII, p94)

k any alg. closed field, A, B algebras /k, M a simple $A \otimes B$-module, of f.d. /k.

Then $M = M_1 \otimes_k M_2$, M_1 a simple A-module
M_2 a simple B-module

<u>Pf</u> (of classical result)

We have M as an A-module or as a B-module. So if we pick an irred A-submodule P of M (M f.d.) we get $\mathrm{Hom}_A(P,M)$ endowed with the structure of a B-module.

Pick $R \subseteq \mathrm{Hom}_A(P,M)$ a simple B-module.

Then $P \otimes R \hookrightarrow P \otimes \mathrm{Hom}_A(P,M) \to M$ an $A \otimes B$-HM. This is iso □

<u>Pf of thm</u>

Now say $G = G_1 \times G_2$
$K = K_1 \times K_2$, $K_i \subset G_i$, $K \leq G$. Then $\mathcal{H}(G,K) \cong \mathcal{H}(G_1,K_1) \otimes \mathcal{H}(G_2,K_2)$.

<u>W a G-module → W gets an $\mathcal{H}(G)$-module structure</u>

$\mathcal{H}(G_1,K_1) \otimes \mathcal{H}(G_2,K_2)$

Pick K s.t. $W^K \neq 0$. Then W^K f.d. /$\mathbb{C}$ & W^K is an $\mathcal{H}(G,K)$-module

Then there exists (by the lemma) an $\mathcal{H}(G_1, K_1)$-module $W_1(K_1)$
& an $\mathcal{H}(G_2, K_2)$-module $W_2(K_2)$, irreducible

& $a_K : W^K \xrightarrow{\sim} W_1(K_1) \otimes W_2(K_2)$

If $\begin{array}{ccccc} K' & = & K_1' & \times & K_2' \\ \cap & & \cap & & \cap \\ K & = & K_1 & \times & K_2 \end{array}$ then we get $W_i(K_i')$

Set $W_1 = \varinjlim_{K_1'} W_1(K_1')$, $W_2 = \varinjlim_{K_2'} W_2(K_2')$

Tensor products commute with direct limits

$$W = W_1 \otimes W_2 = \varinjlim \left(W_1(K_1') \otimes W_2(K_2') \right)$$

W_i are also irreducible. $\square$

Finally we want to understand the case $G = GL_2(\mathbb{A}^\infty) = \prod' GL_2(\mathbb{Q}_v)$

Note that if we have a $\prod_{v \in S} GL_2(\mathbb{Q}_v)$-module W, irred admiss,

then $W = \bigotimes_{v \in S} W_v$.

with W_v an irred admiss $GL_2(\mathbb{Q}_v)$-module.

ecture 6
on22nd Feb '93
2:15pm

Tensor products of infinite families of v /$\mathbb{C}$

Say we are given $\{W_\lambda\}_{\lambda \in \Lambda}$ & (i) a finite subset $\Lambda_0 \subset \Lambda$
(ii) For each $\lambda \in \Lambda \setminus \Lambda_0$ an $x_\lambda \in W_\lambda$, $x_\lambda \neq 0$.

Say S is a finite subset of Λ containing Λ_0

Set $W_S = \bigotimes_{\lambda \in S} W_\lambda$.

If $S \subseteq S'$ define $f_{S,S'} : W_S \to W_{S'}$ by $f_{S,S'}\left(\bigotimes_{\lambda \in S} w_\lambda\right) = \left(\bigotimes_{\lambda \in S} w_\lambda\right) \otimes \left(\bigotimes_{\lambda \in S' \setminus S} x_\lambda\right)$.

Def$^{\underline{n}}$ $\bigotimes_{x_\lambda} W_\lambda = \varinjlim_{S} W_S$

We can change x_λ to $a_\lambda x_\lambda$, $a_\lambda \in \mathbb{C}^\times$.

$\bigotimes W_\lambda$ only depends on the $\mathbb{C}$-vector spaces generated by the x_λ.

It makes sense to talk about $\otimes w_\lambda$ so long as $w_\lambda = x_\lambda$ for all but a finite number of λ.

The same ideas work for:-

Algebras Given $\{A_\lambda\}_{\lambda\in\Omega}$ with an idempotent $e_\lambda \in A_\lambda$ for all but a finite number of λ.

We can give $\bigotimes_{e_\lambda} A_\lambda$ the structure of an algebra by defining

$$(\otimes a_\lambda).(\otimes b_\lambda) = \otimes \underbrace{a_\lambda b_\lambda}_{= e_\lambda \text{ for all but finitely many } \lambda.}$$

Take now Λ = the set of finite places of $\mathbb{Q}$.

For $v \in \Lambda$ set $K_v = GL_2(\mathbb{Z}_v)$ & define $e_v = e_{K_v} \in \mathcal{H}(GL_2(\mathbb{Q}_v))$,

e_{K_v} = (char fn of K_v) / vol(K_v).

We get $\bigotimes_{e_v} \mathcal{H}(GL_2(\mathbb{Q}_v))$

Remark $\bigotimes_{e_v} \mathcal{H}(GL_2(\mathbb{Q}_v)) \cong \mathcal{H}(GL_2(\mathbb{A}^\infty))$ (he said canonical isomorphism)

This seems to be because $\bigotimes_{v\in S} \mathcal{H}(GL_2(\mathbb{Q}_v)) \xrightarrow{\sim} \mathcal{H}(\prod_{v\in S} GL_2(\mathbb{Q}_v))$.

Say $J = \prod_v J_v$ is cpct open in $GL_2(\mathbb{A}^\infty)$

Then $J_v = K_v$ for all but a finite no. of v.

$\mathcal{H}(GL_2(\mathbb{A}^\infty), J) \cong \bigotimes_{e_v} \mathcal{H}(GL_2(\mathbb{Q}_v), J_v)$

He might have fixed some compatible system of measures so * works.

Next note that if we are given $\forall v$ an admiss irred $GL_2(\mathbb{Q}_v)$-module W_v s.t. $\dim_{\mathbb{C}} W_v^{K_v} = 1$ for all but a finite no. of v.

Pick $x_v \in W_v^{K_v}$, $0 \neq x_v$, for these v s.t. $\dim_{\mathbb{C}} W_v^{K_v} = 1$.

Then $W = \bigotimes_{x_v} W_v$ is an irred admiss $GL_2(\mathbb{A}^\infty)$-module.

↑
not too hard to check, evidently

However, the converse is also true:-

Thm Let W be an irred admiss $GL_2(\mathbb{A}^\infty)$-module. For each finite v, there exists an irred. admiss. $GL_2(\mathbb{Q}_v)$-module W_v s.t.

(i) $\dim_{\mathbb{C}} W_v^{K_v} = 1$ for all but a finite no. of v

(ii) $W = \otimes W_v$ relative to x_v, where for all but finitely many v, $0 \neq x_v \in W_v^{K_v}$.

Moreover, the factors W_v are uniquely determined by W.

Pf This is rather a miraculous result, but we've essentially proved it already. Something about isotypic cpts does uniqueness. ~~Use~~ We'll see that it's easier to pass to the Hecke algebras to prove this thm.

W is a module over $\mathcal{H}(GL_2(\mathbb{A}^\infty))$. Choose a cpct open $J = \prod_v J_v$ of $GL_2(\mathbb{A}^\infty)$ s.t. $W^J \neq \{0\}$. Of course, $J_v = K_v$ for all but finitely many v.

$$W^J \text{ is f.d.}/\mathbb{C} \text{ and an irred } \underset{\underset{\displaystyle\otimes_v \mathcal{H}(GL_2(\mathbb{Q}_v), J_v)}{\parallel}}{\mathcal{H}(GL_2(\mathbb{A}^\infty), J)}\text{-module}$$

$$\text{Hence } \bigotimes_{v \in S} \mathcal{H}(GL_2(\mathbb{Q}_v), J_v) \text{ acts on } W^J$$

If S is sufficiently large, & S contains v s.t. $J_v \neq K_v$, then W^J will be irreducible $/ \bigotimes_{v\in S} \mathcal{H}(GL_2(\mathbb{Q}_v), J_v)$ as for $v \notin S$ everything acts as scalars.

Hence by our result for finitely many things, $W^J = \bigotimes_{v \in S} W_v(J_v)$,

$W_v(J_v)$ an irred admiss $\mathcal{H}(GL_2(\mathbb{Q}_v), J_v)$-module.

Now pass to the inductive limit. ■

Also note that $\dim_{\mathbb{C}} W_v(J_v) = 1$ when $J_v = K_v$ □

Tony saved his bacon on this next one

Tony Scholl defined Hecke algebras at infinity.

$$\mathcal{H}_\infty = \mathcal{H}(\mathfrak{g}_\infty, K_\infty), \quad K_\infty = O_2(\mathbb{R})$$

Tony explained all this. It's a nasty tensor product of lots of measures with a universal enveloping algebra. We get $\{e\} \subseteq \mathcal{H}_\infty$

& $\mathcal{H}_\infty = \bigcup e\mathcal{H}_\infty e$.

John's conscience is clear (He won't give the details) He hopes Tony's is also.

Def^{n} $\mathcal{H} = \bigotimes'_v \mathcal{H}_v$ $\mathcal{H}_v = \mathcal{H}_\infty$ if $v=\infty$
$\mathcal{H}(GL_2(\mathbb{Q}_v))$ if $v<\infty$

The modules that Richard has been talking about are tailor-made for this setting. If we have an admissible irreducible $GL_2(\mathbb{A}^\infty) \times (\mathfrak{g}_\infty, K_\infty)$-module (here $\mathfrak{g}_\infty = \mathrm{Lie}(GL_2(\mathbb{R})) = M_2(\mathbb{R})$, & $K_\infty = O_2^{\circ}(\mathbb{R})$.) then we'll be able to factorise it. Oh- here a ~~module~~ $GL_2(\mathbb{A}^\infty) \times (\mathfrak{g}_\infty, K_\infty)$-module is admissible if

(i) V is a smooth $GL_2(\mathbb{A}^\infty)$-module
(ii) V is a $(\mathfrak{g}_\infty, K_\infty)$-module
(iii) The actions above commute
(iv) If ρ is any irred rep of $K = \prod_v K_v$, then $V(\rho)$ is f.d. / $\mathbb{C}$

Via the Hecke algebras, we get a modification of the last thm:

if W is an irred admiss $GL_2(\mathbb{A}^\infty) \times (\mathfrak{g}_\infty, K_\infty)$-module, then for each finite v there exists W_v s.t. $W = \otimes W_v$. He's just rubbed it all off but I'm sure it's clear what John is saying.

The point of all this is that the interesting $GL_2(\mathbb{Q}_p)$-modules are the ones that appear in the decomposition $\mathcal{S}_k = \bigoplus W_i$, $W_i = \bigotimes W_{i,v}$.

The space A°

The problem with $\mathcal{S}_k$ is that it was sort-of invented to model modular forms. There are funny non-holomorphic things invented by Maass in the '30s that aren't accounted for. Our action at ∞ is too easy. Also we're not a $(\mathfrak{g}_\infty, K_\infty)$-module yet.

There's problems with A°. Normalisations vary, just as in $\mathcal{S}_k$ case. He hopes what he's written down is correct. He's sure one of the experts will correct him otherwise.

Define $A^\circ = \{ \varphi : GL_2(\mathbb{A}) \to \mathbb{C} \mid$ erm... we'll come to this $\}$

We're keen on having $\mathcal{S}_k \subseteq A^\circ$. This won't be immediately obvious!

Write $GL_2(\mathbb{A}) = GL_2(\mathbb{A}^\infty) \times GL_2(\mathbb{R})$
$\xi = (\xi^\infty, \xi_\infty)$

Here are our conditions.

<u>Cond. 1</u> For fixed ξ_∞, $\varphi(\xi)$ is locally cst in ξ^∞, & for a fixed ξ^∞ we have $\varphi(\xi)$ is C^∞ in ξ_∞. (He calls this "φ smooth" but the bloke next to me thinks hes only assured that "φ is smooth in both directions")

<u>Cond. 2</u> φ is left-inv by $GL_2(\mathbb{Q})$

<u>Cond. 3</u> φ is inv on the right by an open subgp $U = U(\varphi)$ in $GL_2(\mathbb{A}^\infty)$

<u>Cond. 4</u> For fixed ξ^∞, the function $\xi_\infty \mapsto \varphi(\xi)$ is bounded on $GL_2(\mathbb{R})$

NB if he just put 'slowly increasing' it would give us A, not A°. A has non-cuspidal things in

In fact cond^ns 1-4 still don't quite force cuspidality - as e.g. cst f^s are still in. So impose

<u>Cond. 5</u> φ is <u>cuspidal</u> ie $\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\xi\right) du = 0 \ \forall \xi \in GL_2(\mathbb{A})$.

<u>Cond. 6</u> [illegible] (behaviour under right translation by $\overset{O_2(\mathbb{R})}{K_\infty}$ & $\mathbb{R}^\times \hookrightarrow GL_2(\mathbb{R})$)
[illegible] φ is ~~[illegible]~~ $K_\infty \mathbb{R}^\times$-finite

~~[illegible] $\mathbb{R}^\times \subset GL_2(\mathbb{R})$ acts by right translation~~

Lecture 7
Tues 23^rd Feb '93
9:30am

He must continue on his quest to define A°, despite Brian Birches cry of "come back, John!"

<u>Cond. 7</u> ⌈ If $\mathfrak{g}_\infty$ = Lie algebra of $GL_2(\mathbb{R})$ ie $\mathfrak{g}_\infty = M_2(\mathbb{R})$, then

$$(X.\varphi)(\xi) = \frac{d}{dt}\varphi(\xi \exp(tX))\Big|_{t=0}$$

& extend by $\mathbb{C}$-linearity to a rep^n of $\mathfrak{g}_{\infty,\mathbb{C}}$

Set $\mathfrak{U}_\infty$ = Universal enveloping algebra of $\mathfrak{g}_{\infty,\mathbb{C}}$

Then $$\mathfrak{g}_{\infty,\mathbb{C}} \hookrightarrow \mathfrak{U}_\infty$$ $$X \longmapsto X'$$

and $\mathfrak{z}_\infty$ = centre of $\mathfrak{U}_\infty$ ⌋

φ is $\mathfrak{z}_\infty$-finite

He now ~~[illegible]~~ wants to tell us why this is a reasonable def^n. Actually, he firstly wants to tell us why it's a def^n of anything at all, i.e. he wants to check the def^n makes sense. There's just 1 point: $(X.\varphi)$ is bounded for $X \in \mathfrak{g}_\infty$.

<u>Lemma</u> Assume $\eta : GL_2(\mathbb{R}) \to \mathbb{C}$ is C^∞ & satisfies (i) K_∞-finite
(ii) $\mathfrak{z}_\infty$-finite.

Then $\exists$ C^∞-f^n $\alpha : GL_2(\mathbb{R}) \to \mathbb{C}$ s.t. (i) α has cpct support
(ii) α is K_∞-inv't on the right
(iii) $\eta * \alpha = \eta$

Pf Richard talked about it (in a more general setting) $\square$

Now define $\eta(\xi_\infty) = \varphi(\xi^\infty, \xi_\infty)$

Then $(X.\varphi)(\xi) = \frac{d}{dt}\left(\eta(\xi_\infty \exp(tX))\right)\Big|_{t=0}$

$$= \left(\int_{GL_2(\mathbb{R})} \eta(u) \frac{d}{dt}(u^{-1}\xi_\infty \exp(tX))\Big|_{t=0} du \right.$$

This (evidently) justifies the fact that $(X.\varphi)$ satisfies 4). Evidently.

So we have some daft space A°. We also had S_k. John asserts that $S_k \subseteq A^\circ$. The thing is, there was no Lie algebra action on S_k. We'll have to show that elts of S_k are $\mathfrak{z}_\infty$-finite.

<u>Def</u> $A^\circ_k = \{ \varphi \in A^\circ \mid \varphi(\xi\tau) = \varphi(\xi) j(\tau, i)^{-k} (\det \tau)^{k/2} \ \forall \tau \in U_\infty = \mathbb{R}^\times SO_2(\mathbb{R}) \}$

Recall for $\xi \in GL_2(\mathbb{A}^\infty)$ we're writing $\xi = (\xi^\infty, \xi_\infty)$, $\xi_\infty \in GL_2(\mathbb{R})$

Define $\mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$; $z = \xi_\infty i = x + iy \in \mathcal{H}^\pm$

$$\xi_\infty = \begin{pmatrix} \operatorname{sgn} y . |y|^{1/2} & x/|y|^{1/2} \\ 0 & |y|^{-1/2} \end{pmatrix} . r \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where $r \in \mathbb{R}_{>0}$ & $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

Write $\Phi(\xi^\infty, z) = \varphi(\xi) j(\xi_\infty, i)(\det \xi_\infty)^{-k/2}$

$(z = \xi_\infty i)$ Here $j(\xi_\infty, i) = |y|^{-1/2} r e^{-i\theta}$, $\det(\xi_\infty) = r^2(\operatorname{sgn} y)$

<u>Formula</u> $\varphi(\xi) = \Phi(\xi^\infty, z) y^{k/2} e^{ik\theta}$, $\varphi \in A^\circ_k$

$\mathfrak{g}_{\infty,\mathbb{C}}$ has a basis $\underset{\overset{\shortparallel}{J}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\overset{\shortparallel}{H}}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}, \underset{\overset{\shortparallel}{X_+}}{\tfrac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}}, \underset{\overset{\shortparallel}{X_-}}{\tfrac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}}$

We'll put dashes on things if they're in $\mathcal{U}_\infty$ - e.g. $J \in \mathfrak{g}_\infty$, $J' \in \mathcal{U}_\infty$

he put a bracket

$$(\exp tJ) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}, \quad \frac{d}{dt}\left(\varphi(\xi) \exp(tJ))\right) = 0 \quad \therefore (J.\varphi) = 0$$

<u>Action of H</u> $\quad H = -iA, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$(H.\varphi)(\xi) = (-i)(A.\varphi)(\xi) = (-i)\frac{d}{dt}\varphi(\xi\exp(tA))\Big|_{t=0}$$

Now $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\exp(tA) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$

$$(H.\varphi)(\xi) = (-i)\frac{d}{dt}\left(\varphi\left(\xi\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right)\right)\Big|_{t=0}$$

$$= -i\frac{d}{dt}\left(\Phi(\xi^\infty, z)\, y^{k/2} e^{ik(t+\theta)}\right)\Big|_{t=0}$$

φ $$= k\varphi(\xi)$$

$$\left(\varphi(\xi) = \Phi(\xi^\infty, z)\, y^{k/2} e^{ik\theta}\right)$$

<u>Action of X_+ & X_-</u>

$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X_+ = \frac{1}{2}(U + iV)$, $X_- = \frac{1}{2}(U - iV)$

$$(U\varphi)(\xi) = \frac{d}{dt}\left(\varphi(\xi\exp(tU))\right)\Big|_{t=0}$$

$U^2 = I$

$$\therefore \exp(tU) = \sum_{h=0}^{\infty}\frac{t^{2h}}{(2h)!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{h=0}^{\infty}\frac{t^{2h+1}}{(2h+1)!}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

So $(U.\varphi)(\xi) = \frac{d}{dt}\varphi(\xi\exp(tU))\Big|_{t=0}$

$$\& \quad \varphi(\xi\exp(tU)) = \Phi\left(\xi^\infty, \xi_\infty\exp(tU)(i)\right) j(\xi_\infty\exp(tU), i)^{-k}$$

$$\times\lambda \quad (?!)$$

$$\frac{d}{dt} j(\xi_\infty\exp(tU), i)^{-k}\Big|_{t=0} = k j(\xi_\infty, i)^{-k} e^{2i\theta}$$

$$\xi_\infty\exp(tU)(i) = x + y\,\frac{(1-e^{4t})\cos\theta\sin\theta + ie^{2t}}{1-(1-e^{4t})\sin^2\theta}$$

(this is all an exercise)

$$\therefore (U.\varphi)(\xi) = k(\overset{2\theta}{\cos} + i\sin 2\theta)\varphi(\xi) + \left(2y\cos 2\theta\frac{\partial}{\partial y} - 2y\sin 2\theta\frac{\partial}{\partial x}\right)\Phi(\xi^\infty, z)$$

$$j(\xi_\infty, i)^{-k}(\det \xi_\infty)^{k/2}$$

A totally similar calculation gives us $V.\varphi$. $V^2 = I$ so can do $\exp(tV)$ explicitly, & get

$$(V\varphi)(\xi) = k(\sin 2\theta - i\cos 2\theta)\varphi(\xi) + j(\xi_\infty, i)^{-k}(\det \xi_\infty)^{k/2}\underbrace{\left(2y \sin 2\theta\, \partial/\partial y + 2y\cos 2\theta\, \partial/\partial x\right)}_{\Phi(\xi^\infty, z)}$$

$$X_+.\varphi = \tfrac{1}{2}(U.\varphi + iV.\varphi)$$

$$= j(\xi_\infty, i)^{-k}(\det \xi_\infty)^{k/2} e^{2i\theta}\left(y(\partial/\partial y + i\,\partial/\partial x) + k\right)\Phi(\xi^\infty, z)$$

& $(X_-.\varphi)(\xi) = j(\xi_\infty, i)^{-k}(\det \xi_\infty)^{k/2} e^{-2i\theta}\left(y(\partial/\partial y - i\,\partial/\partial x)\right)\Phi(\xi^\infty, z)$

Now the Cauchy-Riemann equns give

<u>Consequence</u> $\Phi(\xi^\infty, z)$ is holo. as a f of $z \Leftrightarrow (X_-.\varphi) = 0$

Recall that there was some holomorphicity condn in $\mathcal{S}_k$ so this is good-looking stuff.

<u>Thm</u> $\mathcal{S}_k \subseteq A^\circ_k \subseteq A^\circ$. In fact, $\mathcal{S}_k = \{\varphi \in A^\circ_k \mid (X_-.\varphi) = 0\}$

<u>Pf</u> RHS $\subseteq$ LHS by defn of $\mathcal{S}_k$. The problem for $\supseteq$ is $\mathfrak{z}_\infty$-finiteness.

The question: what is $\mathfrak{z}_\infty$? Well J is in the centre of $\mathfrak{g}_\infty$ so clearly $J' \in \mathfrak{z}_\infty$. Also the Casimir operator $\Omega' = H'^2 - 2H' + 1 + 4X'_+X'_-$.

§ Note $[H', X'_+] = 2X'_+$, $[H', X'_-] = 2X'_-$, $[X'_+, X'_-] = H'$

& we see that $[\Omega', \{\text{basis elts}\}] = 0$ so indeed $\Omega' \in \mathfrak{z}_\infty$

In fact $\mathfrak{z}_\infty = \mathbb{C}[\Omega', J']$ & he'll talk about this in his next lecture, & then finish the pf of the thm.

<u>Lecture 8</u>
<u>Tues 23rd Feb '93</u>
<u>2:30pm</u>

Recall the survivor's party, 8:30pm, 104 Mawson Road. Near railway station.

To finish off he's just gonna quote thms etc.

He's trying to show $\mathcal{S}_k = \{\varphi \in A^\circ_k \mid X_-.\varphi = 0\}$ & he's done $\supseteq$.

We need to check out $\mathfrak{z}_\infty$ action on $\mathcal{S}_k$

John's defined $\Omega' = H'^2 - 2H' + \underset{\substack{\uparrow \\ \text{algebra } 1}}{1} + 4X'_+X'_-$ Tony's Casimir operator, as it happens, was $D = \dfrac{\Omega'}{2} - 1$.

$\Omega' \in \mathfrak{z}_\infty$.

Note $H'.\varphi = k\varphi$ & $X'_-.\varphi = 0$, so $\Omega'.\varphi = (k-1)^2\varphi$ & $J'.\varphi = 0$

So clearly, $\mathfrak{z}_\infty = \mathbb{C}[\Omega', J'] \Rightarrow S_k \subseteq A^\circ_k$ & we're home.

So it remains to prove that $\mathfrak{z}_\infty = \mathbb{C}[\Omega', J']$.

We will use a theorem of Harish-Chandra which is probably true in much greater generality.

Write $\mathcal{T} \subseteq \mathfrak{g}_{\infty,\mathbb{C}}$ = vector space of diagonal matrices

C_2 acts on $\mathcal{T}$ by interchanging the diagonal elts

Set $\mathcal{T}^* = \mathrm{Hom}(\mathcal{T}, \mathbb{C})$

<u>Thm</u> (H-C) $\mathfrak{z}_\infty \xrightarrow{\sim} (\text{Polynomial } f^{n}\text{s on } \mathcal{T}^*)^{C_2}$. $\square$

$$\begin{tikzcd}[column sep=small]
(\text{Polynomial } f^{n}\text{s on } \mathcal{T}^*)^{C_2} \arrow[d, phantom, "\parallel"] & & \\
\mathbb{C}[X_1, X_2]^{C_2} = \mathbb{C}[X_1+X_2, & X_1X_2] & \\
J' \arrow[u, leftrightarrow] & \frac{1}{4}(J'^2 - \Omega') \arrow[u, leftrightarrow] &
\end{tikzcd}$$

If we believe all that then we're clearly done. $\square$

$S_k \subseteq A^\circ_k \subseteq A^\circ$

He wants to give us some facts about A before talking a bit about the more arithmetic S_k.

A° is a $GL_2(\mathbb{A}^\infty) \times (\mathfrak{g}_\infty, K_\infty)$-module

$\mathcal{H} = \bigotimes_v \mathcal{H}_v$

Def: An automorphic rep of $\mathcal{H}$ is an irred subquotient of the rep of $\mathcal{H}$ on A.

Henniart wrote sthg up recently & John is cribbing off this. This is where his facts are from

Fact An automorphic rep^n of $\mathcal{H}$ is admissible.

He seems to be writing A for A° now.

If $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$

$$A(\chi) = \left\{ \varphi \in A \;\middle|\; \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\varphi = \chi(z)\varphi \right\}$$

Thm (a) Every auto. rep. of $\mathcal{H}$ occurs in $A(\chi)$ for some χ

(b) For fixed χ, $A(\chi)$ is a direct sum of irred admis $\mathcal{H}$-modules, each occuring with multiplicity 1. Each $W_\cdot$ which occurs has infinite dimension.

$$W = \otimes_v W_v$$

Strong multiplicity 1 thm

Let π_1, π_2 be irred auto. reps of $\mathcal{H}$.

Say $\pi_1 = \otimes_v \pi_{1,v}$, $\pi_2 = \otimes_v \pi_{2,v}$

Then $\pi_1 \cong \pi_2 \iff \pi_{1,v} \cong \pi_{2,v}$ for all but a finite no. of v.

He doesn't really want to talk about non-holomorphic forms & stuff. He does want to look more at S_k & get some arithmetic facts out.

§ $S_k \subseteq A^0$
$\quad \parallel$
$\oplus W_i$, W_i irred admis $GL_2(\mathbb{A}^\infty)$-modules

Take W to be one of these W_is.

$W = \otimes_v W_v$, W_v an irred admis $GL_2(\mathbb{Q}_v)$-module, $v \nmid \infty$
$(\mathfrak{g}_\infty, K_\infty)$-module, $v \mid \infty$

W_∞? Well, $W \subseteq S_k$. We have ~~that~~ an action of $SO_2(\mathbb{R})$ & $\mathbb{R}^\times$ which we understand.

$W_{\infty,k} \neq 0$ & $X'_- . \varphi = 0$ $\forall \varphi \in S_k$ $\therefore$ $W_\infty = \mathcal{D}^+_k$, a $(\mathfrak{g}'_\infty, K'_\infty)$-module

Thinking about a bit $\Rightarrow$

Fact $W_\infty = \sigma(\mu_1, \mu_2)$, $\mu_1 = |\cdot|^{k-1/2}$, $\mu_2 = |\cdot|^{1-k/2} . \mathrm{sgn}$

This all appears to be nonsense. There is no action at infinity.

Say W is an irred admis submodule of $\mathcal{S}_k$

Then $W = \otimes W_v$

$\rightsquigarrow f_W$ a primitive form of wt k for $\Gamma_1(N)$, a suitable N.

We appeal to sthg Tony told us to construct f.

We know $\dim_{\mathbb{C}} W_v^{GL_2(\mathbb{Z}_v)} = 1$ for all but a finite no. of v

For $h \geq 0$ define $\underline{K_{p,h}} = \{ \gamma \in GL_2(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^h \}$

__Theorem__ For every p $\exists$ an integer $f_p \geq 0$ s.t. $W_p^{K_{p,h}} = \{0\}$ if $h < f_p$
& $W_p^{K_{p,f_p}}$ has dimension 1 □

Tony proved that bit using Kirillov models.

$\forall p < \infty$ define η_p to be a non-zero elt of $W_p^{K_{p,f_p}}$

It's obvious that $\otimes \eta_p$ makes sense in $W = \underset{v}{\otimes} W_v \subseteq \mathcal{S}_k$

Then $U_1(N) \subseteq GL_2(\hat{\mathbb{Z}})$. If $N = \prod_p p^{f_p}$ then $U_1(N) = \prod_p ~~\otimes~~ K_{p,f_p}$

Say $\eta = \otimes \eta_p$..

The conclusion is that $\eta \in \mathcal{S}_k^{U_1(N)} \cong S_k(\Gamma_1(N))$

$\eta = \varphi_f$ for some $f \in S_k(\Gamma_1(N))$

It's pretty easy to check that f is a primitive form.

He wants to spend the last few minutes talking about another fact - the Taniyama-Weil conjecture ~~or~~ recipe that Tony told John.

$E/\mathbb{Q}$ an elliptic curve. For every $\overset{v:p}{\cancel{p<\infty}}$ we can attach π_v, an irreducible admissible $GL_2(\mathbb{Q}_p)$-module, & $v = \infty$ gives us π_v, an irred admis $(\mathfrak{g}_\infty, K_\infty)$-module ($\pi_\infty = \sigma(\mu_1, \mu_2)$, $k=2$)

They are almost all unramified

π_p, $p < \infty$, is $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, an ℓ-adic repn

If $p \neq \ell$ there's an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p)$ on $V_\ell(E)$.

The usual Grothendieck trick as explained by Martin Taylor gives us a 2 dim^l rep^n of WD_p

$$\sigma(\pi_p) : WD_p \rightarrow GL_2(\mathbb{C}).$$

Then Local Langlands ⇝ attach π_p, an irred admiss rep of $GL_2(\mathbb{Q}_p)$

$$\text{So } E \rightsquigarrow \{V_\ell(E)\} \leftrightarrow \{\sigma(\pi_p)\} \leftrightarrow \{\underset{\pi_v}{\pi_p}\} \leftrightarrow \pi = \otimes \pi_v$$

π is an admissible irred rep of $\mathcal{H}$.

Taniyama-Weil ⇒ π is automorphic.

Tony has this recipe & John will explain it.

π_p unramified ⟷ E has good reduction at p
$\pi_p = \sigma(\mu_1, \mu_2)$ ⟷ E has potential multiplicative red^n at p (Image of I not finite)
(μ_1 unramified ⟷ mult red^n)
π_p ramified princ. series / supercuspidal ⟷ E has potential good reduction at p
(ramified PS ⟷ good red^n / an abelian ext^n / $\mathbb{Q}_p$
supercuspidal ⟷ good red^n over a non-ab ext^n / $\mathbb{Q}_p$

Hopefully ε factors would match up too.

It would be nice if this were true as the Galois ε-factors are nasty to work out.

IV Quaternion Algebras

Richard Taylor

ecture 1
ues 16th Feb '93
11.00 am

The analysis is much easier for quaternion algebras, although they're less familiar objects. There's 3 parts to this course

1) Quaternion algebras generalities (3 lectures, last one is SAT, first 2 are easy)
2) Functional analysis (3 lectures; what he wished the analysts had taught him as an undergraduate)
3) Automorphic forms (2 lectures - trace formula etc).

§1.1 Generalities

D is a <u>quaternion algebra</u> over a field F if

1) D is an F-algebra, associative with a 1 but not nec. commutative
2) D is central over F ie F is the centre of D
3) D is simple ie ∄ non-trivial 2-sided ideals
4) $\dim_F D = 4$

<u>Example</u> (exercise) $M_2(F)$ = 2×2 matrices over F.

This simple example is exceptional in many ways. We say $M_2(F)$ is <u>split</u>.

$M_2(F)$ is to $GL_2(F)$ as D is to weird gps we'll look at later.

<u>Lemma 1.1</u> If A & B are simple F-algebras & if A has centre F then $A \otimes_F B$ is simple. □

6-line proof but he won't waste time. Check out references

<u>References</u> - ~~Quat algs~~
Non-commutative rings - Hestein
Associative Algebras - Pierce
Algebras de Quaternion - Vigneras
Basic no thy - Weil

They get more arithmetic as you go down.

Lemma 1.1 is Lemma 12.4.b of Pierce & Thm 4.1.1 of Hestein.

<u>Cor</u> If E/F is a field extension then $D \otimes_F E = D_E$ is a quaternion algebra over E. □

Note: D_E may be split, even if D isn't.

We say <u>E splits D</u> if D_E is split.

<u>Lemma 2</u> If D is not split, & $\delta \in D \setminus \{0\}$, then δ has a 2-sided inverse

$\delta\xi = 1$
$\xi\delta\xi = \xi$

<u>Pf</u> The map $\begin{array}{c} D \longrightarrow D \\ x \mapsto x\delta \end{array}$ is linear. Say its kernel is I_δ & its image is $D\delta$.

Then $D \to \mathrm{End}_F(D\delta)$ & $D \to \mathrm{End}_F(I_\delta)$ by left multiplication

D is simple, so either these maps are 0 or they're injections.
Hence $\dim D\delta = 0$ or ≥ 2
$\dim I_\delta = 0$ or ≥ 2

However, if one of them has dimension 2 then the map is an iso. and D splits.
They both can't have dimension > 2 $\therefore$ one has dimension 0.
Hence $D\delta = 0$ ($\#$ as $\delta \in D\delta$)
or $I_\delta = 0$ (☺) & $D\delta = D$ so δ has an inverse.
It must be 2-sided by the same argument, or something. □

<u>Cor</u> If D is not split & $\delta \in D \setminus \{0\}$, then $F(\delta)$ is a field □

<u>Cor</u> If F is algebraically closed, then D must be split. □

Somehow the non-split quat. algs are related to the fact that F isn't algebraically closed.

We want to talk about $\prod_{i=1}^{r} E_i$, where E_i/F is a finite field extn.

We will call $\prod_{i=1}^{r} E_i$ a <u>POF</u> i.e. a product of fields.

He doesn't know of a better notation. They're the semisimple commutative f.d. F-algebras but this is worse!

<u>Lemma 3</u> If $E \subseteq D$ is a POF & $E \neq F$ then $\dim_F E = 2$, E is its own centraliser in D, E splits D, & if E is not a field, then D is split.

NB he'll leave it to our imagination as to what "E splits D" means
– I guess $D \otimes_F E \cong M_2(E)$?

<u>Pf</u> Suppose E is a field. Then $4 = \dim_E D \times [E:F]$ $\therefore \dim_E D = [E:F] = 2$
Thus if $\delta \in D \setminus E$, then $D = E \oplus E\delta$, & D is not commutative $\therefore$ δ does not commute with E $\therefore$ E is its own centraliser.
Next note $D \otimes E \longrightarrow \mathrm{End}_E(D)$
$\delta \otimes x : y \mapsto \delta y x$ (E acts on D on the right)

This is $\neq 0$ so it's injective as $D \otimes E$ is simple.
By dimension counting it's iso. $D_E \cong \mathrm{End}_E(D)$.

If E is not a field then it's all an exercise: $\exists\, \delta \in D$ s.t. $\delta^2 = \delta$, $\delta \neq 0, 1$ & hence $D \cong M_2(F)$, $\delta \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. □

So any subfield of D of dimension 2 $(/F)$ splits D. There is a converse

Lemma 4 If E/F is quadratic & splits D then $E \hookrightarrow D$ (best have char $F = 0$)

Pf (If E isn't a field then $E = F \oplus F$ & D must be split already $\therefore$ E embeds diagonally)

Say E is a field then. Set $E = F(\sqrt{d})$, $d \in F$.

E splits $D \Rightarrow \exists\, \delta_1 + \delta_2\sqrt{d} \in D_E$, $\delta_1 + \delta_2\sqrt{d} \neq 0, 1$, & $(\delta_1 + \delta_2\sqrt{d})^2 = \delta_1 + \delta_2\sqrt{d}$

$\Rightarrow \sqrt{d} \in D$. Time is short so he won't go in to details. □

Defn There's a canonical involution $*: D \to D$ defined thus:

If $\delta \in F$, $\delta^* = \delta$

If D is split, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ = adjugate of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If D is not split & $\delta \in D \setminus F$ then $*$ on (the field) $F(\delta)$ is the non-trivial elt of the Galois gp of $F(\delta)/F$.

Remark: $\delta^* = \delta \iff \delta \in F$

Define $\mathrm{Tr}\, \delta = \delta + \delta^* \in F$ - the reduced trace

& the reduced norm $\nu\delta = \delta\delta^* \in F$ - the reduced norm

They fulfil for non-split D what trace & det do for split D

Lemma 5 If E splits D then

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "{\mathrm{tr}, \nu}"'] & D_E \arrow[d, "{\mathrm{trace}, \det}"] \\ F \arrow[r, hook] & E \end{tikzcd}$$

commutes

Pf is an exercise. It's easy - if D is non-split then $\mathrm{tr}\,\delta$ = trace of $\delta \in \mathrm{End}_F(F(\delta))$ is a neat way of doing it □

So it's easy to reduce facts about tr & ν to facts about trace & det. Eg

Cor $\mathrm{tr}(\delta_1 + \delta_2) = \mathrm{tr}\,\delta_1 + \mathrm{tr}\,\delta_2$, $\nu(d_1\delta_2) = \nu(\delta_1)\nu(\delta_2)$, $(\delta_1\delta_2)^* = \delta_2^*\delta_1^*$

Write $D^\times$ = units in D ($= D \setminus \{0\}$ in non-split case
$= GL_2(F)$ in split case)

& $D^1 = \ker \nu : D^\times \to F^\times$

Another elementary but useful fact is

Lemma 6 (Noether - Skolem) (NB he has lemmas & prop's; everyone else has thms & prop's !!)

If M/F a quadratic POF (NB he sometimes uses M & sometimes E. E is usually $\subseteq D$)
& if $\sigma_1, \sigma_2 : M \hookrightarrow D$, then $\exists\, \delta \in D^\times$ s.t. $\delta \sigma_1(x) \delta^{-1} = \sigma_2(x) \quad \forall x \in M$

Pf If M isn't a field then D is split & it's an exercise - any 2 bases are GL_2-equivalent.
If M is a field then D is a $D \otimes_F M$-module in 2 ways:

$$\delta \otimes m \ : \ x \mapsto \delta x \sigma_1(m) \text{ or } \delta x \sigma_2(m)$$

But $D \otimes_F M = M_2(M)$ has a unique module of dimension $2/M$. (exercise)

$\therefore\ \exists$ iso $\varphi : D \to D$ between the 2 actions
ie
$$\varphi(\delta x \sigma_1(m)) = \delta \varphi(x) \sigma_2(m) \quad \forall \delta, x \in D \ \forall m \in M.$$

So $\varphi(\sigma_1(m)) = \sigma_1(m)\varphi(1) = \varphi(1)\sigma_2(m)$

$\exists x \in D$ with $\varphi(x) = 1 \Rightarrow x\varphi(1) = 1 \Rightarrow \varphi(1)$ is invertible (only important if D split, of course)

$\therefore\ \sigma_2(m) = \varphi(1)^{-1} \sigma_1(m) \varphi(1) \quad \forall m \in M$. □

Exercise If $\sigma_1, \sigma_2 : D \hookrightarrow D_E$, E/F finite field extⁿ, then σ_1 & σ_2 are conjugate by an elt of $D_E^\times$.

Rk: $D \otimes_F D \cong \mathrm{End}_F(D)$
$\delta_1 \otimes \delta_2 \mapsto (x \mapsto \delta_1 x \delta_2^{\iota})$. This is a hint too.

Examples of Noether-Skolem thm in action (NB don't need char 0, evidently)

1) If $\#F < \infty$ & D/F is a quaternion algebra then D is split
Pf (exercise) Let M/F be the quadratic extⁿ. If D is not split then $D^\times = \bigcup_{\delta \in D^\times} \delta M^\times \delta^{-1}$. Now count elts on both sides $\Rightarrow \#$

2) $F = \mathbb{C}$ (or $F = \bar{F}$) : any quat alg is split

3) $F = \mathbb{R}$ $\mathcal{D}$ is either split or $\cong \mathbb{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\} = \mathbb{C} \oplus \mathbb{C}j$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- note $j^2 = -1$, $jzj^{-1} = \bar{z}$

Pf - exercise. If $\mathcal{D}$ is not split then $\mathbb{C} \hookrightarrow \mathcal{D}$ & complex conjugation gives 2 embeddings, so Noether - Skolem $\Rightarrow \exists\ \delta \in \mathcal{D}^\times$ s.t. $\delta z \delta^{-1} = \bar{z}$

Then $\mathcal{D} = \mathbb{C} \oplus \mathbb{C}\delta$; $\delta^2 \in$ centre of $\mathcal{D}$ $\therefore$ WLOG $\delta^2 = \pm 1$

$\delta^2 = +1 \Rightarrow \mathcal{D} \cong M_2(\mathbb{R})$

$\delta^2 = -1 \Rightarrow \mathcal{D} \cong \mathbb{H}$.

One final lemma for today. Quat algs can't be split by odd degree field ext's.

Lemma 7 Suppose E/F has odd degree, & $\mathcal{D}_E \cong \mathcal{D}_{E'}$, then $\mathcal{D} \cong \mathcal{D}'$.

Pf Omitted. It's interesting to note that although the proof appears to be elementary, it's so embedded in any book on the subject that it's difficult to extract. Eg:

Sublemma in thm 4.4.5 Herstein

Lemma 13.4 of Pierce

Cor Suppose E/F is odd degree ext^n, & $\mathcal{D}, \mathcal{D}'$ quat alg $/F$ with $\mathcal{D}' \hookrightarrow \mathcal{D}_E$. Then $\mathcal{D} \cong \mathcal{D}'$.

Pf Exercise

Lecture 2
Wed 17th Feb '93
11.00am

Two things he forgot to say / said wrongly last time.

$$\mathbb{H}^1 = \ker(\nu : \mathbb{H}^\times \longrightarrow \mathbb{R}^\times_{>0}) \cong SU(2) \text{ cpct.}$$

Now if we have $\mathcal{D}/F$ & E/F a quadratic ext^n with ~~E/F separable~~ char $F \neq 2$, $E = F(\sqrt{d})$ & E splits $\mathcal{D}$, then $E \hookrightarrow \mathcal{D}$.

The proof is $\left(\underset{\mathcal{D}_E}{\overset{}{\delta_1}} + \delta_2 \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\sqrt{d}}\right)^2 = 0$. Then $(\delta_1 \delta_2)^2 = d$.

1.2 Quaternion algebras over local fields (sketch proofs coming up)

$F/\mathbb{Q}_p$ a finite ext^n & $v : F^\times \twoheadrightarrow \mathbb{Z}$ a valuation

Say $\mathcal{D}/F$ is a non-split quaternion algebra.

Define $v_\mathcal{D} : \mathcal{D}^\times \longrightarrow \mathbb{Z}$ by $v_\mathcal{D} = v \circ \nu$, i.e. $\mathcal{D}^\times \xrightarrow{\nu} F^\times \xrightarrow{v} \mathbb{Z}$

We'll note some easy properties of $v_\mathcal{D}$

- $v_D(xy) = v_D(x) + v_D(y)$
- $v_D(1+x) \geq \min(0, v_D(x))$ (work in $F(\delta)$; $v \circ N_{F(\delta)/F}$ a valⁿ)
- $v_D(x+y) \geq \min(v_D(x), v_D(y))$
- $v_D(x^*) = v_D(x)$

Set $O_D = \{ x \in D \mid v(x) \geq 0 \}$

$m_D = \{ x \in D \mid v(x) > 0 \}$

$O_D^\times = \{ x \in D \mid v(x) = 0 \}$

O_D is free of rank 4 over O_F

(O_D spans D $\therefore$ $O_D \supseteq \Lambda$, Λ rank 4 free / O_F. So either
$O_D \subseteq p^{-i}\Lambda$ for some i (so we're OK)
or O_D has elements of arbitrarily small valuation (Λ cpct))

Next note O_D / m_D. It's a division ~~algebra~~ ring over the residue field of F.

$\therefore$ O_D / m_D is a field extension of O_F / m_F. Say it's got degree f.

Now all 2-sided ideals of O_D are powers of m_D.

We have $m_F O_D = m_D^e$ for some e

$ef = 4$ (exercise) (just as in local field extⁿ case)

Certainly $e \leq 2$. It's slightly trickier to prove $f \leq 2$.

Hence $e = f = 2$.

If E_0/F is the unramified quadratic extension, then $E_0 \hookrightarrow D$ (as above)

Then $\exists j \in D^\times$ s.t. $jxj^{-1} = x^*$ $\forall x \in E_0$

Then $D = E_0 \oplus E_0 j$, $j^2 \in F$. Scaling j on the right by an element of $E_0^\times$ changes j^2 by an elt of $N_{E_0/F} E_0^\times$

Hence WLOG $j^2 = 1$ or π_F. However, if $j^2 = 1$ then by choosing $x \in E_0$ s.t. $x^* = -x$ we see $x + xj \neq 0$, $(x+xj)^2 = 0$, so D is split, formally a contradiction. Hence $j^2 = \pi_F$ is the only choice.

Lemma 8 There is only one non-split quaternion algebra over F. Any quadratic extension of F embeds in D

Pf If $\overset{\text{res char.}}{p} \neq 2$ then $F(\sqrt{\pi_F}) \hookrightarrow D$ for arbitrary π_F & it's quite easy. (there's only 3 field ext's degree 2)

If $p=2$ then the same argument as the above line gives 3 embeddings. The unramified ext^n also embeds. For the rest, just repeat the above argument. It's a bit messy, but works. $\square$

Lemma 9 D/F not split $\Rightarrow D^1$ is cpct (note D/F split $\Rightarrow D^1 = SL_2(F)$ not cpct)

Pf $\mathcal{O}_D^\times$ is cpct because $\underbrace{\mathcal{O}_D^\times \supset}_{\text{finite index}} \underset{\uparrow \text{ cpct}}{1+m_D}$

& D^1 is closed in $\mathcal{O}_D^\times$ $\therefore D^1$ is cpct. $\square$

Remark For those of us who like concrete things, he'll tell us the non-split one explicitly:

$$D = \left\{ \begin{pmatrix} a & b \\ \pi_F \, {}^\sigma b & {}^\sigma a \end{pmatrix} \;\middle|\; a, b \in E_0 \right\}$$

$\sigma \neq 1 \in Gal(E_0/F)$, σ is the Frobenius

$J = \begin{pmatrix} 0 & 1 \\ \pi_F & 0 \end{pmatrix}$

Remark E/F splits $D \Leftrightarrow [E.F]$ is even.

So over a local field, everything is rather simple.

Now say D is any quaternion algebra / F, maybe split.

$\mathcal{O} \subseteq D$ is called an order if $\mathcal{O}/\mathcal{O}_F$ is free of rank 4 & $\mathcal{O}$ is an algebra.

Lemma 10 1) If D is split then any order is conjugate to a subset of $M_2(\mathcal{O}_F)$
2) If D is not split then $\mathcal{O} \subseteq \mathcal{O}_D$

Cor 1) If D is split then any maximal order is conjugate to $M_2(\mathcal{O}_F)$
2) If D is not split then $\mathcal{O}_D$ is the unique maximal order

Pf of lemma (sketch) 1) $\mathcal{O}$ is cpct. $\mathcal{O}$ stabilizes a lattice
2) $x \in \mathcal{O} \Rightarrow v(x) \geq 0$ as x is integral.

1.3 Quaternion algebras over number fields

(Now he'll hardly even sketch the proofs.)

Say $F/\mathbb{Q}$ finite. D/F a quaternion algebra, v a place of F, $D_v = D \otimes F_v / F_v$

$S(D) = \{ v \mid D_v \text{ is not split} \}$

Facts 1) $S(D)$ is finite, it contains no complex places, & $\# S(D)$ is even.

2) Any set S satisfying 1) comes from some quaternion algebra.

3) $S(D)$ determines D.

There's proofs of this in Pierce & Weil.

Facts (not nearly as deep) about orders

$\mathcal{O} \subseteq D$ is an order if $\mathcal{O}$ is an $\mathcal{O}_F$-algebra, $\mathcal{O}$ is f.g as an $\mathcal{O}_F$-module and $\mathcal{O} \otimes_{\mathcal{O}_F} F = D$.

Eg $M_2(\mathcal{O}_F) \subseteq M_2(F)$.

Here are the facts.

Orders exist. In fact, maximal orders exist.

If $\mathcal{O}$ is a fixed maximal order, then $\exists$ bijection

$\mathcal{O}' \longleftrightarrow \{\mathcal{O}'_v\}$ (localisations) Here v is running thro' all finite places.

~~where $\mathcal{O}$~~ & the bijection is between the orders $\mathcal{O}'$ of D and the collections of orders $\mathcal{O}'_v$ of D_v s.t. $\mathcal{O}'_v = \mathcal{O}_v$ for almost all v.

This is easy once you understand lattices

$\mathcal{O}'_v$ is maximal for almost all v.

$\mathcal{O}'$ is maximal $\Leftrightarrow$ $\mathcal{O}'_v$ is maximal for all v.

This is all an exercise if you understand lattices

Now fix a maximal order $\mathcal{O}_D$.

If $R/\mathcal{O}_F$ is a commutative algebra, define $G(R) = (\mathcal{O}_D \otimes_{\mathcal{O}_F} R)^\times$

Eg if $\mathcal{O}_D = M_2(\mathcal{O}_F)$, $D = M_2(F)$, then $G(R) = GL_2(R)$.

We have a reduced norm map $\nu : G_D(R) \to R^\times$.

Set $G^1_D = \ker \nu$. Eg $G^1 = SL_2$ in above example.

These are the generalisations of GL_2. John will talk about GL_2 & this is how to generalise it.

$G(\mathbb{A})$ is locally cpct once you've given it the correct topology, which isn't the subspace topology on $\mathcal{O}_D \otimes \mathbb{A}$

The correct topology is the subspace topology under the map inclusion

$$G(\mathbb{A}) \to D^2_{\mathbb{A}}$$
$$x \mapsto (x, x^{-1})$$

$G(\mathbb{A})$ is in fact a restricted direct product $\prod_v' G(F_v)$, restricted w.r.t $G(\mathcal{O}_{F,v})$

Of course, $G(F_v) = D^\times_{F_v}$, $G(\mathbb{A}) = (D \otimes \mathbb{A})^\times$, $G(\mathcal{O}_{F,v}) = \mathcal{O}^\times_{D_v}$.

$G(\mathbb{A})$ is just a generalisation of $GL_2(\mathbb{A})$.

Define $\|\cdot\| : G(\mathbb{A}) \to \mathbb{R}^\times_{>0}$

$$\begin{tikzcd} G(\mathbb{A}) \arrow[r, "\nu"] \arrow[rr, bend right, "\|\cdot\|"'] & \underset{\text{idèles}}{\overset{}{\mathbb{A}^\times}} \arrow[r, "\| \|"] & \mathbb{R}^\times_{>0} \end{tikzcd}$$

Define $G(\mathbb{A})^1 = \ker \|\cdot\|$.

Lemma 11 (c.f. Martin Taylors lemmas in $\mathbb{A}$, $\mathbb{A}^\times$ case)

$$\underset{\overset{\parallel}{D^\times}}{G(F)} \xhookrightarrow[\text{diag}]{} G(\mathbb{A})^1$$ & the image is discrete (product formula ensures $\|G(F)\| = 1$)

Pf To prove discreteness we replace F by an extension E/F which splits D, & we're reduced to the following problem...

We need $GL_2(F) \subset GL_2(\mathbb{A})^1$ discrete.

It will do to show $GL_2(O_F) \subset GL_2(F_\infty)$ is discrete

where $F_\infty = \prod_{v|\infty} F_v = \mathbb{A}_\infty$.

If $x \in O_F$ & $|x|_v < \frac{1}{2}$ $\forall v|\infty$ then $x = 0$. $\square$

Prop^n 12 If D is not split then $G(F)\backslash G(\mathbb{A})^1$ is compact.

Remarks i) Compactness fails in the split case.
ii) Things like finiteness of class group & unit theorem all are related to this result, so there's some content to the proof!

Lecture 3
Thu 18th Feb '93
11:00 am

Aside: more on Haar measure

Say G is a locally cpct top gp. Martin Taylor told us about m $C_c(G) \to \mathbb{R}$
$m({}_g f) = m(f)$ $\forall g \in G$

m had various properties & was unique up to scalar.

Richard wants to talk about measures of Borel sets.

Suffice it to say that the Riess representation thm gives us $m \leftrightarrow \mu$

μ: (Borel subsets of G) $\longrightarrow \mathbb{R}_{\geq 0}^{\cup\{\infty\}}$ a measure

The Borel subsets of G are the σ-field generated by the open sets.
σ-fields are things closed under complements & countable unions, roughly.

We have $\mu(gX) = \mu(X)$ for $g \in G$. Also, $K \subseteq G$ cpct $\Rightarrow \mu(K) < \infty$.

We have $\Delta_G : G \to \mathbb{R}^\times_{>0}$ NB $\mu(X) = \inf_{\substack{U \supseteq X \\ U \text{ open}}} \mu(U)$

Δ_G is a cts HM, defined by

$\mu(Xg^{-1}) = \Delta_G(g)\mu(X)$

G is unimodular $\Leftrightarrow_{\text{defn}} \Delta_G \equiv 1$ e.g. G cpct, G abelian, G discrete
($G = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ a ctrexample)

He's written

G is the points of a reductive algebraic group over a local field or the adele ring. I think that in this case he's asserting $\Delta_G \equiv 1$ again

A reference is Bourbaki, chapter 7 or something. Richard does not know why this above statement is true.

<u>Thm</u> If $H \leq G$ is a closed subgroup, & suppose $\Delta_G|_H = \Delta_H$.

(eg ✓ G & H is unimodular
✗ $G = GL_2(\mathbb{R})$, $\underset{H}{\cancel{B}} = B(\mathbb{R})$)

Then $\exists!$ measure (on Borel sets) of G/H s.t.

1) $\mu(gX) = \mu(X) \quad \forall g \in G$

2) $\displaystyle\int_G \varphi(g)\, \underset{\substack{\text{Fixed} \\ \text{Haar} \\ \text{measure} \\ \text{on } G}}{\overset{}{dg}} = \int_{G/H} \int_H \varphi(gh)\, \underset{\substack{\text{Fixed} \\ \text{Haar} \\ \text{measure on } H}}{dh}\, d\bar{g}$

~~[illegible]~~

~~[illegible]~~ OK in the sense that LHS exists $\Rightarrow \int_H \varphi(gh)\,dh$ exists almost everywhere & is integrable, & LHS = RHS,

~~LHS exists ⇒ RHS exists & equal.~~

& also if $\int_H |\varphi(gh)|\,dh$ exists for almost all g & $\int_{G/H}\int_H |\varphi(gh)|\,dh\,dg$ exists then LHS exists & LHS = RHS.

~~[illegible]~~ $\square$

NB Martin did the case where H was cpct & normal, & it's easier then as f's with cpct support on G/H can be pulled back to f's with cpct support on G.

That's all on Haar measure.

Say D/F is a quat alg / F a number field. Set $\mathbb{A} = \mathbb{A}_F$.

Pick a fixed max^l order $\mathcal{O}_D$

$G_D(F) = G(F) = D^\times$. $G^1 = \ker(\nu : G \to GL_1)$

$G(\mathbb{A})^1 = \ker(G(\mathbb{A}) \xrightarrow{\nu} \mathbb{A}^\times \xrightarrow{\|\ \|} \mathbb{R}^\times_{>0})$

$G(F) \subseteq G(\mathbb{A})^1$ discrete subgp.

Here comes that prop^n again.

Prop^n 12 If D is not split then $G(F)\backslash G(\mathbb{A})^1$ is cpct.

Pf From Weil's book. Slick but unenlightening.

Sublemma V/F a vector space. $V \subseteq V\otimes\mathbb{A}$ is discrete & $V\backslash V\otimes\mathbb{A}$ is cpct.

Pf WLOG $F=\mathbb{Q}$. WLOG $\dim_{\mathbb{Q}} V = 1$. $\mathbb{Q}\backslash\mathbb{A} = \hat{\mathbb{Z}} \times \mathbb{Z}\backslash\mathbb{R}$ $\square$

Cor $\exists$ Haar measure μ on $V\otimes\mathbb{A}$ s.t. $\mu(V\backslash V\otimes\mathbb{A}) = 1$ $\square$

Cor If $C \subseteq V\otimes\mathbb{A}$ is a Borel set with $\mu C > 1$ then $\exists x,y\in C$ with $x-y\in V\setminus\{0\}$.

Pf $\mu C = \int_{V\backslash V\otimes\mathbb{A}} \#\pi^{-1}(x)\,dx$ where $\pi : V\otimes\mathbb{A} \to V\backslash V\otimes\mathbb{A}$

$\therefore \#\pi^{-1}(x) > 1$ for some x. $\square$

Rk If $d\in\mathbb{A}^\times$ then $\mu(dX) = \|d\|^{\dim V}\mu(X)$

~~Pf of prop^n 12~~ Choose $C \subseteq D\otimes\mathbb{A}$ cpct with $\mu C > 1$

& $C' = \{x-y \mid x,y\in C\}$ cpct

$C'' = \{xy \mid x,y\in C\}$ cpct

Then $C''\cap D^\times = \{\gamma_1,\dots,\gamma_r\}$ is finite

$X = \{x\in G(\mathbb{A}) \mid (x,x^{-1}) \in (\bigcup_{i=1}^{r} \gamma_i^{-1}C') \times C'\}$ is cpct (with the $G(\mathbb{A})$-topology not the $D\otimes\mathbb{A}$ one)

Claim: $G(\mathbb{A})^1 \subseteq G(F).X$

Pf Let $d\in G(\mathbb{A})^1$. Then (exercise) $\mu(dC) = \mu(C) > 1$ & so

(μ is a Haar measure for addition, & so we have distributivity!)

$\exists$ $x,y\in C$ & $\delta\in D^\times = D\setminus\{0\}$ s.t. $d(x-y) = \delta$ (D not split)

Also $\mu(Cd^{-1}) > 1$ & so $\exists x',y'\in C$, $\delta'\in D^\times$ s.t. $(x'-y')d^{-1} = \delta'$

Then $\delta'\delta = (x'-y')(x-y) \in C''\cap D^\times$

& $(x-y)^{-1} \in X$

$\overset{\shortparallel}{}$ $(\delta'\delta)^{-1}(x'-y') = \gamma_i^{-1}(x'-y')$ for some $i = 1,2,\dots,r$. & we're home $\square$

There's a proof, for what it's worth.

Exercise Repeat the argument to show that $F^\times \backslash (\mathbb{A}^\times)^1$ is compact and deduce that the class no. of F is finite & Dirichlet's unit theorem for F.

Exercise $\mathcal{O}_D$. Invertible fractional rt ideal $I \subset D$ is

- finite $\mathcal{O}_F$-module that spans D
- $I\mathcal{O}_D \subseteq I$
- $\forall v \quad I_v = \delta_v \mathcal{O}_{D,v}, \ \delta_v \in D_v$

RI $\quad I \sim J$ if $I = \delta J, \ \delta \in D^\times$

Say $\underline{\underline{RIC(\mathcal{O}_D)}}$. $\sim$-classes.

Show $\# RIC, < \infty$

(Hint: Note $RIC(\mathcal{O}_D) \leftrightarrow D^\times \backslash G(\mathbb{A}^\infty) / \prod_{v \nmid \infty} \mathcal{O}_{D,v}^\times$. Bit of a hefty hint, actually)

$$\delta\, \mathcal{O}_D \longleftarrow\!\shortmid \ \delta$$

Exercise $\left\{ \begin{matrix} \text{Conj. classes of} \\ \text{max}^l \text{ orders} \end{matrix} \right\} \longleftrightarrow D^\times \backslash G(\mathbb{A}^\infty) / \prod_{\substack{v \in S(D) \\ v \nmid \infty}} D_v^\times \times \prod_{\substack{v \notin S(D) \\ v \nmid \infty}} \mathcal{O}_{D,v}^\times$,

$$\delta\, \mathcal{O}_D\, \delta^{-1} \longleftarrow\!\shortmid \ \delta$$

Exercise Show # conj classes of max^l orders $< \infty$

Lemma ~~13~~ 12? Whether or not D is split, we have

1) $\mu(G(F) \backslash G(\mathbb{A})^1) < \infty$

2) If v is a place of F then $\exists$ cpct set $X \subseteq G(\mathbb{A})$ s.t. $G(\mathbb{A}) = G(F) X G(F_v)$

Pf D not split first. 1) ✓ 2) $\mathbb{R}^\times_{>0} / |\nu G(F_v)|_v$ is cpct.

D split: Use Iwasawa to reduce to Borel & then use normal results for ideles & adeles. □

Note 1) in case $F = \mathbb{Q}$, $D = M_2(\mathbb{Q})$ translates as $\mu(SL_2(\mathbb{Z}) \backslash \mathfrak{h}) < \infty$.

(IV.14)

<u>Prop</u>

Prop^n 13 (Strong Approx Thm) D/F. If v is a place of F, $v \notin S(D)$, then

$$G(F)G^1(F_v) \subseteq G^1(\mathbb{A}) \text{ is dense.}$$

Note John mentioned this in the $SL_2(\mathbb{A})$ case, $F = \mathbb{Q}$.

There's some generalisation to ~~a~~ simply-connected alg gps, or sthg.

I think he said this meant that every ideal had a totally +ve generator.

surely ∞?

Cor Suppose D is split at v and that O_F has strict class no. one. Let $U \subseteq G(\mathbb{A}^\infty)$ be an open cpct subgp, Y $U = \prod_{v \nmid \infty} O_{F,v}^\times$. Then $G(F)UG(F_\infty) = G(\mathbb{A})$.

Pf Exercise. □

Cor Suppose D is split at some ∞ place, & O_F has strict class no. 1.

Then a) all max^l orders in D are conjugate.
b) # $RIC(O_D) = 1$

Pf Exercise. □

He will omit the pf of prop. 13 because "something's got to give". I think he said that there was a proof in Vigneras.

Lecture 4
Fri 19th Feb '93
11:00 am

There will be no lunch tomorrow because 11 sandwiches isn't enough for the caterer.

2 Functional Analysis

He's changing tack totally today.

2.1 Hilbert spaces

Reference: Gelfand & Vilenkin - Generalised f^{n}s vol 4. chap 1 §2
Wallach - Real reductive gps 1 §8.A.1
appendix

Richard says that Wallach's book has served him well. There's lots of results by Harish-Chandra that aren't really written up anywhere, as far as Richard knows, except in Wallach (lovely) & Harish-Chandra's collected works (much less user-friendly).

H, a Hilbert space is a v.s./$\mathbb{C}$ with $(\,,\,) : H \times H \to \mathbb{C}$

$$(\lambda x + y, z) = \lambda(x,z) + (y,z)$$
$$(x,y) = \overline{(y,x)}$$
$$(x,x) \geq 0 \ \& = 0 \iff x = 0$$

s.t. H is complete w.r.t. $\|\cdot\|$ where

$$\|x - y\| = (x-y, x-y)^{1/2}$$

He will also assume H is separable, i.e. has a ctble dense subset, unless he explicitly says otherwise. This is to keep him out of trouble.

So say H is a (separable) Hilbert space.

If $G \subseteq H$ set $G^\perp = \{ x \in H \mid (x,y) = 0 \ \forall y \in G \}$

G a closed subspace $\Rightarrow H = G \oplus G^\perp$.

An orthonormal basis $\{e_i\}$ is not really a basis. $(e_i, e_j) = \delta_{ij}$ & $\sum \mathbb{C} e_i$ is dense in H.

Then $\{e_i\}$ is necessarily countable.

Also, $x \in H \Rightarrow \sum_{i=1}^{\infty} (x, e_i) e_i$ converges, and converges to x.

Then $(x,y) = \sum_{i=1}^{\infty} (x, e_i)(e_i, y)$

<u>Examples</u>

1) If X is a measure space, $L^2(X) = \{ f: X \to \mathbb{C} \mid f \text{ m'able}, \int_X |f(x)|^2 dx < \infty \}$

$$(f_1, f_2) = \int_X f_1(x) \overline{f_2(x)} \, dx$$

$L^2(X)$ is a Hilbert space, not necessarily separable!

2) H_i, $i = 1, 2, \ldots$ Hilbert spaces.

$\hat{\bigoplus} H_i$ consists of vectors (x_i) s.t. $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$

Then $((x_i), (y_i)) = \sum_i (x_i, y_i)$.

Say $T: H \to H$ is linear.

Write 1) $L_\infty(H)$ for the <u>bounded</u> linear maps,

ie $\{ T \mid \exists A > 0 \text{ s.t. } \|Tx\| \leq A\|x\| \ \forall x \in H \}$.

2) An off-putting thing about this stuff, to this number-theorist (ie Richard) is there's lots of collections of linear maps, each having its own use. Here's another one:

= <u>cpct</u> linear maps. $\to K(H) (\subseteq L_\infty(H)) = \{ T \mid T\{x \in H \mid \|x\| \leq 1\} \text{ ~~is~~ has cpct closure} \}$

If you learn analysis in Cambridge then you these are precisely the kinds of maps you learn about.

Contained in 2) is

3) $L_2(H)$ = <u>Hilbert-Schmidt</u> operators. For some orthonormal basis $\{e_i\}$ we have $\sum_i \|Te_i\|^2 < \infty$.

$\cup\!|$ Hence for all o.n. basis, $\sum_i \|Te_i\|^2 < \infty$.

These little remarks are exercises, if you're brave. Richard tried to do them & got stuck. He then looked in Gelfand & everything got easier. He thinks they're exercises if you're slightly cleverer than him.

4) $L_1(H)$ <u>Trace class</u> (or nuclear: check defn if you're reading a book because things vary) - for some ON basis $\{e_i\}$, $\sum_i \|Te_i\| < \infty$

($\Rightarrow$ true $\forall \{e_i\}$ ON)

$\cup\!|$

5) $FR(H)$ = <u>finite-diml range.</u>

We have $\| \|_\infty$ on L_∞ : $\|T\|_\infty = \sup_{\|x\|=1} \|Tx\|$

$\wedge\!\backslash$ $\cup\!|$

$\| \|_2$ on L_2 : $\|T\|_2 = \left(\sum \|Te_i\|^2\right)^{1/2}$ for any o.n. $\{e_i\}$

$\wedge\!\backslash$

$\| \|_1$ on L_1 : $\|T\|_1 = \sup_{\text{over all } \{e_i\},\{f_i\}} \sum_i |(Te_i, f_i)|$

They're all norms. L_∞ & K are complete w.r.t. $\|\cdot\|_\infty$

L_2 is complete w.r.t. $\|\cdot\|_2$

L_1 is complete w.r.t. $\|\cdot\|_1$.

Also $FR \subseteq L_1$ has $\|\cdot\|_1$-closure L_1 & $\|\cdot\|_2$-closure L_2 & $\|\cdot\|_\infty$-closure K.

Why we're doing all this is that you want to take the trace of a repn & it's not clear how you should do this in the ∞-diml case.

$L_2(H)$ is another Hilbert space: $(T,S) = \sum_i (Te_i, Se_i)$, indpt of $\{e_i\}$.

Given T, $\exists T^*$ s.t. $(Tx,y) = (x, T^*y)$ $\forall x,y \in H$.

T^* is the adjoint of T.

All spaces are preserved under T^*. Norms don't change.

$T, S \in L_\infty \Rightarrow TS \in L_\infty$, & $\|TS\|_\infty \leq \|T\|_\infty \|S\|_\infty$

$T \in L_\infty$, $S \in K \Rightarrow TS$ & $ST \in K$

$T \in L_\infty$, $S \in L_2 \Rightarrow TS, ST \in L_2$, and $\|TS\|_2 \leq \|S\|_2 \|T\|_\infty$

$\|ST\|_2 \leq \|S\|_2 \|T\|_\infty$

$T, S \in L_2 \Rightarrow TS \in L_1$. NB this seems to be about the only way of checking that things are Trace class.

If $T \in L_1(H)$, we define $\operatorname{tr} T = \sum_{i=1}^{\infty} (Te_i, e_i)$

This sum is absolutely convergent, & indpt of $\{e_i\}$.

NB some people define T to be trace class if $\sum (Te_i, e_i)$ is absolutely cgt. This seems to strictly contain L_1, but doesn't seem to have a sensible norm on it.

~~Defn~~ Note $\operatorname{tr} : L_1(H) \to \mathbb{C}$ is cts.

If $T, S \in L_2(H)$, $\operatorname{tr}(TS) = (T, S^*)$

Thm (Spectral thm He's gonna state it in a slightly more general, slightly uglier than usual, form. A good reference is Dieudonné's book.)
(Richard was rather impressed by this thm)

Say H is a Hilbert space, $V \subseteq H$ a dense subspace. (Usually the theorem is stated for $V = H$).

Say $T \in K(H)$ with $T^* = T$ and $TV \subseteq V$.

Let the spectrum of T $= \sigma(T)$ be $\left\{ \lambda \in \mathbb{C} \;\middle|\; \underset{\text{invertible on } H}{T - \lambda \mathrm{Id} \text{ is not}} \right\}$

Define $V_\lambda = \{ v \in V \mid Tv = \lambda v \}$

Then $\sigma(T) \subseteq \mathbb{R}$; the only possible limit point is 0

If $\lambda \neq 0$ then $\dim V_\lambda < \infty$

and $H = (\ker T) \oplus \widehat{\bigoplus_{\substack{\lambda \in \sigma(T) \\ \lambda \neq 0}}} V_\lambda$ and V_0 is dense in $\ker T$. (Recall $V \subseteq H$ dense)

Defn T in the theorem ~~if~~ is called positive if $\sigma(T) \subseteq \mathbb{R}_{\geq 0}$.

2.2 Kernels

X locally cpct, Hausdorff & with a ctble basis of open sets.

μ a measure ~~d~~ on Borel sets of X s.t. A cpct $\Rightarrow \mu(A) < \infty$.

Then $L^2(X)$ is separable, & $C_c(X) \underset{\text{dense}}{\subseteq} L_2(X)$.

See Rudin: Real & complex analysis for this stuff.

If $K \in L^2(X \times X)$ we get $T_K : L^2(X) \to L^2(X)$

$$f \mapsto \int_X K(x,y) f(y)\, dy$$

Then $T_K f \in L^2(X)$ & $\|T_K f\|_2 \le \|K\|_2 \, \|f\|_2$

different $\|\cdot\|_2$ s.

Moreover, T_K is bounded & $\|T_K\|_\infty \le \|K\|_2$.

In fact T_K will be Hilbert-Schmidt, I think he said.

If $\{e_i\}$ is an o.n. basis of $L^2(X)$ then $\{e_i(x), \overline{e_j(y)}\}$ is an o.n. basis of $L^2(X \times X)$.

Thus $K = \sum a_{ij} e_i(x) \overline{e_j(y)}$ in $L^2(X \times X)$

& $T_K e_K = \int \sum_{ij} a_{ij} e_i(x) \overline{e_j(y)} e_k(y)\, dy$

$= \sum_i e_i a_{ik}$, and $\|T_K\|_2 = \|K\|_2 < \infty$

Hence T_K is Hilbert-Schmidt, & hence T_K is cpct.

Also, $T_K^* = T_{K^*}$, where $K^*(x,y) = \overline{K(y,x)}$

$T_{K_1} T_{K_2} = T_{K_1 * K_2}$, where $(K_1 * K_2)(x,y) = \int_X K_1(x,z) K_2(z,y)\, dz$

These are easy exercises.

Prop^n This is the prop^n that there was such a clamour to prove last time. He'll prove it next time.

Suppose $\mu(X) < \infty$.

1) Suppose $K \in C_c(X \times X)$, $K^* = K$, and T_K is positive. Then T_K is trace class, and $\operatorname{tr} T_K = \int_X K(x,x)\, dx$.

This is where the work lies. The statement "T_K is positive" is difficult to check. We can get sthg more though...

2) Suppose $K_1, K_2 \in C_c(X \times X)$, Then $T_{K_1 * K_2}$ is trace class, & $\operatorname{tr} T_{K_1 * K_2} = \int_X (K_1 * K_2)(x,x)\,dx$.

All the work goes into 1), & 2) conversant.

There's a short false pf. He'll tell it us as it may convince us it's true.

False pf 1) $T_K : C_c(X) \to C_c(X)$.

Choose $\lambda_i \in \mathbb{R}_{>0}$ s.t. $\lambda_{i+1} \geq \lambda_i$, $\lambda_i \in \sigma(T)$, & every elt of $\sigma(T) - \{0\}$ occurs lbs with multiplicity $\dim L^2(X)_\lambda$.

Then $\exists\, \varphi_i \in C_c(X)$ with $T_K \varphi_i = \lambda_i \varphi_i$

& $\psi_j \in \ker T_K$, s.t. $\{\varphi_i, \psi_j\}$ is an o.n. basis.

$L^2(X \times X)$ has basis $\{ \varphi_i(x)\overline{\varphi_j(y)}$ + psi's $\}$

$K(x,y) = \sum a_{ij} \varphi_i(x) \overline{\varphi_j(y)} + \dots$ with ψ's in $L^2(X \times X)$

$T_K \varphi_i = \lambda_i \varphi_i$, $T_K \psi_j = 0$

$\underset{\text{(exercise)}}{\Rightarrow}$ $K(x,y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)}$ in $L^2(X \times X)$

The false line of the proof is the next one.

$$\int_X K(x,x)\,dx = \sum_{i=1}^{\infty} \lambda_i \int_X \varphi_i(x)\overline{\varphi_i(x)}\,dx$$

$$= \sum_{i=1}^{\infty} \lambda_i = \operatorname{tr} T_K.$$

The point is that ~~ptwise cgce~~ you haven't even got ptwise cgce: "$\sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\overline{\varphi_i(x)} = K(x,x)$"

may not be true for every x,
because $K = \sum \lambda_i \varphi_i \overline{\varphi_i}$ is an equality
~~of~~ of f's in L^2.

He made a rather improper remark about analysts here, but I shall not record it.

Lecture 5
Sat 20th Feb '93
McKean
2:30pm

Recall last time X loc. cpct Hausdorff. ctble basis, μ a measure on Borel subsets of X, $\mu(K)<\infty$ if K cpct.

$$H=L^2(X). \qquad \begin{array}{ccc} L_2(H) & \xleftarrow{\sim} & L^2(X\times X) \\ T_K & \longleftarrow & K(x,y) \end{array}$$

an iso of Hilbert spaces. $(T_K f)(x) = \int_X K(x,y) f(y)\,dy$

$$T_K^* = T_{K^*} \; ; \; K^*(x,y) = \overline{K(y,x)}$$

$$T_{K_1}T_{K_2} = T_{K_1 * K_2} \quad ; (K_1 * K_2)(x,y) = \int_X K_1(x,z)K_2(z,y)\,dz$$

Prop^n 2 If K_1 & $K_2 \in L^2(X\times X)$ & $K = K_1 * K_2$ then T_K is trace class, &

$$\mathrm{tr}\, T_K = \int_X K(x,x)\,dx.$$

Last time he said he was going to deduce it from a very complicated thing. But in fact this is easy to prove, as he now realises.

Pf $\mathrm{tr}\, T_K = \mathrm{tr}\, T_{K_1}T_{K_2} = (T_{K_1}, T_{K_2}^*) = (T_{K_1}, T_{K_2^*}) = \int_{X\times X} K_1(x,y)\overline{K_2^*(x,y)}\,dxdy$

$$= \int_{X\times X} K_1(x,y)K_2(y,x)\,dxdy.$$

By Fubini's thm, $\int_X (K_1 * K_2)(x,x)\,dx$ exists & equals this.

So we are home. □

2.3 Hilbert & admissible rep^ns

↑ ie rep^ns on Hilbert spaces. Everything seems to be a mess in the literature. There don't seem to be any general theorems - you just prove what you need when you need it. He will try to do stuff in some generality.

$$G = G(\mathbb{A}), G(\mathbb{A})^1, G(F_v) \ldots$$
$$G = G^\infty \times G_\infty.$$

Fix $K_\infty \leq G_\infty$ max^l cpct, $U \leq G^\infty$ fixed cpct open.

$\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$

$\mathfrak{U} = \mathfrak{U}(\mathfrak{g}_\infty)$, $\mathfrak{z}$ = centre of $\mathfrak{U}$.

$f: G \to \mathbb{C}$ is called smooth if given $g = g^\infty \times g_\infty \in G$ $\exists$ nhd U of g^∞

$$\begin{tikzcd} U \times G_\infty \arrow[r, "\text{proj}"] \arrow[dr, "f"'] & G_\infty \arrow[d, dashed, "\text{smooth map}"] \\ & \mathbb{C} \end{tikzcd}$$

s.t.

This may well just be "f loc cst @ finite places & smooth @ ∞ places".

① $\pi : G \longrightarrow GL(H)$, H a Hilbert space, & $\pi : G \times H \longrightarrow H$ cts.

Wallach calls this a Hilbert rep^n

If $\operatorname{im} \pi \subseteq$ unitary autos then π is unitary.

Also assume that ~~K~~ $\pi K_\infty \subseteq$ unitary autos - this can always be achieved by varying the inner product without varying the topology - see e.g. Wallach 1.4.8

If $\varphi \in C_c(G)$ then we can define $\pi(\varphi) : H \to H$ s.t.

$$(\pi(\varphi)w, w') = \int_G \varphi(g)(\pi(g)w, w')\, dg \quad \forall w, w' \in H$$

② $G^\infty \times (\mathfrak{g}_\infty, K_\infty)$-module

V a vector space. $\pi : \begin{cases} G^\infty \times K_\infty \longrightarrow \mathrm{Aut}(V) \\ \mathfrak{g}_\infty \longrightarrow \mathrm{End}(V) \end{cases}$

s.t.

a) $\langle gv \mid g \in U \times K_\infty \rangle_{\mathbb{C}}$ is f.d

b) $X \in \mathrm{Lie}(K_\infty) \Rightarrow \pi(X)v = \frac{d}{dt}\left(\pi(\exp(tX))v\right)\Big|_{t=0}$

c) $X \in \mathfrak{g}_\infty$, $g \in G^\infty \times K_\infty$, then

$$\pi(g)\pi(X)\pi(g^{-1}) = \pi(g_\infty X g_\infty^{-1}).$$

Aside If one had a Hilbert space rep^n H of $G(F_v)$, v finite
then for $v \in H$ we say v is smooth if $g \mapsto \pi(g)v$ is smooth
v is finite if finite under U.

These are the same thing if $v < \infty$. For $v | \infty$ they might be different.

$\left.\begin{array}{l} v \text{ smooth} \\ v\ K_\infty\text{-finite} \end{array}\right\}$ need both!

Sometimes you need to have just 1 or the other - eg v just smooth
smoothness is invt under the group action of G
K-finiteness isn't - it turns into gKg^{-1}-finiteness!

③ Say $\pi: G \to \mathrm{Aut}(V)$, V a top. v.s., $G \times V \to V$ cts

A <u>smooth repⁿ</u> is one s.t. $\forall v \in V$, the map $G \to V$, $g \mapsto \pi(g)v$ is smooth

People usually assume V is a Fréchet space or something.
He doesn't really want to talk about all this.

④ If (π, H) is a Hilbert repⁿ, $H = \hat{\bigoplus}_{\sigma} H(\sigma)$, σ f.d. irred repⁿˢ of K_∞

$$H(\sigma) = \sum_{\theta: \sigma \to H} \mathrm{Im}(\theta)$$

↙ = Wallach.
See e.g. [W] 1.4.7

π is <u>admissible</u> if $\dim H(\sigma)^W < \infty$ $\forall \sigma$ $\forall W \subseteq G^\infty$ open cpct.

⑤ V a $G^\infty \times (\mathfrak{g}_\infty, K_\infty)$-module. $V = \bigoplus V(\sigma)$ as above.

V is <u>admissible</u> if $\dim V(\sigma)^W < \infty$ $\forall \sigma, W$ as above.

⌈ NB we get $\pi: \mathfrak{g}_\infty \to \mathrm{End}(V)$ for free in ③ ; $\pi(X)v = \frac{d}{dt}\left(\pi(\exp(tX))v\right)\Big|_{t=0}$ ⌋

<u>Lemma 3</u> 1) V a smooth repⁿ ⇒ $V^\circ = \{ v \in V \mid \dim \langle K_\infty v \rangle < \infty \}$ is a $G^\infty \times (\mathfrak{g}_\infty, K_\infty)$-module.

2) H a Hilbert repⁿ. Set $H^\infty = \{ v \in H \mid g \mapsto \pi(g)v \text{ smooth} \}$! This will be a Fréchet space (once you put some norms on it)

If $G = G_\infty$ then H^∞ can be given the structure of a smooth repⁿ.

3) If H a Hilbert repⁿ, define $H^\circ = (H^\infty)^\circ$. Then $H^\circ = G^\infty \times (\mathfrak{g}_\infty, K_\infty)$-module, & H° is dense in H.

4) H is admissible ⇔ H° is admissible.

In this case, $H^\circ = \{ v \in H \mid \langle K_\infty U v \rangle \text{ is f.d.} \}$

He'll sketch some pfs

Pf 1) You just need to check that K_∞-finiteness is preserved by $\mathfrak{g}_\infty$

But $\mathfrak{g}_\infty \otimes \langle K_\infty v \rangle \to V$ has f.d. image, preserved by K_∞.

2) [W] 1.6.4. You see H^∞ will probably be dense in H so you need a new norm & you take some sequence of norms somehow.

3) H° is dense in H is the only tricky part.

Anyway, $H^\circ \supseteq \bigoplus_\sigma H(\sigma)$ (probably equality holds here ?!)

It will do to show $H(\sigma) \cap H^\infty$ is dense in $H(\sigma)$.

Choose $\varphi_i \in C_c^\infty(G)$ s.t. $\mathrm{supp}(\varphi_i) \supseteq \mathrm{supp}(\varphi_{i+1})$, φ_i +ve real-valued.

$$\bigcap \mathrm{supp}(\varphi_i) = \{1\}$$

$$\int \varphi_i = 1 \ ; \ \varphi_i(x) = \varphi_i(x^{-1})$$

$$\varphi_i(kgk^{-1}) = \varphi_i(g) \ \forall k \in K_\infty, g \in G$$

$\pi(\varphi_i) H(\sigma) \subseteq H(\sigma)$

$\pi(\varphi_i) H \subseteq H^\infty$

If $v \in H$ then $\pi(\varphi_i) v \to v$ ☺

4) There's only the problem that some $H(\sigma)$ may grow.

Pick $W \subseteq G^\infty$ open cpct, $\overset{\text{f.d.}}{\sigma}$

$$\begin{tikzcd} \text{Then } H(\sigma)^W & \supseteq H^\circ(\sigma)^W & \supseteq (H(\sigma) \cap H^\infty)^W \arrow[ll, bend left, "\text{dense}"] \end{tikzcd}$$

so $H(\sigma)^W$ must also be f.d. □

He hopes he's given us the idea.

Lemma 4 (he won't prove this one) $V \subseteq H^\circ$ invt subspace, V is $\mathfrak{z}$-finite

$$\Rightarrow \overline{V} \text{ is } G\text{-invt}$$

Pf Wallach - use 3.4.9 & 1.6.6. It's quite deep. Use the fact that there's some elliptic differential operator so elts of V are well-behaved. □

<u>Lemma 5</u> Suppose H is unitary irred, $V \subseteq H$ dense G-inv^t subspace
$T: V \to H$ commutes with G-action
& suppose $\exists S\ V \to H$ s.t. $(Tx, y) = (x, Sy)\ \forall x, y \in V$.
Then T is a scalar

Pf [W] 1.2.2

These last 2 lemmas & the next one are due to Harish-Chandra.

<u>Lemma 6</u> V a f.g. $(\mathfrak{g}_\infty, K_\infty)$-module (ie $\exists v_1, .., v_n$ s.t. only submodule $\ni v_i\ \forall i$ is V) consisting of $\mathfrak{z}$-finite vectors. Then V is admissible. This is the case if V is irreducible. □

Pf [W] 3.4.9 □

ecture 6
Mon 22^nd Feb '93
11:00am

Last time Richard was talking about the relationship between various kinds of rep's of G,
G = eg $G_0(\mathbb{A})$, $G_0(F_v)$. we wanted $G = G^\infty \times G_\infty$, $K_\infty \subseteq G_\infty$ max^l cpct, $\mathfrak{g}_\infty = \text{Lie}(G_\infty)$.

We had Hilbert rep's & smooth rep's & sometimes admissible rep's & stuff,
& we had a dictionary $\underset{\text{Hilbert rep}^n}{\overset{\uparrow}{H}} \to \underset{G^\infty \times (\mathfrak{g}_\infty, K_\infty)\text{-module}}{\overset{\uparrow}{H^\circ}}$ & with some $\mathfrak{z}$-finiteness we could get back, or something

We did some funky lemmas last time eg Schur's lemma. Note that in Lemma 6 last bit we use the amazing fact that Schur's lemma holds if $\dim V < 2^{\aleph_0}$.

<u>Cor 7</u> H irreducible unitary $\Rightarrow \mathfrak{z}$ acts by a character χ_H on H°.
↳ Hilbert rep^n for G.
$\chi_H: \mathfrak{z} \to \mathbb{C}$ is the <u>infindesimal character</u>

Pf Use lemma 5 applied to H^∞.
If $z \in \mathfrak{z}$; $T = \pi(z)$, $S = \pi(z^*)$ where $*: \mathcal{U} \to \mathcal{U}$
& on $\mathfrak{g}_{\infty\mathbb{C}}$ $*$ is $x \mapsto -\bar{x}$
(note $(xy)^* = y^* x^*$) □

(See eg Wallach 1.6.5)

NB Richard doesn't know what happens if you remove the unitary cond^n. He has no feeling for the subject, really. I don't think Labesse knows either but he does look deep in thought. There's no "lemmas for the algebraists" - all the results have nasty analysis words in like unitary. ∴

<u>Cor 8</u> H irred unitary G_∞-module $\Rightarrow H$ admissible (NB Labesse says also true if for G_v, but we're not sure for G)

Pf H° is $\mathfrak{z}$-finite by cor 7. If $0 \neq v \in H^\circ$, let $V = \mathcal{U}\langle K_\infty v\rangle \subseteq H^\circ$.
Lemma 6 $\Rightarrow V$ admissible
Lemma 4 $\Rightarrow \bar{V}$ is G_∞-invariant $\therefore \bar{V} = H$; $\underset{\text{f.d.}}{V(\sigma)} \overset{\text{dense}}{\subseteq} H(\sigma)$ σ irred rep of K_∞
$\therefore H(\sigma) = V(\sigma)$ f.d. □

He had hoped to get everything from ~~to~~ lemmas 4,5,6, all in Wallach. He now realises he needs more than lemma 4 for the next bit, so he'll have to ~~be~~ resort to Harish-Chandra. NB it may not be so bad - he only realised all this this morning & has just resorted to Harish-Chandra for completeness /desparation.

Lemma 9 V is a smooth repⁿ of G_∞, $v \in V^\circ$ & v is $\mathfrak{z}$-finite.
Then $\exists \varphi \in C_c^\infty(G_\infty)$ s.t. $\pi(\varphi)v = v$

Pf Harish-Chandra : repⁿs of semisimple Lie gps on Banach spaces I □

Cor 10 Say H a Hilbert space repⁿ of G. Then H° irred admis $\Leftrightarrow$ H irred admis.

Pf ($\Rightarrow$) clear. ($\Leftarrow$) Suppose H irred. Use the trick that John used this morning.

If $W \subseteq H^\circ$ is a proper submodule, then $\exists V \subseteq W$ an irred submodule.

⌈The proof of this uses Zorn: for σ some irred rep of $K_\infty \times K^\infty$, ~~open~~ (K° open cpt in G^∞) we have $W(\sigma) \neq (0)$ but it's f.d. so choose $V \subseteq W$ minˡ s.t. $V(\sigma) \neq 0$. ⌋

$\mathfrak{z}$ acts by scalars on V (Schur)

$\overline{V}$ is invt $\therefore \overline{V} = H \therefore V(\sigma) \underset{\emptyset\ \text{dense}}{\subseteq} \underset{\underset{\text{fd}}{\uparrow}}{H(\sigma)} \therefore V(\sigma) = H(\sigma) \therefore V = H^\circ = \bigoplus H(\sigma)$. □

Cor 11 If H is a Hilbert repⁿ of G, & suppose H° is admissible & $\mathfrak{z}$-finite

$$\left\{ \begin{array}{c} \text{submodules} \\ V \subseteq H^\circ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{closed invt subspaces} \\ H^1 \subseteq H \end{array} \right\}$$

$$\begin{array}{ccc} V & \longmapsto & \overline{V} \\ (H^1)^\circ & \longleftarrow\!\shortmid & H^1 \end{array} \text{ (invt by lemma 4 or sthg)}$$

□

Lemma 12 H_1, H_2 admissible unitary. If H_1 irred & $H_1^\circ \cong H_2^\circ$ then $H_1 \cong H_2$ & even the inner product is preserved

Pf easy - see e.g. Wallach chapter III. Use the adjoint map ; if $\varphi : H_1^\circ \to H_2^\circ$ we get $\varphi : H_1(\sigma) \to H_2(\sigma)$ & then $\varphi^* : H_2(\sigma) \to H_1(\sigma)$
$\varphi^* \cdot \varphi$ $H_1^\circ \circlearrowleft$ & Schur $\Rightarrow$ scalar. WLOG it's 1 So φ preserves inner product; extend by continuity & check the action's preserved. He'll spare us the details to save time.

Lemma 13 $G = G(F_v)$, v finite or infinite, H irred admiss, $\varphi \in C_c^\infty(G)$

Then $\pi(\varphi)$ is trace class, & in fact $\operatorname{tr} \pi(\varphi)$ only depends on the corresponding rep π° (so we can write $\operatorname{tr} \pi^\circ(\varphi)$).

Pf $v \nmid \infty$, exercise (note we haven't assumed H is unitary)
$v \mid \infty$: [W] 8.1.2.

Lemma 14 (not really about rep^ns but it's analytic so he'll throw it in here)

If $G = G(F_v)$, $\varphi \in C_c^\infty(G)$, then $\exists \varphi_i, \chi_i \in C_c(G)$ for $i = 1..r$ s.t.

$$\varphi = \sum_{i=1}^{r} \varphi_i * \chi_i \qquad \left(\varphi_i * \chi_i(g) = \int_G \varphi_i(h)\, \chi_i(h^{-1}g)\, dh \right)$$

Pf $v \nmid \infty$: exercise
$v \mid \infty$ Jean-Pierre did it so ask him. □

Lemma 15 Suppose π is an irred admiss. unitary rep^n of $G(\mathbb{A})$. Let π° be the associated $G(\mathbb{A}^\infty) \times (\mathfrak{g}_\infty, K_\infty)$-module. Then

$\pi^\circ = \bigotimes_v{}' \pi_v^\circ$ with π_v° irred admiss, uniquely determined, & unramified almost everywhere.

He'd rather hoped John would have told us about this bit this morning, but it appears it'll be this afternoon.

Then $\pi_v^\circ = (\pi_v)^\circ$ for some (!ly determined) irred admiss unitary rep^n π_v of $G(F_v)$ (quick pf: embed π_v° in π & take its closure, ~~then~~ use earlier facts)

Then $\pi = \hat{\bigotimes}_v \pi_v$, & if $\varphi = \prod \varphi_v \in C_c^\infty(G(\mathbb{A}))$, then

$$\operatorname{tr} \pi(\varphi) = \prod_v \underbrace{\operatorname{tr} \pi_v^\circ(\varphi_v)}_{=1 \text{ a.e., I think because } \varphi_v = \text{char fn of } K_v \text{ a.e., or sthg.}}$$

□

He'll tell us what $\hat{\bigotimes}_v \pi_v$ is: π_v has an o.n. basis $\{e_i^v\}$, & for almost all v we have $\mathbb{C} e_0^v = \pi_v^{G(\mathcal{O}_{F_v})}$ & $\|e_i^v\| = 1$.

$$\text{Then } \left\{ \hat{\bigotimes_v} e_{i(v)}^v \;\middle|\; \begin{array}{l} i : \{\text{places}\} \to \mathbb{Z}_{\geq 0} \\ i(v) = 0 \text{ for almost all } v \end{array} \right\} \text{ is an o.n. basis for } \hat{\bigotimes_v} \pi_v .$$

That concludes the analytic results. Now he'll do sthg which, to him at least, seems more interesting.

3. Automorphic forms on $G_D(F)\backslash G_D(\mathbb{A})$

In this section $F/\mathbb{Q}$ is a finite ext & D/F is always a non-split quat alg.

3.1 $L^2(G(F)\backslash G(\mathbb{A})^1)$

Set $X = G(F)\backslash G(\mathbb{A})^1$ & $H = L^2(X)$.

Then fix an iso $G(\mathbb{A}) = G(\mathbb{A})^1 \times \mathbb{R}^\times_{>0}$ ($\mathbb{R}^\times_{>0}$ embeds diagonally at ∞)

$G(\mathbb{A})$ acts in a unitary way on $L^2(X)$:

(def of action) $(R(g)f)(h) = f(hg)$ (here $g \in G(\mathbb{A})^1$, $f \in L^2(X)$, $h \in X$)

& R is trivial on $\mathbb{R}^\times_{>0}$.

If $\varphi \in C_c^\infty(G(\mathbb{A}))$ then define $\underbrace{(R(\varphi)}_{\text{"Hecke operator"}} f)(h) = \int_{G(\mathbb{A})} \varphi(g) f(hg)\,dg$

Note ~~$R(\varphi)$~~ $R(\varphi)^* = R(\varphi^*)$ where $\varphi^*(g) = \overline{\varphi(g^{-1})}$

$R(\varphi_1)R(\varphi_2) = R(\varphi_1 * \varphi_2)$; $(\varphi_1 * \varphi_2)(g) = \int_{G(\mathbb{A})} \varphi_1(h)\varphi_2(h^{-1}g)\,dh$

Define $K_\varphi : X \times X \to \mathbb{C}$ by

$$K_\varphi(g,h) = \sum_{\gamma \in G(F)} \varphi(g^{-1}\gamma h)$$ (note defined for X not just $G(\mathbb{A})^1$)

- If g, h lie in some cpct then the sum is finite $\therefore K_\varphi \in C(X \times X)$.

<u>Lemma 1</u> If D is not split then $K_\varphi \in C_c(X \times X)$ (note this is false if D split, & that's what makes the whole theory harder)

and $T_{K_\varphi} = R(\varphi)$.

In particular, $R(\varphi)$ is Hilbert-Schmidt & hence cpct.

~~Pf~~ $(R(\varphi)f)(g) = \int_{G(\mathbb{A})^1} \varphi(h) f(gh)\,dh$ (he's probably integrated out the $\mathbb{R}^\times_{>0}$ fibre).

$= \int_{G(\mathbb{A})^1} f(h)\varphi(g^{-1}h)\,dh$ (change of variable)

He might be assuming $\varphi \in C_c^\infty(G(\mathbb{A})^1)$.
He thinks this is WLOG.

$$\therefore (R(\varphi)f)(g) = \int_{G(F)\backslash G(\mathbb{A})^1} \sum_{\gamma\in G(F)} f(\gamma h)\,\varphi(g^{-1}\gamma h)\,dh$$

$$= \int_{G(F)\backslash G(\mathbb{A})^1} f(h)\,K_\varphi(g,h)\,dh \qquad \square$$

This afternoon he'll discuss the ramifications of this point.

Lecture 7
Mon 22nd Feb
3.45pm

Recall ~~F~~ F a no. field, D/F a non-split quat alg, G s.t $G(F) = D^\times$

We're looking at $L^2(G(F)\backslash G(\mathbb{A})^1)$

He should remark that this all ties up with what John's doing:

$\varphi: GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) \to \mathbb{C}$ gives us $S_k \subseteq L^2(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})^1)$

& the analysis he was doing earlier relates the analytic L^2 with the more algebraic S_k.

Now if $\varphi \in C_c(G(\mathbb{A}))$ we can construct $\tilde{\varphi}$ by $\tilde{\varphi}(g) = \int_{\mathbb{R}^\times_{>0}} \varphi(gz)\,d^\times z$

so because of this remark which is a slightly more rigorous ~~remark~~ version of a remark he made earlier, we reduce ourselves to the case $\varphi \in C_c(G(\mathbb{A})^1)$.

Then define $(R(\varphi)f)(h) = \int_{G(\mathbb{A})^1} \varphi(g) f(hg)\,dg$

& $K_\varphi = \sum_{\gamma\in G(F)} \varphi(g^{-1}\gamma h) \in C(X\times X)$

Lemma 1: (D not split) $T_{K_\varphi} = R(\varphi)$ & so $R(\varphi)$ is H-S & cpct.

We will try & understand the L^2 space like John's done S_k: questions to ask are ~~are~~ is it ss? Can we decompose irred bits into local bits?

Prop^n 2 (D not split) $L^2(G(F)\backslash G(\mathbb{A})^1) = \hat{\bigoplus} \pi^{m_\pi}$, π irred unitary rep^n of $G(\mathbb{A})^1$, π distinct, $m_\pi \in \mathbb{Z}_{>0}$ (NB. $m_\pi \leq 1$ or sthg - multiplicity 1) (we'll just show $m_\pi < \infty$)

Pf will be a variant on the pf John gave this morning - we dont have admissibility though ∴ He thinks there is a variant of the pf which will prove admissibility.

<u>Step 1</u> Suppose $0 \neq f \in H$. Then $\exists \varphi \in C_c(G(\mathbb{A})^1)$ s.t. $\varphi = \varphi^*$ and $R(\varphi) f \neq 0$.

Pf Let U be an open nhd of 1 in $G(\mathbb{A})^1$ s.t. $u \in U \Rightarrow \|\pi(u) f - f\| < \frac{1}{2}\|f\|$

Choose $\varphi \in C_c(G(\mathbb{A})^1)$ s.t.

- supp $\varphi \subseteq U$
- $\varphi^* = \varphi$
- φ +ve real-valued
- $\int \varphi = 1$

eg choose U_1 s.t. $U_1 U_1 \subseteq U$, & φ_1 s.t. supp $\varphi_1 = U_1$, $\int \varphi_1 = 1$, φ_1 +ve real-valued, & set $\varphi = \varphi_1^* * \varphi_1$.

Now
$$\|R(\varphi) f - f\| = \Big\| \int_{G(\mathbb{A})^1} \varphi(g) R(g) f \, dg - f \Big\|$$
$$= \Big\| \int_{G(\mathbb{A})^1} \varphi(g) (R(g) f - f) \, dg$$
$$\leq \int_U \varphi(g) . \| R(g) f - f\| \, dg \leq 1 \cdot \tfrac{1}{2}\|f\|$$

$\therefore R(\varphi) f \neq 0.$ $\square$

<u>Step 2</u> Suppose $0 \neq H_1 \subseteq H$ is a closed invt subspace; then H_1 contains a closed irred. subspace.

Pf Choose $0 \neq f \in H_1$ & φ as in step 1. Then $R(\varphi)|_{H_1}$ is non-zero & self-adjoint cpct.

Let $V \subseteq H_1$ be an eigenspace for $R(\varphi)$ with a non-zero eigenvalue. Then by the spectral theorem, $\dim V < \infty$.

Now, let $H_0 \subseteq H_1$ be a min^t closed invt subspace with $H_0 \cap V \neq (0)$. (Zorns lemma)

He claims H_0 is irred. If not then $H_0 = H_2 \oplus H_3$ with H_2, H_3 closed invt.

Then $H_0 \cap V = (H_2 \cap V) \oplus (H_3 \cap V)$ as $R(\varphi)$ preserves H_2 & H_3
$\therefore$ for $i = 2$ or 3 we have $H_i \cap V = H_0 \cap V$ so $H_0 = H_i$. $\square$

Now things are rather easy.

<u>Step 3</u> $H = \hat{\oplus}\pi$, π irreducible

<u>Pf</u> Take a maxl set of $\{W_i\}$, W_i closed invt subspace s.t. $\hat{\oplus} W_i \subseteq H$. (Zorn)
Then $H = H_1 \oplus (\hat{\oplus} W_i)$. If $H_1 \neq (0)$ then $\exists\, H_0 \subseteq H_1$ irred, &

$$\{W_i\} \cup \{H_0\} \supsetneqq \{W_i\} \qquad \# \qquad \square$$

<u>Step 4</u> No π occurs with infinite multiplicity.

<u>Pf</u> If it did, then choose $0 \neq f \in \pi \hookrightarrow H$.

Choose $\varphi \in C_c(G(\mathbb{A}^1))$ s.t. $R(\varphi) f \neq 0$

Let λ be a non-zero eigenvalue of $R(\varphi)|_\pi$. If π occurs infinitely often then the λ-eigenspace of $R(\varphi)$ is ∞-diml. $\square$

$\square$

Note there was some content in all of this - that $R(\varphi)$ was cpct. In the split case it's false. In fact

<u>Rk</u> The result is false for $L^2(GL_2(F)\backslash GL_2(\mathbb{A})^1)$.

However, $L^2_{\text{cusp}}(GL_2(F)\backslash GL_2(\mathbb{A})^1) = \hat{\oplus}\, \pi^{m_\pi}$, π irred, $m_\pi < \infty$ in this case,

where L^2_{cusp} fs have the added property that $\int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix}1&u\\0&1\end{pmatrix}g\right)du = 0$ for

almost all g (NB he doesn't put "for all g" as these L^2-fs are only defined up to some f which is 0 a.e.). The error is "the theory of Eisenstein series"

<u>Cor</u> $L^2(G(F)\backslash G(\mathbb{A})^1) = \hat{\oplus} H^\alpha$, where $\alpha : \mathfrak{z} \to \mathbb{C}$, & $\mathfrak{z}$ acts on $(H^\alpha)^\circ$ via α.
Here $\mathfrak{z}$ is the centre of $\mathcal{U}(\text{Lie}\, G_\infty)$ $\square$

There's some debate now as to whether $\int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix}1&u\\0&1\end{pmatrix}g\right)du = 0$ makes sense for almost all g.
Richard has never really thought about it before. He thinks Fubini's thm kills it. Brian Birch thinks everything is OK. GL_2 is, of course, really John's problem.

<u>Prop 3</u> Suppose D is not split. Then H^α is admissible $\forall \alpha$

<u>Pf</u> ~~we have~~ Say $U \subseteq G(\mathbb{A}^\infty)$ an open cpct subgp. Look at H^α as a $U \times G(F_\infty)^1$-module.
Apply the argument of prop 2

Then $H^{\alpha} = \hat{\bigoplus} \pi^{m_\pi}$, π distinct irred unitary reps of $U \times G(F_\infty)^1$, $m_\pi < \infty$

$\therefore (H^{\alpha})^U = \hat{\oplus} \pi^{m_\pi}$, π distinct irred unitary rep of $G(F_\infty)^1$, $m_\pi < \infty$.

I think the idea is irred unitary $\rightsquigarrow$ K-finite vectors are admissible $\rightarrow$ only finitely many choices (δ or sthg)

$\exists$ only finitely many π with infinitesimal char α

$\therefore$ the sum is finite

each π is admissible

$\therefore (H^{\alpha})^U(\sigma)$ is f.d. for all irred reps of K_∞, σ

$\therefore$ H^{α} admissible

~~Rk~~ Cor Any irred constituent π of $L^2(G(F)\backslash G(\mathbb{A})^1)$ is admissible (& factorisable)

Rk Prop 3 + cor remain true for $L^2_{cusp}(GL_2(F)\backslash GL_2(\mathbb{A})^1)$

So we've shown precisely what John's doing in the holomorphic cusp form case

Prop 4 (Trace formula)

If D is not split, $L^2(G(F)\backslash G(\mathbb{A})) = \hat{\bigoplus}\limits_{\pi \text{ irred (distinct)}} \pi^{m_\pi}$

$\prod\limits_{v} \varphi_v = \varphi \in C_c^\infty(G(\mathbb{A})^1)$

$\uparrow$ group v/s together

Then $\sum\limits_{\pi} m_\pi \operatorname{tr} \pi(\varphi) = \operatorname{tr} R(\varphi)$

(Here $[\gamma]$ = conj class of γ in $G(F)$)

$$\xrightarrow{\hspace{3cm}} = \sum_{[\gamma] \subseteq G(F)} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1)\, O_\gamma(\varphi)$$

An infinite sum, but it's absolutely cgt.

Here G_γ = centraliser of γ in G, &

$$O_\gamma(\varphi) = \int_{G_\gamma(\mathbb{A})^1\backslash G(\mathbb{A})^1} \varphi(g^{-1}\gamma g)\, dg$$ is an orbital integral.

We've got to make sure our measures (are normalised?) correctly to get an equality. Fix a Haar measure on $G(\mathbb{A})^1$. Pts of $G_\gamma(F)$ have measure 1. Use the same Haar measure twice for $G_\gamma(\mathbb{A})^1$.

So $\operatorname{tr} \pi(\varphi) = \prod_v \operatorname{tr} \pi_v(\varphi|_v)$,almost all 1

$$O_\gamma(\varphi) = \prod_v \int_{G_\gamma(F_v)\backslash G(F_v)} \varphi(g^{-1}\gamma g)\, dg \text{ ,(almost all 1)}$$

The way to think about this equality is

$$\text{"Trace of Hecke operator} = \sum_{\substack{\text{quadratic}\\ \text{fields}}} \text{orbital integrals}$$

Lecture 8
Tues 23rd Feb '93
~~1:30pm~~
11:00am

Today he wants to talk about the proof of propn 4, the trace formula.

$\ulcorner$ **Propn 4** If D not split, $L^2(G(F)\backslash G(\mathbb{A})^1) = \hat{\bigoplus} \pi^{m_\pi}$ & $\varphi \in C_c^\infty(G(\mathbb{A})^1)$

$$\text{then } \underset{\substack{\shortparallel \\ \operatorname{tr} R(\varphi)}}{\sum m_\pi \operatorname{tr} \pi(\varphi)} = \sum_{[\gamma] \subset G(F)} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F)^1)\, O_\gamma(\varphi)$$

& both sides abs cgt. $\lrcorner$

Pf WLOG $\varphi = \varphi_1 * \varphi_2$ (as any φ is $\sum_i^f \varphi_1 * \varphi_2$), $\varphi_i \in C_c(G(\mathbb{A})^1)$ (Recall its tricky at infinity - a thm of Labesse+?)

Then $R(\varphi) = R(\varphi_1) R(\varphi_2)$ is trace class, and

$$\operatorname{tr} R(\varphi) = \int_X K_\varphi(x,x)\, dx = \int_X \sum_{\gamma \in G(F)} \varphi(x^{-1}\gamma x)\, dx$$

$$= \int_X \sum_{[\gamma]} \sum_{\delta \in [\gamma]} \varphi(x^{-1}\delta x)\, dx$$

Everything is abs. cgt, so by Fubini's thm we can interchange Σ & $\int$:

$$= \sum_{[\gamma]} \int_{G(F)\backslash G(\mathbb{A})^1} \sum_{\delta \in [\gamma]} \varphi(x^{-1}\delta x)\, dx$$

$$= \sum_{[\gamma]} \int_{G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1} \varphi(x^{-1}\delta x)\, dx \quad \text{(by abs cgce \& stuff)}$$

$$= \sum_{[\gamma]} \int_{G_\gamma(\mathbb{A})^1\backslash G(\mathbb{A})^1} \int_{G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1} \varphi(x^{-1}y^{-1}\gamma y x)\, dy\, dx$$

$$= \sum_{[\gamma]} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A})^1) \int_{G_\gamma(\mathbb{A})^1\backslash G(\mathbb{A})^1} \varphi(x^{-1}\gamma x)\, dx$$

$\uparrow$ indpt of γ, as we'd expect. $\square$

Tony will be explaining an application of the trace formula later in the week

3.2 Automorphic Forms

He stresses again that D is not split.

Fix $\chi : \mathbb{R}^\times_{>0} \to \mathbb{C}^\times$. Then

Def: $\mathcal{A}_\chi = \left\{ \varphi : G(F) \backslash G(\mathbb{A}) \longrightarrow \mathbb{C} \;\middle|\; \right.$
- φ smooth
- φ is $U.K_\infty$-finite, here $K_\infty \subseteq G(F_\infty)$ max^l cpct & $U \subseteq G(\mathbb{A}^\infty)$ open cpct
- φ is $\mathfrak{z}$-finite — here $\mathfrak{g} = (\text{Lie}\, G(F_\infty))_{\mathbb{C}}$, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, $\mathfrak{z}$ = centre of $\mathcal{U}$
- $\varphi(gz) = \chi(z)\varphi(g)$ $\forall z \in \mathbb{R}^\times_{>0}$ — here $\mathbb{R}^\times_{>0} \hookrightarrow F^\times_\infty \subseteq G(F_\infty)$

If you take the sum over all unitary chars χ you get something analagous to A^0 or A or something. Boundedness & stuff all comes from compactness. φ will be bounded on $G(\mathbb{A})^1$ (& slowly increasing on $G(\mathbb{A})$ if χ isn't unitary) (bounded on $G(\mathbb{A})$ if χ is unitary)
Also $\nexists$ cuspidal cond^n as $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ has no analogue in $G(\mathbb{A})$.

Set $\mathcal{A} = \mathcal{A}_{\substack{\text{trivial}\\ \text{char}}}$. Then $\mathcal{A}_\chi = \mathcal{A} \otimes \chi_0 \|\nu\|^{1/2d}$, $d = [F:\mathbb{Q}]$

~~[illegible]~~ $\longleftarrow\!\shortmid\; \varphi \otimes 1$
$\varphi \chi_0 \|\nu\|^{1/2d}$

Rk

$$L^2(G(F)\backslash G(\mathbb{A})^1)^{0,\mathfrak{z}\text{-finite}} = \mathcal{A}$$
$$\|$$
$$\hat{\bigoplus} \pi^{m_\pi}$$

& $\pi \rightsquigarrow \pi^0$ UK_∞-finite vectors (no smoothing necessary, recall)

$$\Rightarrow \mathcal{A} = \bigoplus (\pi^0)^{m_\pi} \quad (\text{alg direct sum})$$

Also, π^0 irred admiss $\Rightarrow \pi^0 = \bigotimes'_v \pi^0_v$

$\alpha : \mathfrak{z} \to \mathbb{C}$; $\mathcal{A} = \bigoplus \mathcal{A}^\alpha$, $\mathcal{A}^\alpha$ admissible

Write $\text{Aut}(G_D(\mathbb{A}))$ = <u>the automorphic rep's of $G_D(\mathbb{A})$</u> for the set of automorphic rep's π^0 occuring in any $\mathcal{A}_\chi$.

<u>Rk</u> In GL_2 case, we can define $\mathcal{A}^\circ_\chi$ - cuspidal automorphic forms on $GL_2(\mathbb{A})$ & all the remarks above remain true.

To the cond's in our def^n of $\mathcal{A}_\chi$ above you would add 1) φ cuspidal
2) φ slowly increasing

Then John's A° is $\bigoplus\limits_{\substack{\chi \\ \text{unitary}}} \mathcal{A}^\circ_\chi$

Write $\mathrm{Aut}^\circ(GL_2(\mathbb{A}))$.

<u>Theorem 5</u> (Jacquet - Langlands)

they only sketched the details of the trace formula they needed. Arthur completed the pf. While he was doing this, a chap called something like Munizo also proved the thm in another way.

The thm says that it's really rather similar whether you use D or GL_2.

1) F a local field, $F \neq \mathbb{C}$, D the non-split quat alg $/F$; then J-L define an injection "JL" from irred admiss reps of $D^\times$ to irred admiss reps of $GL_2(F)$ s.t.

a) the image consists of all discrete series reps:

ie $v|\infty$ ✓ $v\nmid\infty$ only get special or supercuspidal

b) $(v\nmid\infty)$ $JL(\chi \circ \nu) = \sigma(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$

c) $(v|\infty)$ $(\mathbb{H} \hookrightarrow M_2(\mathbb{C})$ standard$)$

$$JL(\mathrm{symm}^{k-2}(\mathrm{std}) \otimes \nu^s) = \sigma(|t|^{s+k-3/2}, |t|^{s-1/2}(\mathrm{sgn}\, t)^k)$$

here $k \in \mathbb{Z}_{\geq 2}$, $s \in \mathbb{C}$, & these are all the irred admiss reps of $D^\times$

Note that i) he hasn't defined it completely yet if $v\nmid\infty$, but see d)
ii) it looks nicer if you use the Local Langlands $n=2$ & look at reps of WD_F instead

d) The global part of the thm holds as well
(this tells us sthg about the supercuspidal case, $v\nmid\infty$)

2) If F is a global field, D/F a non-split quat alg.

$$\pi = \bigotimes_v{}' \pi_v \in \mathrm{Aut}(G_D(\mathbb{A})).$$

Then a) $m_\pi = 1$

b) either $\pi = \chi . \det$ for some χ, or
$\exists JL(\pi) \in \mathrm{Aut}^\circ(GL_2(\mathbb{A}))$ (note $^\circ$ - cuspidal auto reps)

$$\text{s.t. } JL(\pi) = \Big(\bigotimes_{v \notin S(D)}{}' \pi_v\Big) \otimes \Big(\bigotimes_{v \in S(D)} JL(\pi_v)\Big)$$

c) The image of JL is all elts ρ of $\mathrm{Aut}^\circ(GL_2(\mathbb{A}))$ s.t. ρ_v is discrete series $\forall v \in S(D)$.

<u>Thm 6</u> (JL + Jacquet - Shalika)

Suppose $\pi_1, .., \pi_r$ & $\pi_1', .., \pi_r' \in \mathrm{Aut}(G_D(\mathbb{A}))$ are ∞-diml.

Suppose $\exists$ finite set S of bad primes containing all ∞ primes & all v s.t. $\pi_{i,v}$ or $\pi_{i,v}'$ ramifies, & all bad primes of D too, but maybe even bigger than this

Then $\forall v \notin S$ $\quad \pi_{i,v} : \{\alpha_{i,v}, \beta_{i,v}\} \subseteq \mathbb{C}^\times$

$$\pi_{i,v}' : \{\alpha_{i,v}', \beta_{i,v}'\} \subseteq \mathbb{C}^\times$$

If also $\forall v \notin S$, $\mathrm{diag}(\alpha_{1,v}, \beta_{1,v}, \dots, \alpha_{r,v}, \beta_{r,v})$ & $\mathrm{diag}(\alpha_{1,v}', \beta_{1,v}', \dots, \alpha_{r,v}', \beta_{r,v}')$ are conjugate in $GL_{2r}(\mathbb{C})$.

Then $\pi_1, .., \pi_r$ is a permutation of $\pi_1', .., \pi_r'$. $\square$

The same is true for $\mathrm{Aut}^\circ(GL_2(\mathbb{A}))$ $r = 1$ is Strong Multiplicity 1.
I think he said sthg about proving it for GL_2 & then using JL corresp.

<u>3. Examples</u>

This is just a big exercise. If you understand things it shouldn't be too hard.

$F = \mathbb{Q}$. a) $S(D) = \{\infty, p\}$

$$\bigoplus_{\substack{\pi \in \mathrm{Aut}(G_D(\mathbb{A})) \\ \pi_\infty \text{ trivial}}} \pi^\infty = \Big\{ f : D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times \to \mathbb{C} \;\Big|\; f \text{ right invt by some open subgp} \Big\}$$

Also $$S = \bigoplus_{\substack{\pi \text{ as above s.t.} \\ \pi_p \text{ is trivial on } D_p^{(2)}}} \pi^{\infty,p} = \Big\{ f : D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / D_p^{(2)} \to \mathbb{C} \;\Big|\; f \text{ right invt by some open subgp} \Big\}$$

$D_p^{(2)}$ = elts of even val^n in $D_p^\times$

If $x \in D_p^\times$ s.t. $v_{D_p}(x)=1$, then $R(x)^2=1$ (as x^2 & [illegible] has even val^n)

$$S = \underset{\underset{R(x)=1}{\uparrow}}{S^+} \oplus \underset{\underset{R(x)=-1}{\uparrow}}{S^-}$$

$$\text{If } U = \prod_{q \neq p} \mathcal{O}_{D,q}^\times \text{ then } S^U = \{ f : \overset{\overset{\text{ie locally free} \qquad \text{ie fractional ideals}}{\text{right invertible ideal classes}}}{RIC}(\mathcal{O}_D) \to \mathbb{C} \}$$

$$\& \ (S^+)^U = \{ f : \underset{\text{conj classes of max}^l \text{ orders}}{MO(D)} \to \mathbb{C} \}$$

After some discussion with Fröhlich, Richard now believes that for $I, J \in RIC(\mathcal{O}_D)$, even if $I \not\cong J$ we have $I \oplus \mathcal{O}_D \cong J \oplus \mathcal{O}_D$

Use the Jacquet-Langlands thm to deduce

$$S^U \big/_{\substack{\text{constant}\\ \text{maps}}} \cong S_2(\Gamma_0(p)) \supseteq T_p \ ; \ T_p^2 = 1 \ \& \ \text{~~[illegible]~~} \ (S^+)^U \big/_{\substack{\text{cst}\\ \text{maps}}} \cong S_2(\Gamma_0(p))^+.$$

These maps are Hecke equivariant: T_q, S_q for $q \neq p$
$q \neq p$ S_q acts trivially on both sides

$$(\underset{\underset{S^U}{\cap}}{T_q f})(\underset{\underset{\text{ideal class}}{\uparrow}}{[I]}) = \sum_{\substack{J \subseteq I \\ [I:J]=q^2}} f([J])$$

↑ this sum is over $q+1$ things.

It may well be (Brandt matrices) ~~[illegible]~~ that this stuff above was known before JL. JL is really a rather beautiful way of seeing it.

eg $p = \underline{11}$; $\dim_{\mathbb{C}} S^U = 2$; get $[\mathcal{O}_D]$ & $[I]$

$[\mathcal{O}_D] - [I]$ is an eigenclass & it's the one that survives when you mod out by csts

$$T_q \text{ has eigenvalue } a_q : a_q = \#\left\{ \begin{array}{c} J \subseteq \mathcal{O}_D \\ [\mathcal{O}_D : J] = q^2 \\ J \sim \mathcal{O}_D \end{array} \right\} - \#\left\{ \begin{array}{c} J \subseteq \mathcal{O}_D \\ [\mathcal{O}_D : J] = q^2 \\ J \not\sim \mathcal{O}_D \end{array} \right\}$$

so we get the eigenvalues explicitly in terms of the arithmetic of D

We also know ...

$y^2+y=x^3-x^2$ is related to all this.

So $y^2+y=x^3-x^2$ has $q - a_q$ pts / $\mathbb{F}_q$

$$\therefore \quad \left| \frac{q+1}{2} - \#\left\{ \begin{array}{l} J \leq O_D \\ [O_D : J] = q^2 \\ J \sim O_D \end{array} \right\} \right| < \sqrt{q}$$

He had intended to say sthg about the indefinite case, but he's got no time so he'll stop now.

V Orbital Integrals

Richard Taylor

ecture 1
ues 23rd Feb '93
4:00

Tomorrow at 9:30 it's Richard V
" Wed 11 it's VII Peter Schneider
2:30 V
4:00 VI (Tony)

Thu : 9:30 VII
11:00 VI
→ 1:30 VIII (Labesse)
3:00 VI (Tony)

Fri 9:30 V (final RT)
11:00 VI (final Tony)
2:30 VIII
4:00 IX

Sat 9:30 VIII
11:00 IX
2:00 VIII
3:30 IX
} Eisenstein series

Peter Schneider has lost a lecture.

Thursday @ 4:30 Mill Lane rm 9 Frances ~~Carwen~~ Kirwain will be talking about quotients of varieties or sthg & Atiyah-Jones conjecture.

In these final lectures we'll be understanding how the trace formula helps us understand base change.

$$\begin{tikzcd} \pi \text{ of } GL_2(\mathbb{A}_F) \arrow[r, squiggly] & \text{rep}^n\ \rho \text{ of } Gal(\bar{F}/F) \arrow[d] \\ \pi_E & \rho|_{Gal(\bar{F}/E)} \text{ of } E/F. \arrow[l, dashed, "?"'] \end{tikzcd}$$

The construction $\pi \to \pi_E$ is base change.

It has implications for the Artin conjecture. (M. Harrison)

Maten Taylor did enough for GL_1 to convince us that it's true in this case:

$$\begin{tikzcd}
\chi \text{ a grössenchar of } \mathbb{A}_F^\times \text{ of type } A_0 \arrow[r] & \chi_\ell \text{, } \ell\text{-adic char of } Gal(\bar{F}/F) \arrow[d] \\
\chi \circ N_{E/F} \text{ a g.c. of } \mathbb{A}_E^\times & \chi_\ell|_{Gal(\bar{F}/E)} \arrow[l]
\end{tikzcd}$$

To understand the case $n=2$, & maybe general n? we use this trace formula.

Orbital integrals

4 lectures:
1) Norms & σ-conjugacy
2) Matching orbital integrals
3) Pf
4) Geometry of orbits.

He'll "look after the bad places".

Norms & σ-conjugacy

Say F is a finite extn of $\mathbb{Q}$ or $\mathbb{Q}_p$ or $\mathbb{R}$

E/F finite ; $[E:F]=\ell>2$ prime (Langlands also treats $\ell=2$)

$E=F^\ell$ (not a field!) or E a field, E/F Galois, $G(E/F)=\langle\sigma\rangle$, $\sigma^\ell=1$

If $E=F^\ell$ then σ cyclically permutes the coordinates.

He wants to consider D/F a quaternion algebra, split or non-split.

$tr: D\to F$, $\nu: D^\times \to F^\times$

Defn 1) $x\in D$ is **central** $\Leftrightarrow$ $x\in F$
2) $x\in D$ is **regular** $\Leftrightarrow$ $T^2-(tr\,x)T+\nu(x)$ has distinct roots ($\Leftrightarrow F(x)$ is a quadratic field extn of F, or sthg)
3) $x\in D$ is **semi-simple** $\Leftrightarrow$ x central or regular.

Note that D not split $\Rightarrow$ all elts of D are ss.

3½) $\underline{D_E}=D\otimes_F E$ & $\underline{M_E}=M\otimes_F E$ if $M\subseteq D$.

4) If $x, y \in D$, then x, y are <u>conjugate</u> $\Leftrightarrow \exists\, \delta \in D^\times$ s.t. $\delta x \delta^{-1} = y$ ie $x \sim y$

<u>Lemma 1</u>

1) $x, y \in D$ are conjugate in $D_E \Leftrightarrow x \sim y$ in D
2) $x \in D_E$ & $x \sim {}^\sigma x \Rightarrow x \sim y$, $y \in D$.

<u>Pf</u> This is easy but important.
Cases.

1) i) x central $\Rightarrow x = y$
ii) x not ss. Then D is split & $x \sim \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \sim y$
iii) x regular. Then by the Noether-Skolem thm, $F(x) \hookrightarrow D$ by $x \mapsto x$ or $x \mapsto y$

Noether-Skolem $\overset{\text{for } D}{\Rightarrow}$ $x \sim y$.

2) i) x central $\Rightarrow x = {}^\sigma x \Rightarrow x \in F$
ii) x not ss $\Rightarrow x \sim \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ s.t. ${}^\sigma\alpha = \alpha$
iii) x regular. Then $E(x)$ splits D_E $\therefore$ $E(x)$ splits D $\therefore$ $F(x)$ splits D (as odd degree ext's don't affect anything)

$\therefore F(x) \hookrightarrow D$. $\Box$

He's not at all sure that this is a sensible way to do things!

<u>Def^n</u> 1) If $x \in D_E^\times$ then $\underline{Nx} = x\,{}^\sigma x \ldots {}^{\sigma^{l-1}}x$

Note ${}^\sigma(Nx) = x^{-1}(Nx)\,x$. Thus by the lemma above, Nx is conjugate to a unique conjugacy class $[Nx]$ in $D^\times$.

2) $x \in D_E^\times$ is <u>σ-regular</u> $\Leftrightarrow$ $[Nx]$ regular
<u>σ-ss</u> $[Nx]$ ss

3) $x, y \in D_E^\times$. Say x & y are <u>σ-conjugate</u>, $\underline{x \sim_\sigma y}$, iff $\exists\, g \in D_E^\times$ s.t. $x = g^{-1} y ({}^\sigma g)$.

4) If $x \in D_E^\times$ we define the <u>σ-centraliser</u>

$$C_x^\sigma = \{ g \in D_E^\times \mid g^{-1} x ({}^\sigma g) = x \}$$

<u>Rks</u> (exercises) 1) $C^\sigma_{h^{-1} x\, {}^\sigma h} = h^{-1} C^\sigma_x h$

2) $N(g^{-1} x\, {}^\sigma g) = g^{-1}(Nx) g$ $\therefore [N(g^{-1} x\, {}^\sigma g)] = [Nx]$.

Lemma 2

or if $D = M_2(F)$ M can be $F \oplus F$

1) Suppose $x \in D_E^\times$ is σ-regular. Then $\exists\, M \subseteq D$ a max^l subfield s.t. x is σ-conjugate to an elt of M_E, & M is unique up to conjugacy. The norm defined above, restricted to $M_E^\times$, is the usual field norm.
If $x \in M_E^\times - E^\times$ then $C_x^\sigma = M^\times$ (not $M_E^\times$!)

2) Suppose $[Nx]$ is central. Then x is σ-conjugate to an elt of $E^\times$. The norm defined above on $E^\times$ is the usual field norm. If $x \in E^\times$ then $C_x^\sigma = D^\times$

Rk For 2) we have ☡ : we use the fact l is odd.
$l=2$ – maybe $[Nx]$ is central & x isn't σ-conjugate to an elt of $E^\times$.

Pf 1) $\exists\, g \in D_E^\times$ s.t. $\underbrace{g^{-1}(Nx)g}_{= N(g^{-1}x\,{}^\sigma g)} \in D^\times$ ∴ WLOG $Nx \in D^\times$.

Let $M = F(Nx)$. Then $x^{-1}(Nx)x = {}^\sigma(Nx) = Nx$ as $Nx \in D^\times$
i.e. x & Nx commute
∴ $x \in M_E$ by results on quaternion algebras.

Now say $g \in C_x^\sigma$ ie $g^{-1}x\,{}^\sigma g = x$.

$$\text{Then } \overset{\displaystyle N(g^{-1}x\,{}^\sigma g)}{\underset{Nx}{\shortparallel}} = g^{-1}(Nx)g \quad \therefore g \in M_E^\times$$

Hence g & x commute ∴ $g^{-1}x\,{}^\sigma g = x \Rightarrow g = {}^\sigma g$ ∴ $g \in M^\times$

2) Slightly more complicated. Define a new action of $\langle\sigma\rangle$ on D_E: write it as a .

$\sigma.\delta = x({}^\sigma\delta)x^{-1}$ Need to check $\sigma^l = id$.

$\sigma^l.\delta = (x\,{}^\sigma x\ldots)({}^{\sigma^l}\delta)(x\,{}^\sigma x\ldots)^{-1} = (Nx)\delta(Nx)^{-1} = \delta$

∴ σ^l is id & we have an action. It's "σ-linear"

Let $D' = \{\delta \in D_E \mid \sigma.\delta = \delta\}$. Hilbert 90 $\Rightarrow D' \otimes_F E = D_E$

Check D' is a quat. alg. Then $D' \cong D$ (This may be Noether Skolem. It may assume l is odd too)
In fact $\exists\, g \in D_E^\times$ s.t. $D' = gDg^{-1}$

This bit has assumed l odd for simplicity.
It's the first time we've assumed l odd.

Hence $g^{-1}x^{\sigma}g$ is σ-centralised by D & hence centralised by D as σ acts trivially on D. Hence it's centralised by D_E by linearity

Hence $g^{-1}x^{\sigma}g \in E^{\times}$. The rest is an exercise. □

This lays bare what's going on in this case.

Cor x & y are σ-ss. Then $[Nx] = [Ny] \Leftrightarrow x \sim_{\sigma} y$

Pf ($\Leftarrow$) ✓ ($\Rightarrow$): $Nx = g^{-1}(Ny)g = N(g^{-1}y^{\sigma}g)$

$\therefore$ WLOG $Nx = Ny$.

Also WLOG $Nx = Ny \in D^{\times}$.

2 cases i) $Nx = Ny \in F^{\times}$ is central.

Then WLOG $x, y \in E^{\times}$. Then Hilbert 90 (for fields) $\Rightarrow x/y = {}^{\sigma}a/a$ for some $a \in E^{\times}$ $\Rightarrow x = a^{-1}y^{\sigma}a$

ii) almost the same: $Nx = Ny$ regular. Use Hilbert 90 on max^{l} subfield; say $Nx = Ny \in M$; Hilbert 90 on $M_E^{\times}$.

If D splits there's a third case but it's all easy. □

Cor (special case of lemma) $x \in D_E^{\times}$, x σ-ss, $Nx \in D^{\times}$ (always force this after conjugation)

then $C^{\sigma}_{x} = C_{Nx}$ ← in $D^{\times}$. This uses l odd. □

NB the penultimate corollary is just Hilbert 90 in a non-abelian case.

Lecture 2
Wed 24th Feb '93
9:30am

2 Matching Orbital Integrals

Need to understand orbital integrals, to understand the trace formula. Global orbital integrals will factorise, so let's do the local situation.

(local)
F a finite ext^{n} of $\mathbb{Q}_p$ or $\mathbb{R}$, E/F cyclic, degree $l > 2$, prime. $\langle\sigma\rangle = \mathrm{Gal}(E/F)$.
E may be F^{l} & then σ acts by permutation

D/F quat. alg. (split or not). Fix Haar measures on $D^{\times}$ & on $D_E^{\times}$ s.t. $\mu(O_D^{\times}) = 1$ & $\mu(O_{D_E}^{\times}) = 1$.

Say $\gamma \in \mathcal{D}^\times$ ss., & $\varphi \in C_c^\infty(\mathcal{D}^\times)$

$$\text{Set } O_\gamma(\varphi) = \int_{\underset{\uparrow \text{ centraliser of } \gamma}{C_\gamma \backslash \mathcal{D}^\times}} \varphi(g^{-1}\gamma g)\, dg$$

$$\parallel$$

$$O_\gamma(\varphi, \mu_{C_\gamma})$$

↑ He'll only put in measures if he's being v. careful.

The map $g \mapsto \varphi(g^{-1}\gamma g)$ is a map $C_\gamma \backslash \mathcal{D}^\times \longrightarrow \mathbb{C}$

$$\updownarrow$$

$[\gamma]$. This $\leftrightarrow$ to the orbit is a homeomorphism (see later)

& the map $[\gamma] \to \mathbb{C}$ is C_c^∞, the $_c$ being because $[\gamma]$ is closed in $\mathcal{D}^\times$.
(the reason for this ↑ is below)

γ regular $\Rightarrow [\gamma] = (\mathrm{tr}, \nu)^{-1}$ of 1 pt in $F \times F^\times$
γ central $\Rightarrow [\gamma] = \{\gamma\}$

Note $O_\gamma(\varphi) = O_{g^{-1}\gamma g}(\varphi)$,

if we match measures on C_γ & $C_{g^{-1}\gamma g}$ ~~then~~ $(= g^{-1} C_\gamma g)$ via the iso $x \mapsto g^{-1}xg$. (well-defined as G is unimodular).

Now say $\delta \in \mathcal{D}_E^\times$ is σ-ss, $\psi \in C_c^\infty(\mathcal{D}_E^\times)$

$$\text{Ⓓ} \quad \underbrace{TO}_{\substack{\text{Twisted}\\ \text{orbital}\\ \text{integral}}}{}_\delta(\psi) = TO_\delta(\psi, \mu_{C_\delta^\sigma}) = \int_{C_\delta^\sigma \backslash \mathcal{D}_E^\times} \psi(g^{-1}\delta\, {}^\sigma g)\, dg$$

Note $C^\sigma_{g^{-1}\delta\,{}^\sigma g} = g^{-1} C^\sigma_\delta g$

$\therefore$ we can choose compatible measures on σ-centralisers of all σ-conjugacy classes of σ, s.t.

$$TO_{g^{-1}\delta\,{}^\sigma g}(\psi) = TO_\delta(\psi)$$

The $\int$ exists, as $[\delta]_\sigma$ is closed & is homeo (justification later) to $C_\delta^\sigma \backslash \mathcal{D}_E^\times$

↑

2 cases: δ σ-reg : $[\delta] = (\mathrm{tr} \circ N, \nu \circ N)^{-1}(pt)$
δ σ-ss but not σ-reg : $[\delta] = N^{-1}(pt)$

Recall we're trying to understand $\mathrm{tr}\, \pi(\varphi)$. Here's a good way of looking at things.

Def^c We say that functions $\varphi \in C_c^\infty(D^\times)$ & $\psi \in C_c^\infty(D_E^\times)$ are associated if

$\forall$ regular $\gamma \in D^\times$,

$$O_\gamma(\varphi) = \begin{cases} 0 & \text{if } [\gamma] \neq [N\delta] \text{ some } \delta \in D_E^\times \\ TO_\delta(\psi) & \text{if } [\gamma] = [N\delta],\ \delta \in D_E^\times \end{cases}$$

We say they are strongly associated (highly non-standard) notation) if true $\forall$ ss γ.

NB associated $\Leftrightarrow$ strongly associated (use the theory of germs, or sthg)

You see, Tony was going to prove some theorem under an "associated" assumption. However, Richard + Peter Schneider get a "strongly associated" result, so if Labesse does too then it makes Tony's life easier.

A word on measures: If $\gamma = N\delta$ then $C_\gamma = C^\sigma_\delta$ so take the same measures.

Fix $M_1, .., M_r$ representatives of the conjugacy classes of max^l subfields (or POFs) in D (NB F local $\Rightarrow$ only finitely many). Fix Haar measures on each $M_i^\times$ & extend to all centralisers & σ-centralisers by conjugacy.

Thm 3 (clearly the best we can do)

1) If $\psi \in C_c^\infty(D_E^\times)$ then $\exists \varphi \in C_c^\infty(D^\times)$ associated to ψ

2) If $\varphi \in C_c^\infty(D^\times)$ & $O_\gamma(\varphi) = 0$ whenever γ is regular semisimple and $[\gamma]$ is not a norm then $\exists \psi \in C_c^\infty(D_E^\times)$ associated to φ.

Rk It doesn't set up a bijection - it's many-to-many.

Proof uses

Lemma 5. If $\theta \in C_c^\infty(A \times B)$ then $\int_B \theta(x,y)\,dy \in C_c^\infty(A)$

Pf exercise - for p-adic case reduce to $\theta = \mathrm{char}_{A' \times B'}$.

Cor If $\theta \in C_c^\infty(D^\times \times D^\times)$ then $\int_{D^\times} \theta(xy^{-1}, y)\,dy \in C_c^\infty(D^\times)$ $\square$

Pf of thm now. There's 2 cases. $E = F^l$ & E is a field.

Pf of thm

① <u>Case 1</u> $\lfloor E = F^\ell$. Then $D_E \ni \delta = (\delta_1,..,\delta_\ell)$; $\sigma(\delta_1,..,\delta_\ell) = (\delta_2,.,\delta_\ell,\delta_1)$

1) Then $\delta \sim_\sigma (\delta_1\delta_2..\delta_\ell, 1, 1., 1)$

via $(1, \delta_2\delta_3..\delta_\ell, \delta_3..\delta_\ell, ..., \delta_\ell)$

Consider only then $\delta = (\gamma, 1, .., 1)$

Then $N\delta = (\gamma, \gamma, .., \gamma) \in D^\times$ Say γ is regular. Set $M = F(\gamma)$.

Then $TO_\delta(\psi) = \int_{M^\times \backslash (D^\times)^\ell} \psi(g_1^{-1}\gamma g_2, g_2^{-1}g_3, .., g_\ell^{-1}g_1)\, dg_1 . dg_\ell$

$\uparrow$ diagonal embedding

Now set $h_i = g_i^{-1}g_1$; $i = 2,.,\ell$

$$= \int_{M^\times \backslash D^\times} \int_{(D^\times)^{\ell-1}} \psi(g_1^{-1}\gamma g_1 h_2^{-1}, h_2h_3^{-1}, ..\, h_{\ell-1}h_\ell^{-1}, h_\ell)\, dh_2..dh_\ell\, dg_1$$

Set $\varphi(x) = \int_{(D^\times)^{\ell-1}} \psi(xh_2^{-1}, h_2h_3^{-1}, .., h_\ell)\, dh_2 .\, dh_\ell \in C_c^\infty(D^\times)$

Then $TO_\delta(\psi) = O_\gamma(\varphi)$ ie φ is associated to ψ.

⌈ NB if γ was central then write $D^\times$ instead of $M^\times$ & $\int_{M^\times \backslash D^\times} \rightsquigarrow$ "evaluate"

& so we're in fact proved they're strongly associated. ⌋

⌈ Remark. If $\psi = \psi_1 \times .. \times \psi_\ell$ then $\varphi = \psi_1 * .. * \psi_\ell$.. ⌋

Now do

② <u>Case 1</u> 2) Everything is a norm here so "$[\gamma]$ is not a norm" doesn't ever apply.

Given $\varphi \in C_c^\infty(D^\times)$ write $\varphi = \sum_{i=1}^{r} \varphi_1^{(i)} * .. * \varphi_\ell^{(i)}$ with $\varphi_j^{(i)} \in C_c^\infty(D^\times)$

$v \nmid \infty$ is an exercise : $r=1$, $\varphi_1 = \varphi$, $\varphi_2 .. \varphi_\ell$ char. fn of small open cpct subgp (normalised).

$v | \infty$ is a thm: Dixmier + Malliavin, Bull. S. Math 102 (1978) pp 307-330

We don't really ever need $v|\infty$ but it makes the exposition neater.

<u>Case 2</u> E a field

Assume F is p-adic (as l is odd)

<u>Reduction</u>. It suffices to prove that if $\delta \in D_E^\times$ is σ-ss & if $\gamma = N\delta \in D^\times$ then $\exists$ open & closed nhds W of δ & V of γ s.t.

a) V is invt (under conj) & W is in σ-invt

[= 'conj classes'

b) $x \in W \Rightarrow [Nx] \subseteq V$
$y \in V \Rightarrow \exists$ ~~Rddd~~ $x \in W$ with $[Nx] = [y]$

c) If $\psi \in C_c^\infty(W)$ then $\exists \varphi \in C_c^\infty(V)$ associated to ψ

d) If $\varphi \in C_c^\infty(V)$ then $\exists \psi \in C_c^\infty(W)$ associated to φ

Note that all elts of V are norms so the non-norm cond^n has gone.
Note also that he's not assuming δ is σ-regular, because if he ~~br~~ threw away all central elts, what's left wouldn't be closed, & he wants a cptness argument to finish it.

<u>Proof that reduction $\Rightarrow$ thm</u>

1) Show that if $\psi \in C_c^\infty(D_E^\times)$ then $\exists \varphi \in C_c^\infty(D^\times)$ associated to ψ

Pf Let Ω = image of $(\mathrm{supp}(\psi))$ under the map $(tr, \nu) \circ N : D_E^\times \to F \times F^\times$. Then Ω is cpct. Recall $M_1 \ldots M_r$ represent max^l subfields of D.

If $\delta \in (M_i \otimes E)^\times$ then choose nhds W_δ of δ & $V_{N\delta}$ of $N\delta$ as above.

Choose $\delta_1, \ldots, \delta_s$ s.t. W_{δ_i} cover $\mathrm{supp}\,\psi \cap E^\times$.

Choose $\delta_{s+1}, \ldots, \delta_t$ s.t. $(tr, \nu) V_{N\delta_i}$ cover $\underbrace{(tr,\nu) \circ N(\mathrm{supp}\,\psi - \bigcup_1^s W_{\delta_i})}_{\text{cpct}}$

(use the fact that (tr, ν) is open away from central elts (exercise))

We may assume W_{δ_i} are all disjoint (NB in the $\mathbb{R}$ case, of course, we cant. Some partition of unity trick will probably do it though)

By the reduction $\exists \varphi_i$ on $V_{N\delta_i}$ associated to $\psi|_{W_{\delta_i}}$. Let $\varphi = \sum_{i=1}^t \varphi_i$.

It's an exercise to show φ associated to ψ.

2) $\varphi \rightsquigarrow \psi$ is an exercise (its exactly the same)

He's sorry this was a bit nasty. He hadn't realised (tr, ν) wasn't open at the central elts 'till this morning, so had to patch a pf up.

ecture 3
Ned 24th Feb '93
2:30pm

Recall we're trying to show for the reduction of the problem.

We're matching an orbital integral with a twisted orbital integral.

Recall for $\delta \in D_E^\times$, $\gamma = N\delta \in D^\times$ ss we're trying to show $\exists$ open & closed nhds W of δ & V of γ s.t.

1) V is inv't, W is σ-inv't
2) $x \in W \Rightarrow [Nx] \subseteq V$, $y \in V \Rightarrow \exists x \in W$ s.t. $Nx = y$
3) $\psi \in C_c^\infty(W) \Rightarrow \exists \varphi \in C_c^\infty(V)$ assoc to ψ
4) $\varphi \in C_c^\infty(V) \Rightarrow \exists \psi \in C_c^\infty(W)$ assoc to φ

This is a local condition. Note that everything is a norm.

There's an attack that works but we'll do 2 cases - regular & central. For GL_d there's more cases but hey 2 is small.

Case 2a γ regular. Set $M = F(\gamma)$, $\delta \in M_E^\times$.

We need some geometric facts about orbits which he'll just state.

Propⁿ 4. $\exists$ open & closed nhds $U, \tilde{U}$ of 1 in $M^\times$ s.t.

a) $U \to \tilde{U}$, $t \mapsto t^\ell$ is a homeo (log & exp converge nr. 1)
b) $\gamma\tilde{U}$ consists of regular elts

the meat of the propⁿ

c) $M^\times \backslash D^\times \times \tilde{U} \xrightarrow{\quad} V \subseteq D^\times$
$(x, t^\ell) \longmapsto x^{-1}(\gamma t^\ell)x$ is a homeo onto an open + closed set $V \subseteq D^\times$

d) $M^\times \backslash D_E^\times \times U \longrightarrow W \subseteq D_E^\times$
$(x, t) \longmapsto x^{-1}(\delta t)^\sigma x$ is a homeo onto an open + closed set $W \subseteq D_E^\times$

He'll prove this & things like it on Friday. □

The V & W are the V & W we need. $N(x^{-1}(\delta t)^\sigma x) = x^{-1}\underset{\gamma, t \text{ commute}}{\gamma t^\ell} x \sim$ elt of V

So let's check 3). 4) is just the same.

Say $\psi \in C_c^\infty(W)$. We want φ. Note that if $t \in U$,

$$TO_{\delta t}(\psi) = \int_{M^\times \backslash D_E^\times} \psi(x^{-1}\delta t^\sigma x)\,dx \in C_c^\infty(U)$$ (as we're near a regular elt)

Choose $\theta \in C_c^\infty(M^\times \backslash D^\times)$ s.t. $\int \theta = 1$. Define $\varphi \in C_c^\infty(V)$ by

$$\varphi(x^{-1}(\gamma t^l) x) = \theta(x)\, TO_{\delta t}(\psi)$$

↖ smooth of cpct support.

Then $\int_{M^\times \backslash D^\times} \varphi(x^{-1}(\gamma t^l) x)\, dx = TO_{\delta t}(\psi)$.

Hence φ is associated to ψ.

4) works in exactly the same way: integrate out the fibres in the product structure & define everything how you'd expect.

That's the easier case. (near a regular elt the orbital $\int$ works with any reasonable fn you like)

The only thing left is
to prove the reduction in the case γ is $= N\delta$, $\delta \in E^\times$

Case 2b) $\gamma = N\delta$, $\delta \in E^\times$

Similar but more complicated.

Propn 6 a) $\exists$ open + closed invt nhds $U, \tilde{U}$ of 1 in $D^\times$ st.

$$\begin{aligned} U &\to \tilde{U} \\ t &\mapsto t^l \end{aligned}$$

is a homeo (note $D^\times$ not ab but still have log, exp)

b) $\exists$ cts section s of the map $D_E^\times \to {}_{D^\times}\backslash D_E^\times$

s.t. $\gamma \tilde{U} \times {}_{D^\times}\backslash D_E^\times \longrightarrow W \subseteq D_E^\times$

$$(\gamma t^l,\ x) \longmapsto s(x)^{-1} \delta t\, \sigma s(x)$$

is a homeo onto W which is open & closed. Take $V = \gamma \tilde{U}$. $\square$ Pf on Fri again.

Then again the reduction follows with this V & W.
Recall δ, γ are central. 1), 2) of redn are easy

3) If $\psi \in C_c^\infty(W)$ then let's get φ.

Say $t \in U$ regular, $M = F(t)$

$TO_{\delta t}(\psi) = \int_{M^\times \backslash D_E^\times} \psi(x^{-1} \delta t \sigma x)\, dx = \ldots$ see next page.

$$TO_{\delta t}(\psi) = \int_{M^\times\backslash D_E^\times} \psi(x^{-1}\delta t^\sigma x)\,dx = \int_{D^\times\backslash D_E^\times}\int_{M^\times\backslash D^\times} \psi(x^{-1}y^{-1}\delta t y^\sigma x)\,dy\,dx$$

$$= \int_{D^\times\backslash D_E^\times}\int_{M^\times\backslash D^\times} \psi(s(x)^{-1}y^{-1}\delta t y^\sigma s(x))\,dy\,dx$$

Note that because of s we can swap $\int$s around.
Note everything is abs cgt.

$$= \int_{M^\times\backslash D^\times}\int_{D^\times\backslash D_E^\times} \psi(s(x)^{-1}\delta(\gamma^{-1}y^{-1}\gamma t^\ell y)^{1/\ell}\,{}^\sigma s(x))\,dx\,dy$$

$\uparrow$ makes sense by propn

$$= \int_{M^\times\backslash D^\times} \varphi(y^{-1}\gamma t^\ell y)\,dy$$

where $\varphi(z) = \int_{D^\times\backslash D_E^\times} \psi(s(x)^{-1}\delta(\gamma^{-1}z)^{1/\ell}\,{}^\sigma s(x))\,dx \in C(V)$ & infact $\in C_c^\infty(V)$ by lemma.

So the orbital $\int$s of ψ match those of φ.

Let's do 4). for completeness.

We have $\varphi \in C_c^\infty(V)$. Choose $\theta \in C_c^\infty(D^\times\backslash D_E^\times)$ with $\int\theta = 1$

Define $\psi \in C_c^\infty(W)$ by $\psi(s(x)^{-1}\delta t\,{}^\sigma s(x)) = \varphi(\gamma t^\ell)\theta(x)$

This works: $TO_{\delta t}(\psi) = O_{\gamma t^\ell}(\varphi)$. $\square$

$\uparrow$ always a norm.

We have done very well today.

It remains to do prop^n 4 & 6.

Let's do some geometry. The crux of both props is the openness of the maps. We need some p-adic analysis. We need a p-adic inverse function thm.

Say $F/\mathbb{Q}_p$. Check out Serres book Lie Algebras & Lie Groups (Benjamin Lecture Note series 1965)

The key is that these maps are (locally) analytic so $\exists$ power series expansions.

If $U \subseteq F^m$ is open $(F/\mathbb{Q}_p)$

then $\varphi\ U \to F^n$ is called <u>analytic at $x \in U$</u> if $\exists$ power series

$$\sum_{\underset{\sim}{i}} \underset{\underset{\underset{\sim}{a}_{\underset{\sim}{i}} \in F^n}{\uparrow}}{\underset{\sim}{a}_{\underset{\sim}{i}}} T^{\underset{\sim}{i}} \in F^n[[T_1,.., T_m]]$$

where $\underset{\sim}{i}$ runs thru $\underset{\sim}{i} = (i_1,.., i_m) \in \mathbb{Z}^m_{\geq 0}$

& $T^{\underset{\sim}{i}} = \prod_{j=1}^{m} T_j^{i_j}$

s.t. for all $\underset{\sim}{h}$ in some nhd of $\underset{\sim}{0}$ in F^m, the power series converges at $\underset{\sim}{h}$ and

$$\varphi(\underset{\sim}{x}+\underset{\sim}{h}) = \sum_{\underset{\sim}{i}} \underset{\sim}{a}_{\underset{\sim}{i}} \underset{\sim}{h}^{\underset{\sim}{i}}$$

NB the power series converges in some nhd of $\underset{\sim}{0}$ iff $\exists$ $\varepsilon > 0$ s.t. $\underset{\sim}{a}_{\underset{\sim}{i}} \varepsilon^{|\underset{\sim}{i}|} \to 0$ as $\underset{\sim}{i} \to \infty$; $|\underset{\sim}{i}| = \sum_j i_j$

<u>Rk</u> φ analytic at $\underset{\sim}{x} \Rightarrow \varphi$ analytic in a nhd of $\underset{\sim}{x}$ (Serre page LG2·4)

(is analytic at $\underset{\sim}{x}$ $\forall \underset{\sim}{x} \in U$)

If φ analytic on U we can differentiate:

$D\varphi : U \times F^m \to F^n$

$D\varphi_{\underset{\sim}{x}} \in \mathrm{Hom}(F^m, F^n)$ given by $(D\varphi_{\underset{\sim}{x}})(y) = \underset{\sim}{a}_{(1,0,..,0)} y_1 + .. + \underset{\sim}{a}_{(0,..,0,1)} y_m$

where $\underset{\sim}{a}_{\underset{\sim}{i}}$ = coeffts of power series of φ at $\underset{\sim}{x}$

Note that $D(\varphi_1 \circ \varphi_2) = D\varphi_1 \circ D\varphi_2$

What we need is

<u>Propn 7</u> If $U \subseteq F^n$ is open & $\varphi : U \to F^n$, & φ is analytic at x with $D\varphi_{\underset{\sim}{x}}$ an iso, then $\exists$ nhds $V \subseteq U$ of $\underset{\sim}{x}$ & W of $\varphi\underset{\sim}{x}$ s.t. φ is a bijection from V to W, & the inverse $(\varphi|_V)^{-1}$ is analytic

(this implies φ $V \to W$ is a homeo) $\square$

See e.g. Serre LG 2.10. Alternatively try it as a difficult exercise. $n=1$ isn't too bad. Believe Serre if you have an iota of common sense.

Cor If $\varphi : \underset{F^m}{\underset{\cap}{U}} \to F^n$ is analytic at $\underset{\sim}{x}$ & $D\varphi_{\underset{\sim}{x}}$ has rank n, then

$\exists$ nhds $V \subseteq U$ of $\underset{\sim}{x}$ & W of $\varphi \underset{\sim}{x}$ s.t.

1) $\varphi|_V$ is open

2) $\exists$ analytic $s : W \to V$ s.t. $\varphi \circ s = id$.

Pf Usual analytic trick. Consider $\tilde{\varphi} : U \to F^m$
$\underset{\sim}{y} \mapsto (\varphi \underset{\sim}{y}, A\underset{\sim}{y})$

where $A \in \mathrm{Hom}(F^m, F^{m-n})$ is chosen s.t.

$D\varphi_{\underset{\sim}{x}} \oplus A$ is invertible as a linear map $F^m \to F^m$. $\square$

That's all the general nonsense we need. On Friday he'll prove props 4 & 6.

Lecture 4 Today he's gonna talk about things like
Fri 26th Feb '93
9:30am Prop 4: $\exists$ clopen nhds U & $\tilde{U}$ of 1 in $M^\times$ (γ regular, $M = F(\gamma)$, $\gamma = N\delta$) s.t

a) $\begin{array}{l} U \to \tilde{U} \text{ homeo} \\ t \mapsto t^l \end{array}$ b) $\gamma\tilde{U}$ is only regular elts

c) $\begin{array}{l} M^\times \backslash D^\times \times \tilde{U} \longrightarrow V \subseteq D^\times \\ x \; , \; t^l \mapsto x^{-1}\gamma t^l x \end{array}$ is a homeo onto clopen V

d) $\begin{array}{l} M^\times \backslash D_E^\times \times U \longrightarrow W \subseteq D_E^\times \\ x \; , \; t \mapsto x^{-1}\delta t^\sigma x \end{array}$ is a homeo onto clopen W.

Pf. Choose $U, \tilde{U}$ as in a) s.t $\gamma\tilde{U}$ contains no central elts & s.t. $x \in \gamma\tilde{U} \Rightarrow x^* \notin \gamma\tilde{U}$.
Use log & exp to ensure a) & the fact that near a poly with distinct roots the roots are cts w.r.t. the coeffts.

Now ensure, say, d) (a bit harder than c))

d) $\begin{array}{l} M^\times \backslash D_E^\times \times U \longrightarrow D_E^\times \\ (x \; , t) \mapsto x^{-1}\delta t^\sigma x \end{array}$ is cts
injective

$\Big($ $x^{-1}\delta t^\sigma x = x'^{-1}\delta t'^\sigma x'$ $\overset{\text{by taking norm, I guess}}{\Rightarrow}$ $x^{-1}\underbrace{\gamma t^l}_{\in\gamma\tilde{U}} x = x'^{-1}\underbrace{\gamma t'^l}_{\in\gamma\tilde{U}} x'$

$\therefore \gamma t^l = \gamma t'^l \; \therefore t = t'$

$\therefore x x'^{-1} \in M^\times$

$\therefore M^\times x = M^\times x'$ $\Big)$

& we need to show it's open.

It will do to show $\mathcal{D}_E^{\times} \times U \to \mathcal{D}_E^{\times}$ is open

It will do to show that the ~~[illegible]~~ derivative $D_E \times M \to D_E$ is surjective at all pts (x,t)

$$(a,b) \longmapsto x^{-1}(\delta a + \delta t \, \sigma(bx^{-1}) - (bx^{-1})\delta t)^{\sigma} x$$

Hence we just need to show that $D_E \twoheadrightarrow \delta M^{\backslash D_E}$

$$b \mapsto m^{\sigma}b - bm \quad , \quad m = \delta t \in \delta M \setminus \{0\}$$

This is an exercise

c) End up needing to show $\mathcal{D} \twoheadrightarrow M^{\backslash}\mathcal{D}$ (NB also need to check maps are closed but there's an easy reason why this is so)

$b \mapsto [m, b]$ is surjective. ☐

He now wants to talk about propn 6. Happy Birthday Danny. Is Danny here?

<u>**Propn 6**</u> $\gamma \in F^{\times}$, $\gamma = N\delta$, $\delta \in E^{\times}$. Then $\exists$ clopen invt nhds $U, \tilde{U}$ of $1 \in \mathcal{D}^{\times}$ s.t.

$$\begin{aligned} U &\to \tilde{U} \\ t &\mapsto t^{\ell} \end{aligned} \quad \text{is a homeo}$$

Also $\exists$ cts sections to the map $\mathcal{D}_E^{\times} \to \mathcal{D}^{\times \backslash \mathcal{D}_E^{\times}}$ s.t. the map

$$\begin{aligned} \gamma\tilde{U} \times \mathcal{D}^{\times\backslash\mathcal{D}_E^{\times}} &\twoheadrightarrow W \subseteq \mathcal{D}_E^{\times} \\ \gamma t^{\ell} \quad x \quad &\longmapsto s(x)^{-1} \delta t \, \sigma_{s(x)} \end{aligned} \quad \text{is a homeo onto clopen } W$$

<u>Pf</u>

a) Select A & $\tilde{A} = \ell A$ open & closed subsets of $\mathcal{D}_E$ on which exp converges with inverse log. We can replace them with the union of all their conjugates so WLOG A & $\tilde{A}$ are conjugation-invt.

$$\text{Let } U_E = \exp A, \ \tilde{U}_E = \exp(\tilde{A})$$

$$U = U_E \cap \mathcal{D}^{\times}, \quad \tilde{U} = \tilde{U}_E \cap \mathcal{D}^{\times}$$

Then $U, \tilde{U}$ have the first property

b) Things are easier once you assert sthg is a p-adic analytic manifold but Richard will try to skirt round this.

Find nhds Y of 1 in $\mathcal{D}_E^{\times}$ & B of 0 in $\mathcal{D}^{\backslash D_E}$ & a decomposition $D_E = \mathcal{D} \oplus \mathcal{D}^{\backslash D_E}$ s.t.

$$\begin{aligned} B &\to \mathcal{D}^{\times}\backslash\mathcal{D}^{\times} Y \\ b &\mapsto \mathcal{D}^{\times}\exp(b) \end{aligned} \quad \text{is a homeo with inverse } t$$

defn of t

<u>Pf</u> see next page.

Consider $D^\times \times D\backslash D_E^\times \longrightarrow D_E^\times$
$a, b \mapsto a \exp b$

$\exists$ nhds A_1 of 1, B_1 of 0 & Y_1 of 1 s.t. $A_1 \times B_1 \to Y_1$ is a homeo.

We can shrink these nhds to A, B, Y s.t. $A \times B \to Y$ is a homeo,
& $(YY^{-1} \cap D^\times)A \subseteq A_1$.

Let $(r, t)\ Y \to A \times B$ denote the inverse. Then $t(gy) = t(y)$ $\forall g \in D^\times$; $y, gy \in Y$.

Thus $t\ D^\times\backslash D^\times Y \longrightarrow B$ with inverse $b \mapsto D^\times \exp b$ are both homeos.

$D^\times\backslash D_E^\times = \coprod D^\times\backslash D^\times Y_i \eta_i$, $\quad Y_i \subseteq Y$

$s: D^\times\backslash D_E^\times \longrightarrow D_E^\times$,

$D^\times Y \eta_i$

$D^\times y \eta_i \longmapsto (\exp t(y))\eta_i$.

We want $\gamma \tilde{U} \times D^\times\backslash D_E^\times \to W \subseteq D_E^\times$ is a homeo onto clopen image.

(He's not sure he would have spotted it if he hadn't seen it in Arthur-Clozel. It's all quite easy though, once you've seen it) $\downarrow$(this)

i) injective. Say $s(x)^{-1}\delta t^\sigma s(x) = s(x')^{-1}\delta t'^\sigma s(x')$. Then we get

$$s(x)^{-1}\gamma t^\ell s(x) = s(x')^{-1}\gamma (t')^\ell s(x')$$

$$\Rightarrow \gamma (s(x)^{-1} t\, s(x))^\ell = \gamma (s(x')^{-1} t' s(x'))^\ell$$

$$\Rightarrow s(x)^{-1} t s(x) = s(x')^{-1} t' s(x')$$

$$\Rightarrow s(x)^{-1}\,\sigma s(x) = s(x')^{-1}\,\sigma s(x')$$ (neat trick!) (use 1st line I think!)
(Recall δ is central)

$$\Rightarrow s(x')s(x)^{-1} \in D^\times \Rightarrow x = x' \Rightarrow t = t'$$

ii) open. It will do to check

$$\gamma t^\ell \quad D^\times(\exp b)\eta_i \longrightarrow s((\exp b)\eta_i)^{-1}\delta t^\sigma s((\exp b)\eta_i)$$
$$\wr \qquad \wr$$
$$U \times B \longrightarrow D_E^\times$$
$$t, b \longmapsto \eta_i^{-1}\exp(b)^{-1}\delta t^\sigma \exp(b)^\sigma \eta_i$$

$s(\exp(b)\eta_i) = \exp(b)\eta_i$. Use inverse fn thm. It all boils down to ...

showing (the idea is we differentiate ! & use p-adic thms)

$$\mathcal{D}_E \to \mathcal{D} \setminus \mathcal{D}_E$$

$$\beta \mapsto \beta t - t^\sigma \beta \quad \text{is surjective.}$$

Well, RT couldn't, but he could show it was surjective at $t=1$, & surjectivity is an open cond^n & an inv^t cond^n. We have to shrink our nbds (U) if necessary.

(Richard feels it ought to be surjective.)

Surjective @ 1 is enough though. □

Richard has done these because he feels it's the heart of the pf that things are associated.

NB in general these p-adic things have singularities & then things are more difficult. We were a bit lucky with our centralizers.

VI · Base Change

Tony Scholl

ecture 1
Wed 24th Feb '93
4:00pm

In these 4 lectures Tony will explain a lot of the ideas behind base change for GL_2
unless he catches Lecture-shrinking disease

He'll spend a lot of this lecture explaining the statement of the thm
It's debatable whether base change is 1 word or 2.

He'll firstly spend a while talking about GL_1.

§1 Intro, statement of results

"Base change" for GL_1 is rather easy:

If E/F is a finite ext of local fields or number fields, write

$$C_E = \begin{cases} E^* & E \text{ local} \\ J_E/E^* & E \text{ global} \end{cases} \qquad \text{Similarly } C_F.$$

Recall we have

$$\begin{tikzcd} \theta_F : C_F \arrow[r] & Gal(\bar{F}/F)^{ab} \\ \theta_E : C_E \arrow[r] \arrow[u, "N_{E/F}"] & Gal(\bar{F}/E)^{ab} \arrow[u, "\text{restriction}"'] \end{tikzcd}$$

commuting.

This is essentially base change. Here are the 'details' ie rephrase it in a more representation-theoretic way.
Suppose $\rho : Gal(\bar{F}/F) \to \mathbb{C}^*$. Identify this with $\chi : C_F \to \mathbb{C}^*$ by $\chi = \rho \circ \theta_F$.

We have $\rho \rightsquigarrow \rho' = \rho|_{Gal(\bar{F}/E)}$

"Base change" is the corresponding assignment

$$\chi \rightsquigarrow \chi' = \chi \circ N_{E/F} = \rho' \circ \theta_E$$

Now restrict to E/F cyclic, $\langle \sigma \rangle = Gal(E/F)$

Then if $\chi' : C_E \to \mathbb{C}^*$, χ' is of the form $\chi \circ N_{E/F} \iff \chi'^{\sigma} = \chi$.

Hint for this on next page.

Use the exact sequence

$$1 \longrightarrow (1-\sigma)C_E \longrightarrow C_E \xrightarrow{N_{E/F}} C_F \longrightarrow \langle\sigma\rangle \longrightarrow 1$$

since E/F is cyclic.

Hence $\chi' = \chi'^{\sigma} \Leftrightarrow \chi'$ factors thru quotient $C_E \twoheadrightarrow N(C_E)$

Then extend this HM arbitrarily from $N(C_E)$ to C_F.

Now try

$\underline{\underline{GL_n, n \geq 2}}$

<u>Local base change</u> Say F is a local (p-adic) field, E/F a finite extⁿ.

The local Langlands conjecture

$$\begin{pmatrix} \text{Conjugacy classes of} \\ \text{ss HMs} \\ WD_F \to GL_n(\mathbb{C}) \end{pmatrix} \xleftrightarrow{???} \begin{pmatrix} \text{admiss irred} \\ \text{reps of} \\ GL_n(F) \end{pmatrix}$$

$$\Vert$$

$$\mathrm{Hom}_{ss}(WD_F, GL_n(\mathbb{C}))$$

Everyone believes this exists.

Now we have the restriction map $\mathrm{Hom}_{ss}(WD_F, GL_n(\mathbb{C})) \to \mathrm{Hom}_{ss}(WD_E, GL_n(\mathbb{C}))$

& the local Langlands conjecture for E gives us a map

$$\left\{\begin{matrix} \text{admiss irred} \\ \text{reps of} \\ GL_n(F) \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{admiss irred} \\ \text{reps of} \\ GL_n(E) \end{matrix}\right\}$$

So the local Langlands conjecture suggests the existence of a <u>local base change map</u> from reps of GLₙ(F) to reps of GLₙ(E), which we should be able to understand representation-theoretically.

Peter Schneider talked about the unramified case:

$$\left\{\begin{matrix} \text{unram irred} \\ \text{reps of } GL_n(F) \end{matrix}\right\} \longleftrightarrow \left\{\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix} \begin{matrix} \text{ss conj. classes} \\ \text{in } GL_n(\mathbb{C}) \end{matrix}\right\}$$

& we understand Frobenius.

Def: If π is an unramified irred admis rep^n of $GL_n(F)$ with associated parameter $\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{pmatrix}$, the local lifting $\Pi = \pi_E$ of π is the unramified rep^n of $GL_n(E)$ with parameter $\begin{pmatrix}\alpha_1^f & & 0\\ & \ddots & \\ 0 & & \alpha_n^f\end{pmatrix}$, $f =$ residue class degree of E/F.

We can interpret this in terms of base change map on unramified Hecke algebras,

$$\pi \leftrightarrow \Theta_\pi : \mathcal{H} = \mathcal{H}(G(F), K^{\max}) \to \mathbb{C}$$

(given by action on K-fixed vectors)

Then $\Theta_\Pi : \mathcal{H}_E \to \mathbb{C}$ is given by $\Theta_\Pi = \Theta_\pi \circ b_{E/F}$

for $b_{E/F} : \mathcal{H}_E \to \mathcal{H}$ the base change HM.

NB he's not assuming E/F is unramified. Have to keep track of e. In the global case we have the local ext^n unramified a.e. & this is all we need.

L-functions $L(\pi, s) = \prod_i (1-\alpha_i q^{-s})^{-1}$, $L(\Pi, s) = \prod_i (1-\alpha_i^f q^{-fs})^{-1}$

Enough local stuff.

Global base change Say E/F is a finite ext^n of number fields.

Example Say X/F is an elliptic curve, & S a finite set of places

$$\forall v \notin S \ \exists \text{ ell curve } X/k_v \quad \# X(k_v) = 1 + q_v - q_v^{1/2}(\alpha_v + \beta_v),\ \alpha_v \beta_v = 1.$$

$$\rightsquigarrow \pi_v', \text{ unramified rep}^n \text{ of } GL_2(F_v) \text{ with parameter} \begin{pmatrix}\alpha_v & 0\\ 0 & \beta_v\end{pmatrix}$$

Taniyama-Weil (this bit is probably Weil) $\Rightarrow \exists$ cuspidal (if no CM) auto rep^n $\pi = \otimes \pi_v$ of $GL_2(\mathbb{A}_F)$

s.t. $\pi_v' \cong \pi_v \ \forall v \notin S$.

$X/E \rightsquigarrow \forall w|v, v \notin S, \exists \Pi_w'$ unram rep of $GL_2(E_w)$ with parameters $\begin{pmatrix}\alpha_w' & 0\\ 0 & \beta_w'\end{pmatrix}$

& $\alpha_w' = \alpha_v^f$, $\beta_w' = \beta_v^f$, $f = f(w/v)$

$\overset{?}{\rightsquigarrow} \Pi = \otimes \Pi_w$ s.t. $\Pi_w \cong \Pi_w'$ for almost all w

Π_w would be a local lifting of π_v for all $w|v \notin S$.

The relation $\pi \to \Pi$ is indpt of any conjectural rel^{n}ship with Elliptic curves.

Def: $\pi = \otimes \pi_v$, $\Pi = \otimes \Pi_w$ irred auto reps of $GL_n(\mathbb{A}_F)$, $GL_n(\mathbb{A}_E)$ resp.

Say Π is a weak basechange of π if for almost all v, & $\forall w|v$, Π_w is the (unramified) local basechange of π_v

Conjecture (Langlands) Any π has a basechange

(NB He thinks this is what Langlands said. He hasn't put in the word "cuspidal".)

This is an extremely strong conjecture e.g. gives you lots of analytic like properties of non-abelian L-f^ns

Eg $n=2$ gives us X/F ell. curve $\sim$ $L(X, \chi, s) \overset{?}{=} L(X/E, \varphi, s)$ or sthg
E/F ext^n

Thm (Langlands)(based upon important ideas of Saito, Shintani)

If E/F is cyclic of prime degree l, & $\chi_{E/F}$ a non-trivial char of $Gal(E/F) = \langle\sigma\rangle$

(i) Any irred auto rep^n of π of $GL_2(\mathbb{A}_F)$ has a (strong) basechange to E - call it Π.

(ii) If π is cuspidal then so is Π except when $l=2$ & π is obtained from $\theta : \mathbb{A}_E^* / E^* \to \mathbb{C}^*$, $\theta \neq \theta^\sigma$

If π is not cuspidal then neither is Π.

(iii) ~~If~~ Conversely, if Π is a cuspidal irred automorphic rep^n of $GL_2(\mathbb{A}_E)$, then Π is a base change of some π, provided that $\Pi^\sigma \cong \Pi$

(iv) If π, π' are cuspidal, then they have the same basechange $\Leftrightarrow$ $\pi' \cong \pi \otimes (\chi_{E/F} \circ \det)$ for some j.

The meaning of (ii) is that given $\theta = \prod \theta_w : \mathbb{A}_E^* / E^* \to \mathbb{C}^*$, let S include all infinite places & all places of v ramified for E/F or for θ. Then if $w|v \notin S$, $\theta_w(\pi) = \alpha_w$ (unramified).

$$\begin{tikzcd}
\theta_w \arrow[r, leftrightarrow] & \text{1d reps of } W_{E_w} \text{ ie HMs } W_{E_w} \to \mathbb{C}^* \arrow[d, "\text{induction}"] \\
\pi_v & \text{Maps } W_{F_v} \to GL_2(\mathbb{C}) \arrow[l]
\end{tikzcd}$$

Take π_v with parameter $\begin{pmatrix} \alpha_w & 0 \\ 0 & \alpha_{w'} \end{pmatrix}$ if $v = ww'$

$\begin{pmatrix} 0 & 1 \\ \alpha_v & 0 \end{pmatrix}$ if $v = w$ inert

$\exists!\ \pi$ automorphic, local cpt π_v at $v \notin S$.

That was an explanation of the $l=2$ bit of (ii).

Lots of ingredients are necessary for this proof. One that we have not got is a trace formula for GL_2 - we need Eisenstein series for this. We will do a version of the thm for D a non-split quat alg. We'll also avoid the tricky case $l=2$.

Here is the thm that we will prove.

<u>Thm</u> Let G be the gp of invertible elts in a quaternion division algebra D/F, & let E/F be cyclic of prime degree $l > 2$. We have $G(\mathbb{A}_F)$ & $G(\mathbb{A}_E)$.

Then every irred auto rep π of $G(\mathbb{A}_F)$ has a (weak) basechange to E, & π, π' have the same basechange $\Leftrightarrow \pi' = \pi \otimes (\chi^j_{E/F} \circ N_{rd})$ for some j.

Moreover, every Π with $\Pi \cong \Pi^\sigma$ is a basechange.

<u>Rk</u> Since $G(F_v) \cong GL_2(F_v)$ for all but finitely many v, the notion of weak basechange makes sense for G.

<u>ecture 2</u>
<u>Thus 25th Feb '03</u>
<u>11.00 a.m</u>

Recall E/F a cyclic ext^n of no. fields, D = quaternion division algebra $/F$

$\langle \sigma \rangle = \mathrm{Gal}(E/F)$ of order l prime > 2

G = gp of invertible elts of D, $G(F) = D^*$, $G(E) = (D \otimes E)^*$ (G is an alg gp)

$AR(F) = \{$ isom. classes of irred auto reps of $G(\mathbb{A}_F) \}$

$AR(E) = \{$ isom. classes of irred auto reps of $G(\mathbb{A}_E) \}$

Here comes a thm. It's the one we had earlier, I guess

<u>Thm 1</u> (i) If π is in $AR(F)$ then it has a weak basechange $\Pi \in AR(E)$
(ii) π, π' have the same basechange $\Leftrightarrow \pi' \cong \pi \otimes (\chi^\sigma_{E/F} \otimes N_{rd})$
(iii) Any $\Pi \in AR(E)$ is a basechange of some $\pi \Leftrightarrow \Pi \cong \Pi^\sigma$.

We will prove this.

~~Thm 1.1~~

Remarks ① We need the strong multiplicity 1 thm (which has 2 bits)

If $\Pi = \otimes_w \Pi_w$ & $\Pi' = \otimes_w \Pi'$ are in $AR(E)$ & $\Pi_w \cong \Pi_w'$ for almost all w, then $\Pi \cong \Pi'$ ($\Leftrightarrow \Pi'_w \cong \Pi_w$ for all w)

- the proof is a reduction to GL_2 using J-L correspondence.
  The J-L correspondence uses the trace formula, for D & GL_2
  The trace formula for GL_2 uses the theory of Eisenstein series.

So it's a lot of work.

Anyway, it shows that if Π, Π' are weak liftings of π, then $\Pi \cong \Pi'$.

② $G(E) \circlearrowleft^{Gal(E/F) = \langle\sigma\rangle}$; so we get a semidirect product $G'(E) = G(E) \rtimes \langle\sigma\rangle$.

$G(E_v) = \prod_{w|v} G(E_w) \circlearrowleft^{\sigma}$; $G'(E_v) = G(E_v) \rtimes \langle\sigma\rangle$ ($E_v = E \otimes_F F_v = \prod_{w|v} E_w$)

If $\Pi = \otimes \Pi_w \in AR(E)$, write $\Pi_v = \otimes_{w|v} \Pi_w$ which is a rep. of $G(E_v)$

Notation: Π^σ, Π_v^σ

[If Π is a ~~[illegible]~~ basechange of some π, then $\Pi_v^\sigma \cong \Pi_v$ for almost all v, & hence $\Pi \cong \Pi^\sigma$ by strong multiplicity 1]

③ If S = some finite set of primes of F, including all the ones ramified in E or D, set

$$\mathcal{H}_F^S = \bigotimes_{\substack{v \notin S \\ v \text{ finite}}} \mathcal{H}(G(F_v), K_v^{max}),$$ the $\otimes$ of ~~[illegible]~~ the unramified Hecke algebras for all finite $v \notin S$.

Similarly $\mathcal{H}_E^S$. If $\pi \in AR(F)$, unramified at all finite $v \notin S$,

$\pi = \otimes \pi_v$, then for each $v \notin S$ we get a character of the unramified Hecke algebra (on $\pi_v^{K_v}$) & hence a HM

$$\Theta_\pi^S : \mathcal{H}_F^S \to \mathbb{C}$$

(take any non-zero $v \in$ space of π, fixed by $\prod_{v \notin S} K_v$; then

$\pi(f)v = \Theta_\pi^S(f)v$ for $f \in \mathcal{H}_F^S$)

By strong mult 1, Θ_π^S determines π up to iso, & Π is a basechange of $\pi \Leftrightarrow \Theta_\Pi^S = \Theta_\pi^S \circ b_{E/F}$ where $b_{E/F} : \mathcal{H}_E^S \to \mathcal{H}_F^S$ is the base change HM, which is, of course, the $\otimes$ of all the local b's)

This is the form of the thm which we'll attack.

We need some twisted trace formulae.

§2. Trace Formulae

Recall $L^2 = L^2(\underbrace{G(F)\backslash G(\mathbb{A}_F)^1}_{\text{cpct} \quad (D \text{ div alg})}) = L^2(\mathbb{R}^\times_{>0} G(F)\backslash G(\mathbb{A}_F))$

$$\text{Recall } G(\mathbb{A}_F)^1 = \{ x \in G(\mathbb{A}_F) \text{ s.t. } |N_{\text{red}}(x)| = 1 \}$$

It's a unitary rep^n of $G(\mathbb{A}_F)$. &

$L^2 = \hat{\bigoplus}_{\pi} \pi$, summing over pairwise non-isomorphic unitary rep's π. The π that occur are precisely the unitary rep's corresponding to those elts of $AR(F)$ whose central char. is trivial on $\mathbb{R}^\times_{>0}$

à la Richard with the K-finite vector stuff

Note that we have to pass to the L^2 way of thinking to get the trace formula.

NB of course any $\pi \in AR(F)$ can be twisted to make its central char. trivial on $\mathbb{R}^\times_{>0}$

$\widetilde{N}$ $\quad \widetilde{L^2} = L^2(G(E)\backslash G(\mathbb{A}_E)^1) = \hat{\bigoplus}_{\widetilde{\Pi}} \Pi$

$f \in C^\infty_c(G(\mathbb{A}_F))$; $r(f)$ the associated operator on L^2

This next stuff was all done in Richard's lectures.

$r(f)$ is of trace class, represented by kernel $K(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$,

and (formula:)

$$\operatorname{tr} r(f) = \sum_{\{\gamma\}} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F)^1) \times O_\gamma(f)$$

sum over conj. classes in $G(F)$.

where $O_\gamma(f) = \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g)\, dg = \prod_v O_\gamma(f_v)$ where $f = \otimes f_v$.

Analagous formula for $\tilde{L}^2$

Acting on $\tilde{L}^2$ we have σ, acting by $(\sigma\psi)(x) = \psi(\sigma^{-1}x)$

This gives an action of the semidirect product $G'(\mathbb{A}_E)$ on $\tilde{L}^2$.

Let R be this action. Study $R(\varphi\times\sigma) = R(\varphi)R(\sigma)$ for $\varphi \in C_c^\infty(G(\mathbb{A}_E))$.

The functional analysis is identical to the case of $r(f)$ so he will just stick to the formal details.

The kernel of $R(\varphi)$ is $\sum_{\delta\in G(E)} \varphi(x^{-1}\delta y)$, so $R(\varphi\times\sigma)$ has kernel $\sum_{\delta\in G(E)} \varphi(x^{-1}\delta\sigma y) = \tilde{R}(x,y)$, and

$$\mathrm{tr}\, R(\varphi\times\sigma) = \int_{G(E)\backslash G(\mathbb{A}_E)^1} \tilde{R}(x,x)\,dx = \int_{G(E)\backslash G(\mathbb{A}_E)^1} \sum_\delta \varphi(x^{-1}\delta\sigma x)\,dx$$

$$= \sum_{\substack{\{\delta\} \\ \sigma\text{-conj classes} \\ \text{in } G(E)}} \int_{G^\sigma_\delta(E)\backslash G(\mathbb{A}_E)^1} \varphi(x^{-1}\delta\sigma x)\,dx$$

where $G^\sigma_\delta(E) =$ σ-stabilizer of δ in $G(E)$.

Now this implies

$$\mathrm{tr}(R(\varphi\times\sigma)) = \sum_{\substack{\{\delta\} \\ \sigma\text{-conj} \\ \text{classes}}} \mathrm{vol}(G^\sigma_\delta(E)\backslash G^\sigma_\delta(\mathbb{A}_E)^1) \int_{G^\sigma_\delta(\mathbb{A}_E)\backslash G(\mathbb{A}_E)^{?}} \varphi(g^{-1}\delta\sigma g)\,dg$$

could be a 1 here but it just falls off

$$= \sum_{\substack{\{\delta\} \\ \sigma\text{-conj} \\ \text{classes}}} \mathrm{vol}(G^\sigma_\delta(E)\backslash G^\sigma_\delta(\mathbb{A}_E)^1) \prod_v TO_\delta(\varphi_v)$$

if $\varphi = \otimes\varphi_v$, $\varphi_v \in C_c^\infty(G(E_v))$

This nasty thing is the <u>twisted trace formula</u>

It's so nasty we'll discard it.

Recall, from Richards lectures, that φ_v & f_v are <u>associated</u> if

$$TO_\delta(\varphi_v) = O_\gamma(f_v) \text{ whenever } [\gamma] = [N\delta]$$

$$\& \ O_\gamma(f_v) = 0 \text{ if } [\gamma] \text{ is not a norm.}$$

(for all regular elts γ)

So if φ_v, f_v are associated for all v, the corresponding global orbital integrals are equal (for all regular elts γ)

If $\gamma = N\delta$ then $G_\gamma(F) = G^\sigma_\delta(E)$

& also for adelic pts, so volumes are equal.
NB there's an important technicality here about picking sensible Haar measure normalisations to make the fundamental lemma work, or something. He may come back to this later. But he may not.

<u>Thm 2</u> If f_v is associated to φ_v for all v [and for some v, $O_\gamma(f_v) = TO_\delta(\varphi_v) = 0$ whenever γ & $N\delta$ are central]

then $\operatorname{tr} r(f) = \operatorname{tr} R(\varphi \times \sigma)$

Note that the bit in brackets is not necessary but we have to put it in because we haven't analysed central elts quite enough. It's not that difficult to fudge f_v & φ_v so that we lose no info & s.t. the bracketed statement holds.

Note also that the statement is vacuous if we can't form $\otimes\varphi_v$. Fortunately, Peter proved this morning that ($\nwarrow$ ie $\varphi_v \neq 1$ for lots of v)
the unit elts $\varphi_v = 1_{E_v}$ & $f_v = 1_{F_v}$ are ↙ <u>Fundamental Lemma</u>
associated for v unramified in E/F & in D. So we can take f_v, φ_v to be unit elts for almost all v.

↙ yes, he did.

I <u>think</u> he said that now Thm 2 was content-free by the trace formula but now he's well into §3 so I'd best stat that. It's a lot of functional analysis that we need but fortunately it's not too difficult. □

§3 Spectral decomposition at ∞

Lemma 1 Let $\{(\rho, V_\rho)\}$ be a family of pairwise non-isomorphic irred unitary rep^ns of G, a locally cpct gp.

Let $\mathbb{B} \subseteq L^1(G)$ be a dense subalgebra. Suppose $\exists\, c_\rho \in \mathbb{C}$ s.t.

$$(+)\ \sum c_\rho \|\rho(f)\|^2 = 0 \quad (\text{abs cgt})$$

for all $f \in \mathbb{B}$ (Here $\|\rho(f)\| =$ Hilbert-Schmidt norm)

Then $c_\rho = 0$ for all ρ.

Recall $\|\rho(f)\|^2 = \operatorname{tr} \rho(f)\rho(f)^* = \operatorname{tr}(\rho(f * f^*))$ if $f^* = \overline{f(g^{-1})}$

Lecture 3
Thurs 25th Feb '93
4:00pm

Pf Pick ρ_0 & $v_0 \in V_{\rho_0} \setminus \{0\}$. This next bit (intertwining operators) is a trick found in Jacquet-Langlands.

Define $W = \hat{\bigoplus}_{\rho \neq \rho_0} \operatorname{End}_{H\text{-}S}(V_\rho) = \hat{\bigoplus}_{\rho \neq \rho_0} V_\rho \hat{\otimes} V_\rho^*$, G acting by left composition with $(\rho(g))$.

Define $\theta : \mathbb{B} \to W$ by

for $f \in \mathbb{B}$, $\theta(f) = (|c_\rho|^{1/2} \rho(f))_{\rho \neq \rho^0} \in W$

Note that by hypothesis $\sum |c_\rho| \|\rho(f)\|^2 < \infty$

Let $W' \subseteq W$ be the closure of the image of G.

Suppose there was $C > 0$ s.t. $\forall f \in \mathbb{B}$, $\|\rho_0(f)\|^2 < C \sum_{\rho \neq \rho_0} |c_\rho| \|\rho(f)\|^2$ $\circledast$

Then there's a well-defined cts map

$W' \to V_{\rho_0}$ s.t. on the image of $\mathbb{B}$ it's given by

$$\theta(f) \longmapsto \rho_0(f) v_0$$

We must check $\theta(f) = 0 \Rightarrow \rho_0(f) v_0 = 0$ but this is clear because of $\circledast$.
It's also cts for the same reason.

It's also surjective because $\mathbb{B}$ is dense in $L^1(G)$ & $v_0 \neq 0$
(& the image is G-inv^t, I guess).

Composing with the orthog proj^n $W \twoheadrightarrow W'$ we get a G-equivariant cts linear map $W \twoheadrightarrow V_{\rho_0}$ which is impossible as no V_ρ is $\cong V_{\rho_0}$, $\rho \neq \rho_0$.

Hence no C exists s.t. ⊛ holds.

However, $|c_{\rho_0}| \, \|\rho_0(f)\|^2 \le \sum_{\rho \neq \rho_0} |c_\rho| \, \|\rho(f)\|^2$ $\forall f$ by hypothesis.

so So $c_{\rho_0} = 0$. $\square$

<u>Elementary rks about traces</u>

. $\mathrm{tr}(A \otimes B) = \mathrm{tr}\, A . \mathrm{tr}\, B$

. If $V = \hat{\bigotimes}_v V_v$, $A = \otimes A_v$, A_v are trace class operators on V_v,

A_v = projection onto the distinguished 1-dim^l subspace, for almost all v,

then $\mathrm{tr}(\otimes A_v) = \prod_v \mathrm{tr}\, A_v$

. V, $A_1, \ldots, A_l$ endomorphisms of V, $A_1 \otimes .. \otimes A_l \in \mathrm{End}(\otimes^l V)$

& $\sigma : x_1 \otimes .. \otimes x_l \mapsto x_2 \otimes x_3 \otimes \ldots \otimes x_l \otimes x_1$.

Then $\mathrm{tr}(A_1 \otimes . \otimes A_l . \sigma) = \mathrm{tr}(A_1 A_2 . A_l)$

Recall also Thm 2:

$\mathrm{tr}\, r(f) = \mathrm{tr}\, R(\varphi \times \sigma)$; $\otimes f_v = f \in C_c^\infty(G(\mathbb{A}_F))$, $\otimes \varphi_v = \varphi \in C_c^\infty(G(\mathbb{A}_E))$, . f_v, φ_v associated $\forall v$.

He never really told us what $r(f)$ was though. There's actually 2 choices:

f can act on $L^2(G(F) \backslash G(\mathbb{A}_F)^1) = L^2$ in 2 ways. Say $\psi \in L^2$

(i) $(r_1(f)\psi)(x) = \int_{G(\mathbb{A}_F)^1} f(g)\psi(xg)\,dg$

(ii) $(r_0(f)\psi)(x) = \int_{G(\mathbb{A}_F)} f(g)\psi(xg)\,dg$ regarding $\psi \in L^2(\mathbb{R}^\times_{>0} G(F) \backslash G(\mathbb{A}_F))$

Now $r_0(f) = r_1(f^1)$ where $f^1(g) = \int_{\mathbb{R}^*_{>0}} f\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) d^\times a$.

(assume we've been sensible with our Haar measures)

We proved thm 2 for r_0. The association $f \rightsquigarrow f_1$ changes the matching at infinity, unfortunately.

$f \rightsquigarrow f^1$ changes matching at ∞ by a multiple of l.

So the formula for the r_0-action is $\operatorname{tr} r(f) = l \operatorname{tr} R(\varphi\times\sigma)$. ($\varphi, f$ associated everywhere)

Note that the r_0 action is factorizable, the r_1 action isn't.

Now let's decompose the formula according to the decomposition of L^2, $\tilde{L}^2$, then

$$\operatorname{tr} r(f) = \sum_{\pi} \operatorname{tr} \pi(f)$$ (π occuring in L^2) (note we're using multiplicity 1 here)

$R(\varphi\times\sigma)$? Say $\tilde{L}^2 = \hat{\oplus}\, \Pi$. If $\Pi \neq \Pi^{\sigma}$ then $R(\varphi\times\sigma)$ cyclically permutes the spaces $\Pi^{(\sigma^i)}$ & so these spaces do not contribute to the trace.

Now suppose $\Pi = \Pi^{\sigma}$. Then Π is stable under σ, & therefore we can regarding it as a rep^n Π' of $G(\mathbb{A}_E)\rtimes\langle\sigma\rangle$.

Note $\Pi = \Pi^{\sigma} \Rightarrow \Pi \cong \Pi^{\sigma} \Rightarrow \Pi_v \cong \Pi_v^{\sigma}$, so we can extend the action of $G(E_v)$ (in a non unique way!) to a rep^n Π'_v of $G'(E_v)$ on the space of Π_v - but any 2 extensions differ by an lth root of 1 in σ.

For almost all v, <u>choose</u> Π'_v s.t. σ is 1 on the spherical vector, & adjust Π'_v for the other v s.t. $\Pi' \cong \otimes \Pi'_v$ (it's a bit silly as $G(\mathbb{A}) = \prod' G(E_v)$ but $G'(\mathbb{A}) \neq \prod' G'(E_v)$ - I think this is what's going on).

Now look at ∞.

If π is in L^2, $\pi = \pi_\infty \otimes \pi^\infty$, π_∞ a rep^n of $G(F_\infty)$ & π^∞ a rep^n of $G(\mathbb{A}_F^\infty)$.

If Π is in $\tilde{L}^2$ s.t. $\Pi = \Pi^{\sigma}$, then $\Pi_\infty = \rho\otimes\dots\otimes\rho$, for some ρ, & we can take Π'_∞ s.t. σ acts by $x_1\otimes x_2\otimes\dots\otimes x_l \mapsto x_2\otimes x_3\otimes\dots\otimes x_l\otimes x_1$. $G(E_w) = G(F_w)^l$, l odd.

Let $\{\rho\}$ be all unitary irred rep^{n}s of $G(F_\infty)$ which occurs either as a π_∞ or as a factor of a Π_∞.

The trace identity then becomes

$$\operatorname{tr} r(f) = \sum_{\rho} \operatorname{tr} \rho(f_\infty) . a_\rho, \quad a_\rho = \sum_{\substack{\pi \\ \pi_\infty \cong \rho}} \operatorname{tr} \pi^\infty(f^\infty)$$

$$\operatorname{tr} R(\varphi \times \sigma) = \sum_{\rho} \operatorname{tr}(\rho \otimes .. \otimes \rho)(\varphi_\infty \times \sigma)\, b_\rho$$

with $b_\rho = \sum_{\Pi = \Pi^\sigma} \operatorname{tr} \Pi^{\infty'}(\varphi^\infty \times \sigma)$

Also recall $\operatorname{tr} r(f) = l \operatorname{tr} R(\varphi \times \sigma)$.

Choose $\varphi_\infty = \varphi_{\infty,1} \otimes .. \otimes \varphi_{\infty,l}$, $\varphi_{\infty,i} \in C_c^\infty(G(F_\infty))$

$$f_\infty = \varphi_{\infty,1} * .. * \varphi_{\infty,l}$$

These are associated, like in Richard's course. ("∞ is a split place" ?)

$$\therefore \operatorname{tr}(\rho \otimes ... \otimes \rho)(\varphi_\infty \times \sigma) = \operatorname{tr} \rho(\varphi_{\infty,1}) \rho(\varphi_{\infty,2}) ... \rho(\varphi_{\infty,l})$$

$$= \operatorname{tr} \rho(\varphi_{\infty,1} * .. * \varphi_{\infty,l}) = \operatorname{tr} \rho(f_\infty)$$

$\{f\} = C_c^\infty * .. * C_c^\infty = C_c^\infty(G(F_\infty))$ by eg Dixmier-Malliavin (although you can probably get away with much less)

$$\underset{\text{dense}}{\subseteq} L^1(G(F_\infty))$$

So apply lemma 1, using $f_\infty = f * f^*$

$\Rightarrow a_\rho = l b_\rho$ for all ρ , & hence for all ρ we have

$$\textcircled{*} \quad \sum_{\substack{\pi \text{ in } L^2 \\ \pi_\infty \cong \rho}} \operatorname{tr} \pi^\infty(f) = l \sum_{\substack{\Pi \text{ in } \tilde{L}^2 \\ \Pi = \Pi^\sigma \\ \Pi_\infty = \rho \otimes . \otimes \rho}} \operatorname{tr} \Pi^{\infty'}(\varphi \times \sigma)$$

for all $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}_F^\infty))$

$\varphi = \bigotimes_{v \text{ finite}} \varphi_v$ associated everywhere.

Note that this sum is finite: If f is bi-inv by $U \subseteq G(\mathbb{A}_F^\infty)$ & φ is bi-inv by $\tilde{U} \subseteq G(\mathbb{A}_E^\infty)$, then π, Π don't contribute to the sums unless $(\pi^\infty)^U \neq (0)$, $(\Pi^\infty)^{\tilde{U}} \neq (0)$, & the set of such π, Π with fixed cpt at infinity is finite (see eg Richard's course, probably).

§4 Spectral decomposition - finite places

We want to decompose ⊛ overleaf even further.

Let S be a finite set of finite places of F, including all primes ramified in E or D.

$\mathcal{H}^S_E$, $\mathcal{H}^S_F = \mathcal{H}(G(\mathbb{A}_F^{S,\infty}), \text{max}^l \text{ cpct}) = \otimes$ unramified Hecke algs at all finite $v \notin S$.

Θ^S_π is the corresponding char of $\mathcal{H}^S_F$ if π is unramified at all $v \notin S$

<u>Thm 3</u> This thm involves sums over things which have so many cond^n's on them that there's hardly any at all. In fact both sums are typically empty, & the RHS sum has ≤ 1 elt.

Let $\Psi : \mathcal{H}^S_E \to \mathbb{C}$ be a character. Fix ρ.

Then
$$\sum_{\substack{\pi \text{ in } L^2 \\ \pi_\infty \cong \rho,\ \pi^\infty \text{ unramified away from } S \\ \Theta^S_\Pi \circ b_{E/F} = \Psi}} \operatorname{tr} \pi_S(f_S) = \ell \times \sum_{\substack{\Pi = \Pi^\sigma \text{ in } L^2 \\ \Pi_\infty = \rho \otimes \rho. \\ \Pi^\infty \text{ unram away from } S \\ \Theta^S_\Pi = \Psi}} \operatorname{tr} \Pi'_S(\varphi_S \times \sigma)$$

Here, of course, $f_S = \bigotimes_{v \in S} f_v$, $\varphi_S = \bigotimes_{v \in S} \varphi_v$ associated.

Pf next time.

<u>Rk</u> Since Θ^S_Π determines Π by strong mult 1, the RH expression is 0 or has just 1 term.

Now suppose we're given Π. Choose S "large enough" & $\Psi = \Theta^S_\Pi$. Then RHS is 1 term only.

Choose φ_S s.t. $\operatorname{tr} \Pi'_S(\varphi_S \times \sigma) \neq 0$.

Then RHS $\neq 0$ ∴ LHS $\neq 0$ ∴ LHS is not a sum over 0 elts

∴ $\exists\ \pi$ s.t. $\Theta^S_\pi \circ b_{E/F} = \Theta^S_\Pi$.

But this just asserts that Π is a basechange of π. Hence

<u>Cor</u> If $\Pi = \Pi^\sigma$ then Π is the basechange of some π. □

Unfortunately we can't use the same trick going the other way, as typically the LH sum has ≥ 1 elt.

ecture 4
Fr 26th Feb '93
11:00am

Last time, Tony defined $\mathcal{H}_E^S$, $\mathcal{H}_F^S$ to be the $\otimes$ of the unramified Hecke algebras at all finite $v \notin S$.

Recall we're gonna prove

Thm 3 Let $\Psi : \mathcal{H}_E^S \to \mathbb{C}$, ρ a rep^n of $G(F_\infty)$. Then

$$\sum_{\substack{\pi \text{ in } L^2 \\ \pi_\infty = \rho,\ \textcircled{H}^S_\pi \circ b_{E/F} = \Psi}} \operatorname{tr} \pi_S(f_S) = l \sum_{\substack{\Pi = \Pi^\sigma \text{ in } L^2 \\ \Pi_\infty = \rho \otimes \ldots \otimes \rho, \\ \textcircled{H}^S_\Pi = \Psi}} \operatorname{tr} \Pi'_S(\varphi_S \times \sigma)$$

if f_S, φ_S are associated.

[Recall $\Pi = \Pi^\sigma \Rightarrow$ RHS $\neq \sum_\emptyset$ if $\Psi = \textcircled{H}^S_\Pi \Rightarrow \exists \pi$ ☺]

Rk It's easy to find φ_S s.t. $\operatorname{tr} \Pi'_S(\varphi_S \times \sigma) \neq 0$

- since if $(\Pi_S)^{K_S} \neq 0$ (K_S suff. small open cpct $\subseteq \prod_{v \in S} G(E_v)$ $= G_S$)

then the image of $\mathcal{H}(G_S, K_S)$ is the full endo. alg of $\Pi_S^{K_S}$ (since it's irred)

So e.g. can take $\varphi_S \in \mathcal{H}(G_S, K_S)$ s.t. $\Pi_S(\varphi_S) = \Pi'_S(\sigma^{-1})$.

Then $\Pi'_S(\varphi_S \times \sigma) =$ projn onto $(\Pi_S)^{K_S}$, so trace $\neq 0$.

$$\sum_{\substack{\pi \text{ in } L^2 \\ \pi_\infty \cong \rho}} \operatorname{tr} \pi^\infty(f) = \sum_{\substack{\Pi = \Pi^\sigma \text{ in } L^2 \\ \Pi_\infty \cong \rho \otimes \cdot \otimes \rho}} l \operatorname{tr} \Pi'(\varphi \times \sigma)$$

We need a fund. lemma analogue for pf. These things are traditionally called lemmas but they're really thms.

Lemma (See Labesse's lectures) Fix v unramified in D, E; $G(E_v) \cong GL_2(E_v)$
$G(F_v) \cong GL_2(F_v)$

Suppose $\{\pi_v\}$, $\{\Pi'_v\}$ are finite collections of reps (irred, admis) of $G(F_v)$, $G'(E_v)$. Suppose we have the identity

$$\circledast \sum_{\pi_v} c(\pi_v) \operatorname{tr} \pi_v(f_v) = \sum_{\Pi'_v} d(\Pi'_v) \operatorname{tr} \Pi'_v(\varphi_v \times \sigma)$$

for all associated (f_v, φ_v), invt under Iwahori subgps of $G(F_v)$, $G(E_v)$ (certain cts $c(\pi_v)$, $d(\Pi'_v) \in \mathbb{C}$)

Then $\circledast$ holds for all $(b_{E/F}\varphi_v, \varphi_v)$ where $\varphi_v \in$ unramified Hecke algebra of $G(E_v)$. Labesse will prove this this afternoon. $\square$

It's analagous to the Fundamental lemma, which is tricky to prove in this context.

Apply this lemma as follows: pick associated (f_S, φ_S) (& ρ).

Pick $v \notin S$. Define f to be

$$f = f_S \otimes f_v \otimes (\text{unit elts at all } v' \notin S \cup \{v\})$$

$$\& \quad \varphi = \varphi_S \otimes \varphi_v \otimes (\text{unit elts})$$

where f_v, φ_v are as in the lemma (associated & Iwahori-inv't)

Then f, φ are associated everywhere (hypothesis + fund. lemma for unit elts)

$$\Rightarrow \sum_{\pi_\infty = \rho} \mathrm{tr}\, \pi_S(f_S)\, \mathrm{tr}\, \pi_v(f_v) = \sum_{\substack{\Pi = \Pi^\sigma \\ \Pi_\infty = \rho \otimes \cdots \otimes \rho}} l\, \mathrm{tr}\, \Pi'_S(\varphi_S \otimes \sigma)\, \mathrm{tr}\, \Pi'_v(\varphi_v \times \sigma) \quad \textcircled{**}$$

The sets of $\{\pi\}$, $\{\Pi\}$ occuring in both sides with non-zero trace are finite (unramified @ all $v' \notin S \cup \{v\}$, inv't by fixed cpct open at $S \cup \{v\}$)

$\therefore$ applying lemma, get that $\textcircled{**}$ holds also for $\varphi_v \in$ unramified Hecke algebra, $f_v = \varphi_v \circ b_{E/F}$

Now take another place $\notin S \cup \{v\}$ ...

We end up with the identity

$$\sum_{\pi_\infty = \rho} \mathrm{tr}\, \pi_S(f_S)\, \mathrm{tr}\, \pi^S(\varphi^S \circ b_{E/F}) = l \sum_{\substack{\Pi \\ \Pi = \Pi^\sigma \\ \Pi_\infty = \rho \otimes \cdots \otimes \rho}} \mathrm{tr}\, \Pi'_S(\varphi_S \times \sigma)\, \mathrm{tr}\, \Pi'^S(\varphi_S \times \sigma)$$

for all $\varphi^S \in \mathcal{H}^S_E = \varinjlim_{v_1, \ldots, v_s} \bigotimes_i \mathcal{H}(G(E_i), K_{\max})$

$$\text{ie} \sum_{\substack{\pi_\infty = \rho \\ \pi^S \text{ unram}}} \mathrm{tr}\, \pi_S(f_S)\, \Theta^S_\pi(b_{E/F}(\varphi^S)) = l \sum_{\substack{\Pi = \Pi^\sigma \\ \Pi_\infty = \rho \otimes \cdots \otimes \rho \\ \Pi^S \text{ unram}}} \mathrm{tr}\, \Pi'_S(\varphi_S \times \sigma)\, \Theta^S_\Pi(\varphi^S)$$

(rk - we normalised Π'_v for $v \notin S$ s.t. $\sigma = 1$ on spherical vector). The characters of $\mathcal{H}^S_E$ are lin ind, so can decompose the last identity corresponding to chars $\Psi : \mathcal{H}^S_F \to \mathbb{C}$

$\Rightarrow$ Thm 3 $\square$

(but messy formulae!)

Thm 3 is important. If we had the fundamental lemma it would be a bit easier, but we use thm 3 to prove fund. lemma !!

This is nearly all the ingredients.
We do need sthg else though - sthg that it would be inappropriate not to mention:

§5 Application of L-f^ns

π irred auto rep^n of $GL_2(\mathbb{A}_F)$. For all $v \notin S$ (finite set) π_v is unramified, so it corresponds to $t_v = \begin{pmatrix}\alpha_v & 0 \\ 0 & \beta_v\end{pmatrix} \in GL_2(\mathbb{C})$ (up to conjugacy)

$$L(\pi_v, s) = \det(I - q_v^{-s} t_v)^{-1} = (1-\alpha_v q_v^{-s})^{-1}(1-\beta_v q_v^{-s})^{-1}$$

$$= \det(1 - q_v^{-s} \sigma_v(\mathrm{Frob}_v))^{-1}$$

where $\sigma_v : W_F \to GL_2(\mathbb{C})$ is the unramified HM s.t. $\sigma_v(\mathrm{Frob}_v) = t_v$.

$\Rightarrow L^S(\pi, s) = \prod_{v \notin S} L(\pi_v, s)$ (the incomplete L-f^n.)

π cuspidal $\Rightarrow$ an entire f^n of s (Hecke theory as applied by Jacquet, Langlands)

π, π' unram for $v \notin S$ $\Rightarrow$ "Rankin convolution".

$$L^S(\pi \times \pi', s) = \prod_{v \notin S} L(\pi_v \times \pi'_v, s)$$

where $L(\pi_v \times \pi'_v, s) = L(\sigma_v \otimes \sigma'_v, s) = \det(I - q_v^{-s} t_v \otimes t'_v)^{-1}$,

$$t_v \otimes t'_v = \begin{pmatrix} \alpha_v \alpha'_v & & & \\ & \alpha_v \beta'_v & & \\ & & \beta_v \alpha'_v & \\ & & & \beta_v \beta'_v \end{pmatrix}$$

Thm 4 (Jacquet - Shalika) (true for GL_m, GL_n). Assume π, π' are ~~unramified~~ unitary & cuspidal. Then $L^S(\pi \times \pi', s)$ is holomorphic & $\neq 0$ at $s=1$ except if $\pi' \simeq \tilde{\pi}$, when it has a simple pole. □

($\pi' = \tilde{\pi} \to \exists\ \zeta$-f^n in L-f^n, & dividing out gives an L-f^n on sthg in GL_3)

<u>Cor 1</u> Suppose $\pi^{(1)}, \dots, \pi^{(r)}, \pi^{(1)'}, \dots, \pi^{(r)'}$ are $2r$ irred cusp auto rep's of $GL_2(\mathbb{A}_F)$, & S = a (large enough) set of primes.
Suppose that for all $v \notin S$, $\pi_v^{(i)}$, $\pi_v^{(i)'}$ are unramified, & also that

$$(\dagger) \quad \sigma_v^{(1)} \oplus \dots \oplus \sigma_v^{(r)} \cong \sigma_v^{(1)'} \oplus \dots \oplus \sigma_v^{(r)'}.$$

Then $\{\pi^{(i)}\}, \{\pi^{(i)'}\}$ are the same (up to reordering)

⌈Note $(\dagger) \iff$ the matrices $\begin{pmatrix} t_v^{(1)} & & 0 \\ & \ddots & \\ 0 & & t_v^{(r)} \end{pmatrix}$ & $\begin{pmatrix} t_v^{(1)'} & & 0 \\ & \ddots & \\ 0 & & t_v^{(r)'} \end{pmatrix}$ are conjugate in $GL_{2r}(\mathbb{C})$.⌋

<u>Pf</u> Let $\Lambda(s) = \prod_{j=1}^{r} L^S(\tilde{\pi}^{(1)} \times \pi^{(j)}, s)$, $\Lambda'(s) = \prod_{j=1}^{r} L^S(\tilde{\pi}^{(1)} \times \pi^{(j)'}, s)$.

The local factor $\Lambda_v(s)$ (obvious notation)

$$\text{is } \prod_j \det(1 - q_v^{-s} \underbrace{t_v^{(1)-1} \otimes t_v^{(j)}}_{\sigma_v^{(1)}(\mathrm{Frob}^{-1}) \otimes \sigma_v^{(j)}(\mathrm{Frob})})^{-1}$$

$$\Lambda'_v(s) = \prod_j \det(1 - q_v^{-s} \sigma_v^{(1)}(\mathrm{Frob}^{-1}) \otimes \sigma_v^{(j)'}(\mathrm{Frob}))^{-1}$$

$$\underset{\substack{\uparrow \\ \left(\text{note} \sim \text{ to get } \mathrm{Frob}^{-1} \rightsquigarrow \mathrm{Frob}\right)}}{=} L(\tilde{\sigma}_v^{(1)} \otimes (\bigoplus_j \sigma_v^{(j)'}), s) = \Lambda_v(s)$$

$\Lambda(s)$ has a pole at $s=1$ (from $\tilde{\pi}^{(1)} \times \pi^{(1)}$) by thm 4.

So $\Lambda'(s)$ has a pole at $s=1$, hence $\pi^{(1)} \cong \pi^{(j)'}$ for some j (Thm 4)

$\Rightarrow$ result by induction. □

NB Tony has no clue, he has to confess, as to why the thm is true (the $\neq 0$ bit). Classically Rankin proved stuff about the complete L fn so maybe you have to understand bad primes.

This will not stop Tony drawing further corollaries.

<u>Cor 2</u> E/F, G as before, π, π' irred auto rep's of $G(\mathbb{A}_F)$. Assume π, π' each have base change to E, & that the base changes are iso. Then
$\pi' \cong \pi \otimes (\chi^j_{E/F} \otimes N_{\mathrm{red}})$, some j.

<u>Pf</u> If π, π' are 1-dim^l, this is just CFT. If π is ∞-dim^l then π_v is ∞-dim^l for ∞ly many v. π is associated to a <u>cuspidal</u> auto rep^n by J-L correspondence

As π_v, π'_v have the same basechange for almost all v, π' is also ∞-dim^l $\rightsquigarrow$ cuspidal rep^n of $GL_2(\mathbb{A}_F)$. Apply the corollary to

$$\{ \pi \otimes (\chi^j_{E/F} \otimes N_{red}) \}, \; \{ \pi' \otimes (\chi^j_{E/F} \otimes N_{red}) \}$$

It's enough to check $\oplus$. (cor 1)

$\pi_v \leftrightarrow \sigma_v$

$\pi'_v \leftrightarrow \sigma'_v$.

$$\text{Then } \sigma_v \big|_{W_{E_v}} \simeq \sigma'_v \big|_{W_{E_v}}.$$

$$\text{so. } \bigoplus_{j=1}^{\ell} \sigma_v \otimes \chi^j_{E_v/F_v} \cong \bigoplus_{j=1}^{\ell} \sigma'_v \otimes \chi^j_{E_v/F_v} \quad \square$$

The same pf shows that the LHS of thm 3, namely

$$\sum_{\substack{\pi \text{ in } L^2 \\ \pi_\infty = \rho,\ \Theta^S_\pi \circ b_{E/F} = \Psi}} tr\, \pi_S(f_S)$$

is ~~a sum~~ either empty, or a sum over rep^{n}s of the form

$$\pi \otimes (\chi^j_{E/F} \circ N_{red}), \text{ some fixed } \pi.$$

Given π, choose f_S s.t. $tr\,(\pi_S \otimes \chi^j)(f_S) = tr\, \pi_S(f_S) \neq 0$.

Choose f_S with support a suff. small nbd of 1. Then LHS $\neq 0$ $(= \ell\, tr\, \pi_S(f_S))$

So RHS $\neq 0$ ie $\exists \pi$. ▨

[redacted]

VII Fundamental Lemma - 1

Peter Schneider

ecture 1
Wed 24th Feb '93
11:00am

This lecture is really about

Unramified local base change

& it'll be putting together lots of things we've heard before. (Scholl etc)

F a local field, O = integers ~~O is goodness knows what~~, π a uniformiser, k = residue class field, $q = \# k$, ω = normalised disc val^n of $F^\times$.

$G = GL_d(F)$ (hardly any loss of generality here)

$K = GL_d(O)$ = max^l cpct subgp

Normalise dg s.t. $\int_K dg = 1$

A general convention: if $H \subseteq G$ is a closed subgp then normalise dh s.t. $\int_{H \cap K} dh = 1$.

$\mathcal{H}$ = Hecke algebra = K-bi-inv f's on G with cpct support.

$$\varphi * \psi = \int_G \varphi(g)\psi(g^{-1}\_)\,dg$$

1 = classf^n of K.

Later E/F will be an unramified ext^n.

① Say $S \subseteq G$ is the diagonal matrices.

An unramified char of S is $\chi : \underset{(F^\times)^d}{\overset{}{S}} \to \mathbb{C}^\times$ s.t. $\chi|_{(O^\times)^d} = 1$

We have a bijection

$$\text{unram char of } S \xleftarrow{\sim} \mathbb{C}^{\times d}$$

$$\text{via } \underset{\sim}{z} = (z_1, .., z_d) \longmapsto \left(\chi_{\underset{\sim}{z}} : \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_d \end{pmatrix} \longmapsto \prod z_i^{\omega(a_i)}\right)$$

W = subgp of permutation matrices in G $\cong S_d$

W acts on S by conjugation.

W also acts on unram. chars of S. & this corresponds to the permutation action on $\mathbb{C}^{\times d}$.

Think of $\mathbb{C}^{\times d}$ as being the diagonal matrices in $GL_d(\mathbb{C})$.

We have the Jordan normal form:
- any ss elt of $GL_d(\mathbb{C})$ is conjugate to a diagonal elt
- 2 diag ~~elts~~ matrices are conjugate iff they're a permutation of each other.

We get a bijection

$$\left\{\begin{matrix}\text{W-orbits in the}\\ \text{unramified chars of } S\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{semisimple conj.}\\ \text{classes in } GL_d(\mathbb{C})\end{matrix}\right\}$$

There's bijection ①. Here comes another one.

② W_F, the Weil gp of F

An unramified parameter ψ of W_F is (the isom. class of) a semisimple repⁿ

$$\psi : W_F \to GL_d(\mathbb{C}), \quad \text{s.t. } \psi|_{\text{inertia subgp}} = 1$$

However, $W_F / \text{inertia} \cong \mathbb{Z} = \langle \text{Frobenius} \rangle^{\mathbb{Z}}$

& so ψ is determined by $\psi(\text{Frob}) \in \{\text{ss conj classes in } GL_d(\mathbb{C})\}$

Hence

$$\left\{\begin{matrix}\text{unramified}\\ \text{parameters}\\ \text{of } W_F\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{semisimple}\\ \text{conj classes}\\ \text{in } GL_d(\mathbb{C})\end{matrix}\right\}$$

So we get, from ①, & ②, the following bijection:

$$\left\{\begin{matrix}\text{W-orbits in the}\\ \text{unramified}\\ \text{chars of } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unramified}\\ \text{params of}\\ W_F\end{matrix}\right\}$$

$$\chi \longmapsto \left(\text{Frobenius} \mapsto \begin{pmatrix} \chi(s_1) & 0 \\ 0 & \chi(s_d) \end{pmatrix} \right)$$

$$\text{where } s_i = \begin{pmatrix} 1 & & 0 \\ & \pi \ddots & \\ 0 & & 1 \end{pmatrix}$$

What has this got to do with Hecke operators ?

③ If $\Lambda = S / S \cap K = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_d$, $\quad \lambda_i = s_i(S \cap K)$.

Then we get yet another bijection

$$\left\{\begin{matrix}\text{W-orbits in}\\ \text{unram chars}\\ \text{of } S\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \mathrm{Hom}_{alg}(\mathbb{C}[\Lambda]^W, \mathbb{C})$$

$$\chi \longmapsto \alpha_\chi \quad : \alpha_\chi(\Sigma c_\lambda \lambda) = \Sigma c_\lambda \chi(\lambda)$$

Note: $\mathbb{C}[\Lambda]^W \subseteq \mathbb{C}[\Lambda]$ is finite

$$\underset{\substack{\uparrow \\ \text{max'l} \\ \text{ideals}}}{\mathrm{Max}(\mathbb{C}[\Lambda])} \twoheadrightarrow \mathrm{Max}(\mathbb{C}[\Lambda]^W)$$ is surjective (this is some kind of going-up thm)

We also recall the Satake iso (of course, his $\mathcal{H}$ is our $\mathcal{H}(G,K)$) (not our $\mathcal{H}(G)$)

$$\mathcal{S} : \mathcal{H} \longrightarrow \mathbb{C}[\Lambda]^W$$

$$\varphi \longmapsto \left(S \ni s \mapsto \int_N \varphi(su)\,du \times \delta(s)^{1/2} \right);$$ $N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ is not unimodular, & so you put δ in to make it work.

<u>Thm</u> (Satake) $\mathcal{S}$ is an algebra isomorphism... We've seen a pf for GL_2 □

Consequences

ⓐ Set $\sigma_1 = \lambda_1 + .. + \lambda_d, \ldots, \sigma_d = \lambda_1 .. \lambda_d$ be the elementary symmetric polys

Then $\mathbb{C}[\Lambda] = \mathbb{C}[\lambda_1^{\pm 1}, .., \lambda_d^{\pm 1}] = \mathbb{C}[\lambda_1, .., \lambda_d, \underset{\text{fortunately symmetric}}{\sigma_d^{-1}}]$

Hence

$$\mathbb{C}[\Lambda]^W = \mathbb{C}[\sigma_1, .., \sigma_d, \sigma_d^{-1}]$$

So this ring is explicit. Let's make the map explicit too.

$$\text{Define } t_i := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \pi & \\ & & & & \ddots \\ & & & & & \pi \end{pmatrix} \in S$$

$\underbrace{\pi \ldots \pi}_{i \text{ times}}$

Define τ_i = char fn of $Kt_iK \in \mathcal{H} = \mathcal{H}(G,K)$.

Fact $\mathcal{S}(\tau_i) = q^{\frac{1}{2}i(d-i)}\sigma_i$ (easy for $d=2$) (Tony did it)

So the Satake map is now explicit

Hence

$$\begin{tikzcd} ⓑ \left\{\begin{matrix}\text{W-orbits in} \\ \text{unram chars} \\ \text{of } S\end{matrix}\right\} \arrow[r, leftrightarrow] & \mathrm{Hom}_{alg}(\mathbb{C}[\Lambda]^W, \mathbb{C}) \arrow[r, leftrightarrow, "\sim", "\text{new bit ⓑ}"'] & \mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C}) \\ & \left\{\begin{matrix}\text{unramified} \\ \text{params of} \\ W_F\end{matrix}\right\} \arrow[ul, leftrightarrow] \arrow[ur, leftrightarrow, "\circledast"'] & \end{tikzcd}$$

We understand ⊛: If $\alpha \in \mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C})$ then ⊛ sends α to

the repn sending Frobenius to $\begin{pmatrix} z_1 & 0 \\ 0 & z_d \end{pmatrix}$

where $\prod_{i=1}^{d}(X - z_i) = \sum_{j=0}^{d} (-1)^{d-j} q^{-\frac{1}{2}j(d-j)} \alpha(\tau_{d-j}) X^j$

This is rather whizzo.

There's more!

ⓒ Clearly $\mathcal{H}$ is commutative & f.g. $/\mathbb{C}$.

So by Schur's lemma, any simple $\mathcal{H}$-module is 1-dim^l & given by a char in $\mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C})$

ⓓ he's not so sure about any more. He'll postpone it.

④ The unramified rep^n theory of G

A smooth & irred rep^n ~~of~~ V of G is unramified if $V^K \neq \{0\}$.

Fact V^K is a simple $\mathcal{H}$-module via $\varphi * v = \int_G \varphi(g) gv \, dg$

Hence we have a map
(a bijection - Tony showed this when $d=2$)

$$\left\{\begin{array}{l}\text{isom. classes of}\\ \text{unramified}\\ \text{rep's of } G\end{array}\right\} \longleftrightarrow \mathrm{Hom}_{alg}(\mathcal{H}, \mathbb{C})$$

$$V \cong V_\alpha \longleftrightarrow \alpha \text{ s.t. } \varphi * v = \alpha(\varphi) v \text{ for } \varphi \in \mathcal{H}, v \in V^K$$

We've done this when $d=2$ but he wants to talk about

realisation of V_α

Choose the unramified char χ of S corresponding to α. Then

ⓐ via principal series: $I(\chi) :=$ space of all loc. cst f^{n}s $f : G \to \mathbb{C}$ s.t.

$$f(g\underset{\substack{\uparrow\\ \text{unip}\\ \text{otent}}}{n}\underset{\substack{\uparrow\\ \text{diagonal}}}{s}) = \delta(s)^{-1/2} \chi(s^{-1}) f(g)$$

He's addicted to left actions so it's all the other way round. G acts by left translations. Boo.

Fact $I(\chi)$ has a ! irred subquotient V_χ which is unramified & $\cong V_\alpha$.

There's another way

ⓑ via spherical f^{n}s. It uses ⓓ above which was an explanation of spherical f^{n}s so he'll have to omit ⓑ too.
The spherical f^n would have been written Γ_χ.

So we have a bijection (the unramified local Langlands correspondence) :

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Isom classes of}\\ \text{unramified}\\ \text{rep's of } G\end{matrix}\right\} \arrow[r, leftrightarrow] \arrow[rr, leftrightarrow, bend right, "\text{this is the correspondence.}"'] & \mathrm{Hom}_{alg}(\mathcal{H},\mathbb{C}) \arrow[r, leftrightarrow] & \left\{\begin{matrix}\text{unramified}\\ \text{params of}\\ W_F\end{matrix}\right\}
\end{tikzcd}$$

Now say E/F is a finite unramified ext of degree l.

It's all easy.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Isom classes of}\\ \text{unram rep's of}\\ G = GL_d(F)\end{matrix}\right\} \arrow[rr, leftrightarrow] & & \left\{\begin{matrix}\text{unram}\\ \text{params}\\ \text{of } W_F\end{matrix}\right\} \arrow[dd, "\text{restriction}"] \\
& \mathrm{Hom}_{alg}(GL_d(F), GL_d(\mathcal{O}_F)) \arrow[ul] \arrow[ur, leftrightarrow] & \\
\left\{\begin{matrix}\text{Isom classes}\\ \text{of unram rep's}\\ \text{of } G_E = GL_d(F)\otimes E\\ = GL_d(E)\end{matrix}\right\} \arrow[rr, leftrightarrow] & & \left\{\begin{matrix}\text{unram params}\\ \text{of } W_E\end{matrix}\right\} \\
& \mathrm{Hom}_{alg}(\mathcal{H}_E,\mathbb{C}) \arrow[ul, leftrightarrow] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

We get a base change map for Hecke algebras $b : \mathcal{H}_E \to \mathcal{H}$ characterised by

$$\begin{tikzcd}
\begin{matrix}\text{W-orbits in}\\ \text{unram chars}\\ \text{of } S\end{matrix} \arrow[r] & \begin{matrix}\text{W-orbits in}\\ \text{unram chars}\\ \text{of } S_E\end{matrix} \\
\chi \arrow[r, mapsto] & \chi^{l}
\end{tikzcd}$$

(note new Frobenius = (old one)^{l})

This b comes from the map

$$\begin{tikzcd}
\mathbb{C}[\Lambda]^W \arrow[dd] & \mathbb{C}[\Lambda_E]^W = \mathbb{C}[\Lambda]^W \arrow[l] \arrow[dd, "\cong"] \\
\sum c_\lambda \lambda^l & \sum c_\lambda \lambda \arrow[l, mapsto] \\
\mathcal{H} & \mathcal{H}_E \arrow[l, "b"']
\end{tikzcd}$$

Lecture 2
Thus 25th Feb '93
9:30am

He has E/F unramified & $\langle\sigma\rangle = \mathrm{Gal}(E/F)$ although he may call it θ by accident because Kottwitz calls it θ & he's cribbed this off Kottwitz.

We have G & G_E, & if $g \in G_E$ define $Ng = g\,{}^{\sigma}g \cdots {}^{\sigma^{l-1}}g$

This N induces an injection

$$N : \left\{ \begin{matrix} \sigma\text{-conjugacy} \\ \text{classes in } G_E \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{conjugacy} \\ \text{classes in } G \end{matrix} \right\}$$

& also recall we have $b : \mathcal{H}_E \to \mathcal{H}$.

We also have the orbital integrals:

$\gamma \in G$, $\varphi \in \mathcal{H} \rightsquigarrow O_\gamma(\varphi) = \displaystyle\int_{G_\gamma \backslash G} \varphi(x^{-1} y x)\, d\bar{x}$ (Pick a fixed Haar measure)

$g \in G$, $\psi \in \mathcal{H}_E \rightsquigarrow O_{g\sigma}(\psi)$. This is our twisted orbital integral but he will call it O, not TO, as σ-orbits are really just orbits in some semidirect product of G_E & $\langle\sigma\rangle$ or sthg.

$$O_{g\sigma}(\psi) = \int_{G_{g\sigma} \backslash G_E} \psi(y^{-1} g\, {}^{\sigma}y)\, d\bar{y}$$

We only defined this stuff for ss elts. I think he's only going to use it for ss elts.

He's now in a pos^n to state the fundamental lemma.

Fundamental lemma

i) If the orbit $O_\gamma \subseteq G$ is not a norm, then $O_\gamma(b\psi) = 0$ for any $\psi \in \mathcal{H}_E$.

ii) If $O_\gamma = N(O_{g\sigma})$ then $\exists c \in \mathbb{R}^\times_{>0}$ s.t. $O_\gamma(b\psi) = c. O_{g\sigma}(\psi)$ $\forall \psi \in \mathcal{H}_E$.

His job is to prove this for $\psi = 1_E$. Then $b\psi = 1$

Def^ns

Put $X = G/K \hookrightarrow X_E := G_E/K_E$

& put X^γ = fixed pts of γ on X, $(G_\gamma)_x$ = stabilizer of $x \in X^\gamma$ in G_γ

$= G_\gamma \cap yKy^{-1}$ if $x = yK$

$X_E^{g\sigma}$ = fixed pts of $g\sigma$ on X_E

$(G_{g\sigma})_x$ = fixed pts of $x \in X_E^{g\sigma}$ in $G_{g\sigma}$

He needs

Lemma

1) $O_\gamma(1) = \sum_{x \in G_\gamma \backslash X^\gamma} \operatorname{vol}((G_\gamma)_x)^{-1}$ (may be infinite sums, he thinks)

2) $O_{g\sigma}(1_E) = \sum_{x \in G_{g\sigma} \backslash X_E^{g\sigma}} \operatorname{vol}((G_{g\sigma})_x)^{-1}$ (nb both sides of both eqn depend on a choice of Haar measure, so take the same one!) in thing

Pf 2) ⇒ 1) $(E = F)$

2) $O_{g\sigma}(1_E) = \operatorname{vol}\Big(G_{g\sigma} \backslash \bigcup_{y^{-1} g^\sigma y \in K_E} G_{g\sigma} y\Big)$ = ~~vol~~ $\sum_{\substack{y \in G_{g\sigma} \backslash G_E / K_E \\ y^{-1} g^\sigma y \in K_E}} \operatorname{vol}(G_{g\sigma} \backslash G_{g\sigma} y K_E)$

$x = yK_E$, $x \in X^{g\sigma}$

$$= \sum_{x \in G_{g\sigma} \backslash X_E^{g\sigma}} \operatorname{vol}(G_{g\sigma} \backslash \underbrace{G_{g\sigma} y K_E}_{\cong (y^{-1} G_{g\sigma} y \cap K_E) \backslash K_E})$$

$$\cong (G_{g\sigma} \cap yK_E y^{-1}) \backslash \underset{\substack{\uparrow \\ \text{volume } 1}}{yK_E y^{-1}}$$

□

Now we need

Prop^n Assume $X_E^\gamma \neq \emptyset$; then there exists $g \in G_E$ s.t.

a) $Ng = \gamma$
b) g lies in the centre of $G_\gamma(E)$
c) $X_E^\gamma = X_E^g$

Moreover, we then have $X^\gamma \xrightarrow{\sim} X_E^{g\sigma}$ & $G_{g\sigma} = G_\gamma$

We'll prove this in a sec..
We'll now prove the fundamental lemma

Pf of fund lemma

i) O_γ is not a norm $\overset{a)}{\Rightarrow} X^\gamma = \emptyset \Rightarrow O_\gamma(1) = 0$

ii) Assume $O_\gamma = N(O_{g\sigma})$

Case 1) $X_E^\gamma \neq \emptyset$. Apply prop^n

Case 2) $X^\gamma_E = \emptyset \Rightarrow X^\gamma = \emptyset \Rightarrow O_\gamma(1) = 0$

$N(h^{-1} g \sigma h) = h^{-1} N(g) \sigma h = \gamma$

$\Rightarrow$ can assume $Ng = \gamma \Rightarrow X_E^{g\sigma} \subseteq X_E^\gamma$. □

Proof of prop^n

Think of X_E = set of all O_E-~~lattices~~ lattices in E^d

$K_E = GL_d(O_E)$ = stabilizer of $O_E^d \subseteq E^d$

Put C_E = O_E-subalgebra in $M_d(E)$ of all elts y s.t.

- y belongs to the centre of the centralizer of γ in $M_d(E)$
- $y\Lambda \subseteq \Lambda$ for any lattice $\Lambda \in X_E^\gamma$

C_E is commutative ; $\gamma \in G \Rightarrow C_E$ is σ-stable

$X_E^\gamma \neq \emptyset \Rightarrow C_E$ is as an O_E-module finitely generated & free.

(C_E is some conjugate of $M_d(O_E)$)

Put $C = C_E^\sigma$, an O-subalgebra. It's somehow clear that $C \otimes_O O_E \xrightarrow{\sim} C_E$

Claim $C_E^\times \xrightarrow{N} C^\times$ is surjective

Clearly $\gamma \in C^\times$, & hence $\exists g \in C_E^\times$ s.t. $Ng = \gamma$ a) ✓
and also b) ✓ because of defn of C_E.

Moreover, $g^{-1} \in C_E^\times$ too so ~~g^{-1} & g~~ $g^{-1}\Lambda \subseteq \Lambda$, $g\Lambda \subseteq \Lambda$ $\forall \Lambda \in X_E^\gamma$
$\Lambda \subseteq g\Lambda$

$\therefore$ $\Lambda = g\Lambda$ & we get c) ✓

Note $\sigma^{-1}(g) \in C_E^\times \Rightarrow X_E^\gamma \subseteq X_E^{\sigma^{-1}g} \Rightarrow g\sigma(x) = \sigma({}^{\sigma^{-1}}g(x)) = \sigma x$ for $x \in X_E^\gamma$

But $Ng = \gamma \Rightarrow X_E^{g\sigma} \subseteq X_E^\gamma$ - a general fact (easy direct pf)

$\therefore$ $X_E^{g\sigma} = (X_E^\gamma)^\sigma = X^\gamma$

Finally we need to show ~~⊕~~ $G_{g\sigma} = G_\gamma$

Now $N(h^{-1}g\sigma h) = h^{-1}N(g)h$; a) $\Rightarrow G_{g\sigma} \subseteq G_\gamma(E)$

b) $\Rightarrow$ g lies in the centre of $G_\gamma(E)$

$h \in G_{g\sigma}$; $h^{-1}g\sigma h = g$
$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\quad gh^{-1}\sigma h$ $\Rightarrow G_{g\sigma} = G_\gamma(E)^\sigma = G_\gamma$ □ of prop.

So we just have to prove the claim now.

Pf of claim (it's just some generalisation of the CFT norm map being surj)

Consider the filtration
$$\begin{array}{ccccccc} C_E^\times & \supseteq & 1+\pi C_E & \supseteq & 1+\pi^2 C_E & \supseteq & \dots \\ \cup\!\mid & & \cup\!\mid & & & & \\ C^\times & \supseteq & 1+\pi C & \supseteq & 1+\pi^2 C & \supseteq & \dots \end{array}$$

It suffices to prove the assertion for each subquotient!

But ⊛
$$\begin{tikzcd}
(1+\pi^m C_E)/(1+\pi^{m+1} C_E) \arrow[r, phantom, "\cong"] \arrow[ddd, "N"] & \pi^m C_E/\pi^{m+1} C_E \arrow[d, equal] \\
& (\pi^m C/\pi^m C)\otimes_k k_E \arrow[d, phantom, "\cong"] \\
& k_E^r \arrow[d, "\text{trace}"] \\
(1+\pi^m C)/(1+\pi^{m+1} C) \arrow[r, phantom, "\cong"] & k^r
\end{tikzcd}$$
☺

so we just have to deal with the first step.

But here we can use a general fact (proved in e.g. Serre's book)

A is a finite commutative k-algebra ; then

$$(A \otimes_k k_E)^\times \xrightarrow{N} A^\times \text{ is surjective.}$$

So we're home

VIII : Fundamental Lemma II

Jean-Pierre Labesse

Lecture 1
u 25th Feb '93
.30pm

Jean-Pierre is going to chat a bit about the fundamental lemma.

If things don't match then you can't even start - eg $\varphi = \otimes\varphi_v$ etc - you absolutely needed the matching result for 1_E & 1

The fundamental lemma in its full generality may not be needed by e.g. Tony. He will discuss (a variant of) it anyway!

We have a local field F with unif. par ϖ so save confusion with π.

We have $\mathcal{H}(G,K)$, $G = GL_2(F)$, $K = GL_2(\mathcal{O})$

$t = \begin{pmatrix} \varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix}$, $n_1, n_2 \in \mathbb{Z}$. $\dfrac{KtK}{\mathrm{vol}(K)}$ is an interesting elt of $\mathcal{H}(G,K)$.

the spherical Hecke algebra

He wants to do some easy harmonic analysis, but hey - we're beginners!

Say $K(t) = \dfrac{\mathrm{char}(KtK)}{\mathrm{vol}(K)}$

We have $K(t_1)K(t_2) = \sum c(t_1, t_2; t)\, K(t)$

$K(t) = K(\tilde{t})$, $\tilde{t} = \begin{pmatrix} \varpi^{n_2} & 0 \\ 0 & \varpi^{n_1} \end{pmatrix}$

Set $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$. The map $g \mapsto s({}^t g)s^{-1}$ is an anticautomorphism & it preserves $K(t)$.

Hence commutative (?!)

We have $\mathcal{S}$, the Satake transform : if $h \in \mathcal{H}(G,K)$, set

$$(\mathcal{S}h)(m) = \int_{n \in N} h(mn)\, \delta(m)^{1/2}\, dn$$

where $m \in M$ = diagonal matrices, $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

$M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

Now assume m is regular, $m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$, $m_1 \neq m_2$.

$$(\tilde{\mathfrak{S}}h)(m) = \int_{M\backslash G} h(g^{-1}mg)\,\Delta(m)^{1/2}\,dg \quad ; \quad \Delta(m) = |D(m)|,$$

$$D(m) = \det(1 - \mathrm{Ad}\,m \,|\, \mathfrak{g}/\mathfrak{m})$$

He will prove this. Now. To slow him down.

We have the Iwasawa decomposition $G = MNK$; $M\backslash G \rightsquigarrow$ use $dn\,dk$.

Then $(\tilde{\mathfrak{S}}h)(m) = \iint h(n^{-1}mn)\,\Delta(m)^{1/2}\,dk\,dn$ - here vol$(K)=1$

-note $n^{-1}mn = m\underbrace{(m^{-1}n^{-1}m)n}_{n_1} = mn_1$ so $dn_1 = c(m)\,dn$

$$\Delta(m) = |D(m)| = |(1-m^{\alpha})(1-m^{-\alpha})| \quad ; \quad m^{\alpha} = m_1/m_2 \;,\; \delta(m) = |m^{\alpha}|$$

$$\delta(m)^{-1}\Delta(m) = |1-m^{-\alpha}|^2$$

That's enough of that nonsense. It's v. easy calculations. You should see them once in your life.

We have $\mathcal{H}(G,K) \xrightarrow{\mathfrak{S}} \mathcal{H}(M, M\cap K)$

If χ is an unramified char ie a char of $M/M\cap K$,

we get a repⁿ of $\mathcal{H}(G,K)$:

$$h \longmapsto \hat{\mathfrak{S}}h(\chi) = \int_M (\mathfrak{S}h)(m)\,\chi(m)\,dm$$

$$\mathcal{H}(G,K) \to \mathbb{C}$$

~~where~~ So $\hat{\mathfrak{S}}h(\chi) = \displaystyle\int_M \int_{M\backslash G} h(g^{-1}mg)\,\Delta(m)^{1/2}\,\chi(m)\,dg\,dm$

Now given χ we can of course form the principal series I_χ

$I_\chi = \rho(\mu_1,\mu_2)$ in Tony's notation.

We have $\hat{\mathfrak{S}}h(\chi) = \mathrm{tr}\, I_\chi(h) = \mathrm{tr}\, \pi_\chi(h)$

$\pi_\chi = \pi(\mu_1,\mu_2)$ spherical subquotients (ie contains a K-inv^t vector)

Now for f we get $(\mathcal{S}f)(m) = \int_{M\backslash G} f(g^{-1}mg)\, \Delta(m)^{1/2}\, dg$

ie $(\mathcal{S}f)(m) = \int_K \int_N f(k^{-1}mnk)\, \delta(m)^{1/2}\, dndk$

It turns out that $\mathrm{tr}\, I_\chi(f) = (\widehat{\mathcal{S}f})(\chi)$

but the trace of the subquotient may not be equal in general:
$\mathrm{tr}\, I_\chi(f) \neq \mathrm{tr}\, R_\chi(f)$ in general. That's life.

Now I_χ acts by right translations by G on the space of fns on G

$$\varphi(nmx) = \chi(m)\, \delta(n)^{1/2}\, \varphi(x)$$

$$(I_\chi(f)\varphi)(x) = \int_G \varphi(xy)\, f(y)\, dy$$

$I_\chi(f)$ has a kernel on ~~N~~ $L^2(K)$ (note φ is determined by $\varphi|_K$)

NB χ not unitary → rep^n won't be unitary, probably.

$$\int K_{I_\chi(f)}(k_1,k_2)\, \varphi(k_2)\, dk_2 = ((I_\chi(f))\varphi)(k_1)$$

$(I_\chi(f)\varphi)(k_1) = \int_G \varphi(y)\, f(k^{-1}y)\, d\bar{y}$, $y =$ ~~nmk~~ mnk_2
(careful about Haar measure)

$$K_{I_\chi(f)}(k_1,k_2) = \iint f(k_1^{-1}mnk_2)\, \chi(m)\, \delta(m)^{1/2}\, dndm$$

$$(\widehat{\mathcal{S}f})(\chi) = \int_M \int_K \int_N f(k^{-1}mnk)\, \chi(m)\, \delta(m)^{1/2}\, dndkdm.$$

Hence trace of $I_\chi(f)$ = Mellin transform of Satake transform.

Note $\mathrm{tr}\, I_\chi(f_1) I_\chi(f_2) = \mathrm{tr}\, I_\chi(f_2) I_\chi(f_1)$

However $\mathrm{tr}\, I_\chi(f) \neq \mathrm{tr}\, R_\chi(f)$ in general - semisimplification of I_χ is $R_\chi \oplus$ (other stuff).
$\ddot{\frown}$

He's still not quite at the end of his introduction.

Say E/F is a cyclic unramified ext^n.

$\langle\sigma\rangle = \mathrm{Gal}(E/F)$; $G_E = GL_2(E)$.

We can form $G_E \rtimes \langle\sigma\rangle = G'_E$ & $\underset{(E)??}{K \rtimes} \langle\sigma\rangle = K'_E$

Off we go again. $\varphi(nmx) = \varphi(x)\,\chi(m)\,\delta(m)^{1/2}$, $m \in M'_E$

$$\varphi_0(k) \equiv 1 \text{ for } k \in K$$

$$\varphi'_0(k\sigma^r) = 1 \text{ for } k, r \in \mathbb{Z}/\ell\mathbb{Z}$$

Now let's look at $\mathcal{H}(G'_E, K_E)$ (no ' on K_E)

χ extended from M_E to M'_E, by making it trivial on σ, is gonna be called χ'.

$I_{\chi'}(h')$; $h' = h \rtimes \sigma$ = sum of double coset classes translated by σ

$$(\mathcal{S}h')(\chi) = \iint_{M'\backslash G_E} h(g^{-1}m'g)\,\chi(m')\,d\dot{g}$$

You have to be a bit careful with the centraliser

We have $b_{E/F} : \mathcal{H}(G,K) \to \mathcal{H}(G_E, K_E)$, & if $h \in \mathcal{H}(G_F, K_E)$,

$\mathrm{tr}\,\pi_\chi(b_{E/F}h) = \mathrm{tr}\,\pi_{\tilde{\chi}}(h)$ where $\tilde{\chi} = \chi \circ N_{E/F}$

This tells you a lot. But not everything. (w.r.t. the Fundamental Lemma)

$\mathrm{tr}\,R_\chi(bh) = \mathrm{tr}\,R_{\tilde{\chi}}(h)$ implies immediately that

$$TO^\sigma_\delta(h) = O_\gamma(bh) \text{ if } \delta \in M_E, \gamma \in M_F \text{ \& } \gamma = N\delta$$

$O_\gamma(bh) = \int_{M\backslash G} f(x^{-1}\gamma x)\,dx$; $\therefore$ $\underline{\underline{O_\gamma(bh) = \Delta(\gamma)^{1/2}\,\mathcal{S}(bh)(\gamma)}}$

Similarly $TO_\delta(h) = \underline{\underline{\Delta^\sigma(\delta)^{1/2}\,\mathcal{S}h(\delta)}}$

$\Delta(\gamma) = \Delta^\sigma(\delta)$, $\gamma = N\delta$

$\det(1-\delta\sigma \mid \mathfrak{g}_E/\mathfrak{m}_E) = \det(1-N\delta)$ $\downarrow$ $(N\delta = \delta_l \delta_{l-1} \cdot \delta_1)$

$\mathfrak{g}_E/\mathfrak{m}_E \otimes_F \bar{F}$

$$\begin{pmatrix} 1 & & & -\delta_1 \\ -\delta_l & 1 & 0 & \\ & & \ddots & \\ 0 & & -\delta_{l-1} & 1 \end{pmatrix}$$

$\mathfrak{g}/\mathfrak{m} \otimes_F \bar{F}$

If we had to work only with split elements then, we would be done.

However we have to deal with elliptic elts.

There are 2 strategies: the first due to Langlands is to compute explicitly on the somethings. (all the cases?)

- do it for all cases. $O_\gamma(h)$ $TO_\delta(h)$

Langlands ~~dult~~ ~~dit~~ did it for Gl_2 Kottwitz did it (a tour de force) for Gl_3 & in doing so discovered the trick that Peter Schneider told us about this morning.

The 2nd trick is due to Clozel et al & is not to compute but to find enough elts which you know its true for & then deduce the general case.

Tomorrow he'll talk about this.

Technique: work with a subalgebra of the Iwahori-Hecke algebra

BwB, $B =$ Iwahori $= \begin{pmatrix} \mathcal{O}^* & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^* \end{pmatrix}$ & $w \in \left\{ \begin{pmatrix} \varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix}, \begin{pmatrix} 0 & \varpi^{n_1} \\ \varpi^{n_2} & 0 \end{pmatrix} \right.$

We will assume $n_1 > n_2$ & prove a fundamental lemma for this (we will ~~I~~ only have split objects then) & then deduce the general fundamental lemma.

ecture 2
n 26th Feb '93
:30pm

Recall $B(t) = \dfrac{B\begin{pmatrix}\varpi^{n_1} & 0\\ 0 & \varpi^{n_2}\end{pmatrix}B}{\mathrm{Vol}(B)}$, $n_1 > n_2$ integers

$t = \begin{pmatrix}\varpi^{n_1} & 0\\ 0 & \varpi^{n_2}\end{pmatrix}$, $t' = \begin{pmatrix}\varpi^{n_1'} & 0\\ 0 & \varpi^{n_2'}\end{pmatrix}$, $n_1' > n_2'$

Note $B(t)B(t') = B(tt')$ & the $B(t)$ generate a commutative subalgebra of the double coset algebra. (Here B is the Iwahori $= \begin{pmatrix}\mathcal{O}^\times & \mathcal{O}\\ \mathfrak{p} & \mathcal{O}^\times\end{pmatrix}$)

We have E/F a cyclic unramified extn, $l = \deg E/F$

Set $f_t^E = B_E(t)$, $f_{t^l}^F = B_F(t^l)$

Prop^n 1: f_t^E & $f_{t^l}^F$ are associated. He will prove this.

i.e. they "satisfy the fundamental lemma"

Prop^n 2 If π is an irred admiss rep^n of G_F, then

$$\mathrm{tr}\,\pi(f_t) = \begin{cases} 0 \\ \text{unless} \\ \pi \text{ is a subquotient of} \\ \text{an unramified princ. series}\end{cases}$$

2 is easier than 1 so maybe he'll have a crack at 2 first.

Set I_χ = princ. series (χ unramified)

$$\mathrm{tr}\, I_\chi(f_t) = \Delta(t)^{1/2}(\chi(t) + \chi(\tilde{t})) \text{ where } \tilde{t} = \begin{pmatrix}\varpi^{m} & 0\\ 0 & \varpi^{n_1}\end{pmatrix}$$

Of course $\mathcal{H}(G,K) \subseteq \underset{\substack{\uparrow\\ \text{not nec commutative}}}{\mathcal{H}(G,B)}$

Prop^n 2' (twisted version of 2) Π rep ... of G_E

$$\mathrm{tr}(\Pi(f_t^E)\Pi(\sigma)) = \begin{cases} 0 \\ \text{unless} \\ \Pi \text{ is a subquotient} \\ \text{of an unramified p.s.}\end{cases}$$

$$\mathrm{tr}(I_{\tilde\chi}(f_t^E) I_{\tilde\chi}(\sigma)) = \underset{\Delta_F(t^l)^{1/2}}{\underset{\shortparallel}{\Delta_E^\sigma(t)^{1/2}}}(\tilde\chi(t) + \tilde\chi(\tilde t))$$

Notes: 1) $\tilde\chi$ is nowt to do with $\tilde t$
2) We're assuming σ acts trivially on spherical vector.

There are various other sublemmas we'll need. He's gonna prove these prop's.

There's some kind of key that relates everything. This might be

Technical lemma

$$(b,m) \mapsto b^{-1}tmb$$

$$b\in B,\ m\in M\cap K = M\cap B = \left\{\begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{pmatrix}\right\} \ ; \ {}_M\backslash B\times M \xrightarrow{\sim} BtB.$$

This tells us how you can analyse things. If we have this then tackling orbital integrals can be done by small kids.

Note that $\mathbb{B}(t) = \dfrac{1}{\mathrm{Vol}(B)} \operatorname{char}\left\{ b^{-1}tmb \,\middle|\, b\in {}_M\backslash B,\ m\in M\cap B \right\}$

$$O_\gamma(f_t) = \int_{G_\gamma\backslash G} f_t(x^{-1}\gamma x)\,dx$$

$$x^{-1}\gamma x = b^{-1}tmb \quad \therefore \gamma \text{ \& } tm \text{ are conjugate.}$$

So ~~to~~ by computing this orbital integral, we see it's

$$O_\gamma(f_t) = \begin{cases} 0 & \text{unless } \gamma \sim \cancel{tm}\ tm \text{ for some } m\in B\cap M \\ \text{see below} & \text{if } \gamma\sim tm. \end{cases}$$

Assume $\gamma = tm_1$.

$$\int_{M\backslash G} f_t(x^{-1}tm_1x)\,dx = 1/_{\mathrm{vol}(B\cap M)} \quad ; \quad x^{-1}tm_1x = b^{-1}tmb$$

$$\text{WLOG } x^{-1}tm_1x = tm$$

$$\therefore x^{-1}(tm_1)^n x = (tm)^n$$

$$x^{-1}t^nm_1^nx = t^nm^n$$

$t^{-n}x^{-1}t^n = m^nx^{-1}m_1^{-n}$ & this is true $\forall n\in\mathbb{Z}$ & the eigenvalues of t have distinct valuations

$\therefore x\in M$ (otherwise we get unbounded cpct things)

So our integral $= \displaystyle\int_{B\cap M\backslash B} f_t(b^{-1}tm_1b)\,db$ & we've proved his statement about orbital integrals.

Note that they are "just stupid".

It's an exercise to do the twisted ~~prop~~ version now.

We can now get prop 1(modulo the technical lemma again)

- note that we do know the fundamental lemma for M.

So just use the ~~fact that~~ idea $M_E \cap K_E \to M \cap K$

$$m \mapsto mm^{-\sigma}$$

He's now proved prop 1, ~~2~~. $\square$

It's an exercise to prove this for GL_d. $t = \begin{pmatrix} \varpi^{n_1} & & 0 \\ & \ddots & \\ 0 & & \varpi^{n_d} \end{pmatrix}$, $n_1 \geq n_2 \geq .. \geq n_d$

Use Kottwitz' pf that 1 is associated to 1. (Peter Schneider's lecture)

This is also true for B.

To do prop 2 we recall $f_t = \frac{BtB}{\mathrm{vol}\, B}$

$$\mathrm{tr}\, \pi(f_t) = (\text{scalar})\, \mathrm{tr}\, \pi(e_B)\, \pi(t) \pi(e_B) \neq 0$$

$$\Rightarrow \pi^B \neq 0$$

This characterises the subquotients of the unramified principal series.

Recall Tony did this ($V^H \neq 0 \Rightarrow \ldots$)

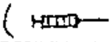
It's in fact an example of a much more general thm, which he may well be about to tell us!

If (π, V) admissible rep of (a quasi split gp) eg GL_n

need this to define B

then $V^B \hookrightarrow V_N = V/V(N)$ (NB $V(N) \hookrightarrow \mathcal{S}(F^\times)$)

$\uparrow$ $V(N) = \{ (\pi(n)-1)v \mid n \in N, v \in V \}$ is the Jacquet module, according to Frazer

If V irred & $V^B \neq 0$ this is iso.
He just needs an injection (!!!!!) ()

He quite clearly knew the proof of this & verbally sketched it. He did it "to lose time".

$$\mathrm{tr}\, I_\chi(f_t) = \int_{M\backslash G}\int_{M} f_t(x^{-1}mx)\,\Delta(m)\,\chi(m)\,dt\,dm\,dg$$

$$f_t(x^{-1}mx) \neq 0 \Rightarrow x \in WB$$

W will just have to remain a mystery.

$$\mathrm{tr}\,\mathbb{1}(f_t) = \Delta(t) = \mathrm{vol}(B(t))$$

$$BtB = \coprod_{b\in B/B\cap tBt^{-1}} btB \quad , \quad \Delta(t) = \# B/_{B\cap tBt^{-1}}$$

$\underline{\mathbb{1}} \oplus \mathrm{St}$ = semisimplification of $\rho(|\cdot|^{1/2}, |\cdot|^{-1/2})$ ← indecomposable but reducible

St ← Steinberg $= \sigma(|\cdot|^{1/2}, |\cdot|^{-1/2})$

$\underline{\mathbb{1}}$ ← $= \pi(|\cdot|^{1/2}, |\cdot|^{-1/2})$

I_χ, $\chi = \delta^{1/2}$

$$\underset{\substack{\uparrow\\ \text{char of } \mathbb{1}\\ = \Delta(t)}}{\Delta(t)^{1/2}\delta(t)^{-1/2}} + \underset{\substack{\uparrow\\ \text{char of St}\\ = 1}}{\Delta(t)^{1/2}\delta(t)^{1/2}} \; ; \text{ note } \Delta(t) = \delta(t)^{-1} = q^{n_1-n_2}$$

$$\mathrm{tr}\,\mathbb{1}(f_t) = \Delta(t) = \mathrm{vol}(B(t))$$

$$\mathrm{tr}\,\mathrm{St}(f_t) = 1$$

Replacing σ-conjugacy by with conjugacy etc gives you lots more formulae.
Use $n_1 > n_2$.

This proves prop^n 2. He's now going to talk about the tricky proposition.

$$b \in M\backslash \underbrace{B\cap tBt}_{H_t}, \quad b = \begin{pmatrix} 1 & \beta\varpi^r \\ \varpi\gamma & 1\end{pmatrix} \qquad r = n_1 - n_2 \qquad t = \begin{pmatrix} \varpi^n & 0 \\ 0 & 1\end{pmatrix}$$

$$m = \begin{pmatrix} \alpha & 0 \\ 0 & \delta\end{pmatrix} \qquad b^{-1}tmb \qquad (??)$$

$$= \frac{1}{1-\beta\gamma\varpi^{r+1}}\begin{pmatrix} \omega^r & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \alpha - \varpi\beta\delta\gamma & -\beta\delta + \varpi^r\alpha\beta \\ \delta\gamma + \varpi^r\gamma\alpha & \delta + \delta\alpha\beta\,\varpi^{2r+1}\end{pmatrix}$$

$${}_{K\cap M}\backslash(H_t \times M\cap K) \xrightarrow{\sim} b^{-1}tBb$$

$$H_t\backslash B \ni b$$

$$M\cap K \backslash B\times M\cap K \longrightarrow BtB$$

$$(b, m) \longmapsto b^{-1}tmb$$

Want volumes preserved by this p-adic analytic morphism.

Compute volumes Jacobian

$$\overset{\text{Jacobian}}{\|}\ \Delta(t\underset{\underset{\text{unit}}{\uparrow}}{m}) = \Delta(t)$$

This proves the technical lemma.

With this computation you're now home. $\mathfrak{g}/\mathfrak{m}$.

A no-computation proof.

He is going to spend the last 2 minutes of this lecture proving the lemma that Tony needed this morning.

$$\sum c(\pi_v)\,\mathrm{tr}\,\pi_v(f) = \sum d(\pi_v)\,\mathrm{tr}\,\Pi_v(\varphi\times\sigma)$$

He's changed f to $f_{t\ell}^{F}$ & φ to f_t^E

Everything is a twist of the trivial or the twist of the Steinberg.

Everything must compensate exactly.

$$\sum_{\pi_v \text{ subquot } I_\chi} c(\pi_v) = \sum_{\substack{\pi_v \text{ subquot}\\ \text{of } I_{\tilde\chi}}} d(\pi_v)$$

Spherical... substitute the f^n in Hecke algebra - def^n of base change $\mathrm{tr}\,\Pi_\chi(bh) = \mathrm{tr}\,\Pi_\chi(h)$... trace in full princ series = that of subquotient... establishes lemma.

This finishes his talks on the Fundamental lemma.

IX Artin's Conjecture

Mike Harrison

ecture 1
Fri 26th Feb '93
4:00pm

Thanks to Karsten for taking notes for this one, which I couldn't attend, unfortunately.

Ad
Labesse: "Louder!"
Mike: "Why don't you move a bit closer?"

Artin L-functions

E/F a finite Galois ext^n of number field

$$\sigma: \mathrm{Gal}(E/F) \to GL(V), \quad V/\mathbb{C} \text{ a f.d. v.s.}$$

For each finite place v of F choose $w|v$ a place of E

We have $I(w|v) \subseteq D(w|v) \subseteq \mathrm{Gal}(E/F)$.

$$\text{Let } P_v(\sigma, X) = \det\Big((I - X\sigma(Fr_w))\Big|_{V^{I(w|v)}}\Big)$$

This is indpt of w. if $w_1|v$, then $\exists\, \tau \in \mathrm{Gal}(E/F)$ s.t. $\tau w = w_1$

$$\text{Then } D(w_1|v) = \tau D(w|v)\tau^{-1}$$
$$I(w_1|v) = \tau I(w|v)\tau^{-1}$$

$$\& \ \{Fr_{w_1}\} = \{\tau Fr_{w_2} \tau^{-1}\}$$

We have $\sigma(\tau): V \xrightarrow{\sim} V$ &

$$\begin{tikzcd} V^{I(w|v)} \arrow[r] \arrow[loop below, "Fr_w"] & V^{I(w_1|v)} \arrow[loop below, "Fr_{w_1}"] \end{tikzcd}$$

Def: $L(\sigma, s) = \prod_{v \text{ finite}} P_v(\sigma, (Nv)^{-s})^{-1}$, the Artin L-function

$P_v(\sigma, X) = (1 - \zeta_1 X) \dots (1 - \zeta_r X)$, the Euler product will converge if $\mathrm{Re}(s) > 1$ or sthg; its unif abs cgt on cpct subsets in this area.

roots of 1

Eg If σ is the trivial rep^n,

$$L(\sigma, s) = \zeta_F(s) = \prod_v (1 - (Nv)^{s})^{-1} = \sum_{\substack{\mathfrak{a} \\ \text{(non-zero) ideals} \\ \text{of } O_F}} 1/(N\mathfrak{a})^s$$

<u>Prop^n 1.1</u> i) If $E' \supseteq E \supseteq F$, & σ is a rep^n of $Gal(E/F)$, & $\tilde{\sigma}$ a rep^n of $Gal(E'/F)$ given by

$$\tilde{\sigma} : Gal(E'/F) \twoheadrightarrow Gal(E/F) \xrightarrow{\sigma} GL(V)$$

Then $L(\tilde{\sigma},s) = L(\sigma,s)$

ii) If σ_1, σ_2 are rep^ns of $Gal(E/F)$, then

$$L(\sigma_1 \otimes \sigma_2, s) = L(\sigma_1,s)L(\sigma_2,s)$$

iii) If $E' \supseteq E \supseteq F$ & σ is a rep^n of $Gal(E'/E)$, (don't need E/F Galois)

then $L(\sigma,s) = L(\mathrm{Ind}_{G(E'/E)}^{G(E'/F)} \sigma, s)$

<u>Pf</u> omitted. i) & ii) are easy - just look at Euler factors.
iii) is a little harder - need places of $E \leftrightarrow$ places of E'.
& It was inspired by abelian L-series:

$$\zeta_{\mathbb{Q}(\mu_n)}(s) = \zeta(s) \prod_{\substack{\chi \text{ mod } n \\ \text{dirichlet} \\ \text{char}}} L(\chi,s) \quad \text{or sthg.}$$

$\square$

Now write $G = Gal(E/F)$. The regular rep^n of G decomposes as

$$\rho_{reg} = \oplus n_i \sigma_i$$

as σ_i runs thru' all irred reps of G, & $n_i = \dim \sigma_i$

Also, $\rho_{reg} = \mathrm{Ind}_{\{1\}}^{G}(\text{trivial})$

$$\Rightarrow L(\rho_{reg}, s) = \zeta_E(s)$$

& hence $$\zeta_E(s) \Big/ \zeta_F(s) = \prod_{\substack{\sigma_i \\ \text{non-trivial}}} L(\sigma_i, s)^{n_i}$$

<u>Question</u> Does ζ_E / ζ_F have an entire ext^n to $\mathbb{C}$?

This would follow from

<u>Conjecture</u> (Artin) If σ is a non-trivial irred Galois rep^n, then $L(\sigma,s)$ has an entire ext^n to $\mathbb{C}$.

Firstly note that there is a meromorphic continuation.

Let $\rho = \chi$ be a 1-dim^l char. of $Gal(E/F)$. Then χ factors thru' $Gal(E_\chi/F)$ where E_χ/F is an abelian ext^n.

Then by global CFT we have $G(E_\chi/F) \cong C_F / N_{E_\chi/F} C_{E_\chi}$

So χ on $G(E_\chi/F) \rightsquigarrow \tilde{\chi}$ on C_F, a finite GC.

We also have an L-f^n for $\tilde{\chi}$. (see below)

<u>Claim</u> $$L(\chi, s) = L(\tilde{\chi}, s) = \prod_{\substack{v \text{ finite} \\ \tilde{\chi} \text{ unramified } @ v}} \left(1 - \tilde{\chi}(\pi)(N_v)^{-s}\right)^{-1}$$

We will justify this claim. Say the conductor of E_χ/F is $\mathfrak{f}_\chi$, the minimal $\mathfrak{f}$ s.t. $E_\chi \subseteq F(\mathfrak{f}_\chi)$ ([illegible]. < this may be some ray class the field or sthg)

The global conductor is the product of local conductors.

Note $F^\times\left(\prod_v U_v^{(n_v)}\right) / F^\times \subseteq N_{E_\chi/F} C_{E_\chi}$

↳ princ. local units or sthg

Here n_v is the power of v in $\mathfrak{f}_\chi$

$\tilde{\chi}$ is faithful, & so [illegible] cond χ is $\mathfrak{f}_\chi$

Then the ramified primes of E_χ/F are the ramified primes of $\tilde{\chi}$.

For $v \nmid \mathfrak{f}_\chi$ we have $\tilde{\chi}(\pi_v) = \chi(F_v)$, ∴ same Euler factor at v

For $v \mid \mathfrak{f}_\chi$ we're ramified @ v. Then $\chi(I_v)$ is non-trivial so $V^{I_v} = 0$ ⇒ both Euler factors at v are 1.

So indeed the two L-series are the same.

Hecke observed that if φ is any GC on C_F, then $L(\varphi,s)$ has a meromorphic continuation to $\mathbb{C}$, which is in fact entire iff $\varphi \neq \|\cdot\|^t$

(Here, of course, $\|(\alpha_v)\| = \prod_v |\alpha_v|_v$ (over all v to make sure $F^\times \subseteq \ker \|\ \|$)

If φ is ~~of (the) exceptional type~~ then L has a simple pole at $t+1$.

So if σ is a 1-dim^l, non-trivial Galois character, or if σ is induced from a non-trivial char, then $L(\sigma,s)$ has an entire continuation.

Thm 1.2 (Brauer) If G is a finite group, & σ is a f.d. virtual rep^n, then $\exists$ subgps H_i of G & (1-dim^l) chars χ_i on H_i, & also $n_i \in \mathbb{Z}$, s.t.

$$\sigma \cong \sum n_i \operatorname{Ind}_{H_i}^{G} \chi_i \qquad \square \qquad \text{(finite sum, of course)}$$

So then $L(\sigma,s) = \prod_i L(\chi_i,s)^{n_i}$

& thus each $L(\sigma,s)$ indeed has a mero. continuation to $\mathbb{C}$, & in fact also satisfies a functional eqn.

Now say φ is a GC on C_F

For $v|\infty$ define $G_v(s) = \begin{cases} 2(2\pi)^{-s}\,\Gamma(s) & v \text{ complex} \\ \pi^{-s/2}\,\Gamma(s/2) & v \text{ real} \end{cases}$

Then we can define the completed L-series

$$\Lambda(\varphi,s) = \Big(\prod_{v|\infty} G_v(s+\underset{\uparrow}{r_v})\Big)\, L(\varphi,s)$$

cst depending on φ_v, $\varphi = \prod \varphi_v$.

Then we have a functional eqn

$$\Lambda(\varphi,s) = \left(W(\varphi)\, d_\varphi^{\frac{1}{2}-s}\right) \Lambda(\varphi^{-1},1-s)$$

where $|W(\varphi)| = 1$ & $d_\varphi = |d_F|\, N^F_{\mathbb{Q}} f_\varphi$ (d is a discriminant)

Mike will now spend a while making a fool of himself trying to reach the top board. Use the stick, Mike.

I think now σ is any rep^n (not nec 1-dim^l) of E/F.

For each place v define $n_v(\sigma)$ thus:

$$v|\infty \qquad n_v(\sigma) = n = \dim(\sigma)$$

$$v \text{ finite} \quad n_v(\sigma) = \sum \frac{|G_v^i|}{|G_v^{i+1}|} \left(\dim V - \dim V^{G_v^i}\right),$$ that well-known integer.

Here V is the space that σ acts on, & $G_v^{(i)}$ is the i^{th} ramification group $\subseteq G_v \subseteq G = Gal(E/F)$. It's only a finite sum, as eventually $G_v^i = \{1\}$ $\therefore \dim V = \dim V^{G_i}$.

If v is a real place, $n_v(\sigma) = n_v^+(\sigma) + n_v^-(\sigma)$, the dimension of the ± 1-eigenspaces for the action of the generator $w \in G_v \subseteq G$

$$\begin{tikzcd} w \arrow[d, dash] & E \arrow[r, dash] & E_w = \mathbb{R} \text{ or } \mathbb{C} \arrow[d, dash] \\ v & F \arrow[r, dash] & F_v = \mathbb{R} \end{tikzcd}$$

Then $$\Lambda(\sigma,s) = \prod_{v \text{ complex}} G_v(s)^{n_v(\sigma)} \prod_{v \text{ real}} \left(G_v(s)^{n_v^+(\sigma)} G_v(s+1)^{n_v^-(\sigma)}\right) L(\sigma,s)$$

(this is probably a def^n)

We have a functional eqn

$$\Lambda(\sigma,s) = \underbrace{\left[W(\sigma) |d_F|^{\dim\sigma} N_{\mathbb{Q}}^F f_\sigma\right]^{\frac{1}{2}-s}}_{\varepsilon(\sigma,s)} \Lambda(\tilde{\sigma}, 1-s) \quad \text{contragredient}$$

where $f_\sigma = \prod_{v \text{ finite}} v^{n_v(\sigma)}$

Now let's write everything in our new$\overset{\text{more}}{\wedge}$modern technology

Say σ_v = restriction of σ to $\underset{D(w/v)}{\overset{}{G_v}} \subseteq G$ for finite places (& infinite ones if you like)

We get a semisimple rep of W_{F_v} of Galois type

$\rightsquigarrow$ WD rep^n with $N=0$.

To f.dim^l ss rep's τ_v of WD_{F_v} we can associate $L(\tau_v, s)$ & $\varepsilon(\tau_v, s)$

It's now 5pm. For σ_v, the local Euler factors of $\Lambda(\sigma,s)$ are $L_v(\sigma_v, s)$
$\varepsilon(\sigma,s)$ are $\varepsilon_v(\sigma_v, s)$ (?)

Now say $n = 2$ or 3 & $F/\mathbb{Q}_p$. Recall Local Langlands (a thm as $n=2,3$)

$$\left\{\begin{array}{c}\text{conjugacy classes}\\ \text{of semisimple } n\text{-dim'l}\\ \text{rep}^{\text{n}}\text{s of } WD_F\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{irred admiss}\\ \text{rep's of } GL_n(F)\end{array}\right\}$$

$$\rho \longmapsto \pi(\rho)$$

- unramified rep's of W_F $\rightsquigarrow$ unramified rep's
Unramified ext's ??

$\rho(Fr_v) \sim \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \longmapsto$ ext^n corr. to α, β

Say $\chi : F^\times \to \mathbb{C}^\times$
$\rho \otimes \chi \longrightarrow \pi(\rho) \otimes (\chi \circ \det)$

$\omega_{\pi(\rho)} = \det \rho$ $\qquad$ $\pi(\rho)\begin{pmatrix} a & & 0\\ & \ddots & \\ 0 & & a\end{pmatrix} = \omega_{\pi(\rho)}(a)\, I$

& $\pi(\tilde{\rho}) = \widetilde{\pi(\rho)}$.

We can then define L- and ε-factors for $\pi(\overset{\rho}{p})$ coming from ρ

We can do similar things for $\mathbb{R}$ & $\mathbb{C}$, replacing WD_F by W_F and $GL_n(F)$ by $(\mathfrak{g}, K)$-module.

If $\dim \sigma = 2$ or 3, $\sigma_v \rightsquigarrow \pi(\sigma_v)$, irred, admiss, unram for almost all v

Define $\pi(\sigma) = \bigotimes_v{}' \pi(\sigma_v)$

We've now got an irred admiss (global) $(\mathfrak{g}, K) \times GL_n(\mathbb{A}^\infty_F)$-module.

Also define $L(\pi(\sigma), s) = \prod L_v(\pi_v(\sigma_v), s) = \prod L_v(\sigma_v, s) = \Lambda(\sigma, s)$

Hence $L(\pi(\sigma), s)$ is entire $\Leftrightarrow \Lambda(\sigma, s)$ is.

Our aim now is to show that $\pi(\sigma)$ is actually an automorphic form.

~~, δ $\chi = \delta^{sh} \chi_0$~~

~~We get $L(s,\chi_0)$ / $L(s+1,\chi_0)$. We've ignored k_1. So if, say, $\chi \neq 1$ & χ K-invt we get equality.~~

~~So the cgce of $M\chi$ is controlled by cgce of L-f's~~

~~Intertwining op for par subgp rep^n L-gp general concept of L-f^n on L-groups. Langlands realised this & defined L-f's on L-gps in the lecture~~

~~Euler products (Yellow lecture notes 1966)~~

~~He guesses he should stop here.~~

(this is the end of another lecture!)

Lecture 2
Sat. 27th Feb '93
11:00am

Yesterday he talked a lot about 1-dim^l rep's & L-f^{n}s. Today he'll do

§2 2-dim^l irred reps

Thm 2.1 If $\sigma \cdot G_F \to GL_2(\mathbb{C})$ (finite image $\therefore$ factors thru Gal(E/F) E/F finite) be an irred cts rep^n.

$$\bar{\sigma} : G_F \xrightarrow{\sigma} GL_2(\mathbb{C}) \xrightarrow{proj} PGL_2(\mathbb{C})$$

Then $\text{Im}(\bar{\sigma})$ is either i) D_n of order $2n$, $n \geq 2$
or ii) A_4, S_4, A_5. $\square$ no time for pf.

Neat little pf though

Cor of pf In the dihedral case, $\sigma(G)$ has an abelian subgp of index 2 ($\exists$ direct pf of this)(probably)
$\therefore$ $\exists$ quadratic ext^n K of F s.t. $\sigma(G_K)$ is abelian

Then $\exists$ non-triv char χ of G_K s.t. $\sigma = \text{Ind}_{G_K}^{G_F} \chi$.

Hence $L(\sigma,s) = L(\chi,s)$ is entire :) $\square$

We want to attack A_4 & S_4. ~~As~~ The methods fail for A_5

$$D_n \cong \text{proj}\left\{ \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

A_4, S_4, A_5 come 'geometrically' from embedding the rotation gp of the tetrahedron, octahedron & icosahedron into $PGL_2(\mathbb{C})$ but he doesn't really understand how!!

The adjoint square

$$\begin{tikzcd} GL_2(\mathbb{C}) \arrow[rr, "A^2\ ?"] \arrow[dr, "proj"'] & & GL_3(\mathbb{C}) \\ & PGL_2(\mathbb{C}) \arrow[ur, "ad\ action"'] & \end{tikzcd}$$

- where ad is the action of $PGL_2(\mathbb{C})$ on its tgt space at 0

$$\mathbb{C}^3 \cong \mathfrak{sl}_2(\mathbb{C}) = \{ M \in M_2(\mathbb{C}) \mid tr\, M = 0 \} \text{ by conjugation}$$

$$x(M) = xMx^{-1}$$

Defn $A^2(\sigma) = $ 3-dim^l rep^n, $\sigma : G_F \to GL_2(\mathbb{C}) \xrightarrow{A^2} GL_3(\mathbb{C})$

$\sigma : G_F \to GL_2(\mathbb{C})$ of tetrahedral type

Let E be the cubic ext^n of F s.t. $\bar{\sigma}(G_E) = V_4 \trianglelefteq A_4$

Lemma 2.2 σ as above. $A^2(\sigma)$ is an irred 3-d repr $\cong Ind_{G_E}^{G_F}(\chi)$, for any non-trivial char χ on G_E

Pf **Claim** $A_4 \cong \bar{\sigma}(G_F)$ is conjugate by $GL_2(\mathbb{C})$ to

$$proj\left\{ < \underset{(12)(34)}{\underset{\updownarrow}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}}, \underset{(14)(23)}{\underset{\updownarrow}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}}, \underset{(123)}{\underset{\updownarrow}{\begin{pmatrix} \frac{1-i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{pmatrix}}} > \right\}$$

Any finite $G \subseteq PGL_2(\mathbb{C})$ has a lift G' in $SL_2(\mathbb{C})$ s.t. $G' \twoheadrightarrow G$

In fact if $G = \widetilde{Im_\rho}$ then $G' = \{\pm \text{ pullbacks to } SL_2(\mathbb{C})\}$

In fact G' is the ! lift to $SL_2(\mathbb{C})$ as same ~~thing~~ next argument shows: any lift $\subseteq G'$ & is even order $\therefore$ contains an elt of order 2 & hence contains $-I$, the only elt of order 2 in $SL_2(\mathbb{C})$.

Elts of order 2 in $\bar{\sigma}(G)$ lift to elts of order 4 (as they can't lift to $-I$)

thing $\leftrightarrow$ (12)(34) is X say. Choose a basis s.t. $X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$Centraliser_{SL_2(\mathbb{C})}(X)$ = diag matrices. If $Y \leftrightarrow (14)(23)$ then

$XY = \pm YX$. If $YX = XY$ then Y is diagonal & $Y = \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ # $\notin SL_2(\mathbb{C})$

$\therefore \begin{pmatrix}i&0\\0&-i\end{pmatrix} Y = -Y \begin{pmatrix}i&0\\0&-i\end{pmatrix}$ & after conjugating by diagonal elts get

$Y = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$

Similarly for Z : $ZXZ^{-1} = \pm Y$ ...

So now we can work out A^2 etc. (see below)

From this explicit repⁿ we can work out $A^2(\sigma)$ easily & indeed it's induced up from these χ s. ☐

3. The Big Thm section

Mike has no idea how to prove any of the thms in this section but they'll be used in §4 to prove cases of Artin..

n = 2 or 3. π is an irred admiss $(\mathfrak{g}_\infty, K_\infty) \times GL_2(\mathbb{A}^F)$-module

$\pi = \bigotimes_v' \pi_v$. By Local Langlands thm we get

$\pi_v = \pi_v(\sigma_v)$ for σ_v some n-dim' repⁿ of WD_{F_v}.

Then $L_v(\pi_v, s) = L(\sigma_v, s)$, $\varepsilon_v(\pi_v, s) = \varepsilon_v(\sigma_v, s)$

⌈ If v is a finite place, $\sigma_v \to (\sigma_v', N)$; $L_v(\sigma_v, s) = \det\left(1 - Fr_v (N_v)^{-s} \Big|_{V_N^{I_v}}\right)$

↑ repⁿ space for σ_v - killed by N

$L(\pi, s) = \prod_v L_v(\sigma_v, s)$, $\varepsilon(\pi, s) = \prod_v \varepsilon_v(\pi_v, s)$.

Not obviously cgt.

Thm 3.1 (J-L, n=2; J-Godement, n=3 - if n>3 it's true but all different as ∄ J-L corresp but you still get an L-fⁿ if you try)

If π is automorphic, then

(i) $L(\pi, s)$ has a meromorphic ~~continuation~~ extension to $\mathbb{C}$ with finitely many poles

(ii) $L(\pi, s) = \varepsilon(\pi, s) L(1-s, \tilde{\pi})$ if $\tilde{\pi}$ is the contragredient

(iii) If π is cuspidal then $L(\pi, s)$ is actually entire. ☐

There's also a wacky converse, which to Mike seems much stronger:

If $\forall$ GC $\omega : G_F \to \mathbb{C}^\times$, $L(s, \omega \otimes \pi)$ & $L(s, \omega^{-1} \otimes \tilde{\pi})$ are entire,
bounded in vertical strips,
& satisfy

$$L(s, \omega\otimes\pi) = \varepsilon(s, \omega\otimes\pi)\, L(1-s, \omega^{-1}\otimes\tilde{\pi})$$

(& possibly 1 or 2 other technical other conditions)

then π is ~~aut~~ cuspidal (& automorphic, I guess)

Let $\sigma : G_F \to GL_{\underset{\llcorner 2 \text{ or } 3}{n}}(\mathbb{C})$ be n-dim^l irred ; then $\pi(\sigma) = \bigotimes_v{}' \pi(\sigma_v)$

$$\pi(\sigma)\otimes\omega = \bigotimes_v{}' \pi(\sigma_v\otimes\omega_v) \quad \& \quad \widetilde{\pi(\sigma)}\otimes\omega = \bigotimes_v{}' \pi(\tilde{\sigma}_v\otimes\omega_v^{-1})$$

If $\sigma = \mathrm{Ind}_{G_E}^{G_F} \chi$, $[E:F] = n$, then σ_v are the corresponding local induced things

$\omega_{E/F} = \omega \circ N_{E/F}$ GC on E

$L(\pi(\sigma)\otimes\omega, s) = L(\chi\,\omega_{E/F}, s)$; $L(\widetilde{\pi(\sigma)}\otimes\omega^{-1}, s) = L(\chi^{-1}\omega_{E/F}^{-1}, s)$

These are entire, bounded on vertical strips & satisfy functional eqn

(provided $\chi . \omega_{E/F} \neq \|\cdot\|^t$ for some ω & some t $(\|\cdot\|_F = \|\ \|_E \circ N_{E/F})$.

$\iff$ χ is not of the form $\mu \circ N_{E/F}$ some μ, GC on F
$\iff$ χ is not the restriction of a char $\tilde{\chi}$ on G_F

If that were true $\nearrow$, then $\underset{\substack{\uparrow \\ \text{1-dim}^l}}{\tilde{\chi}}$ would be a constituent of $\mathrm{Ind}_{G_E}^{G_F} \chi = \underset{\nwarrow \text{irred}}{\sigma}$ #, so indeed these are entire

Jacquet, et al

<u>Thm 3.2</u> (Converse thm) If σ is an irred n-dim^l Galois rep^n induced from a char. on a subgp (monomial) then $\pi(\sigma)$ is cuspidal. ☐

Base change for GL_2 coming up (another big thm)

(there's still 4 thms he wants to state. It's 5 to 12.

<u>Base change</u> If E/F is an extⁿ of no. fields

$\pi = \bigotimes_v' \pi_v(\sigma_v)$ - cuspidal repⁿ on $GL_2(\mathbb{A}_F)$

$\pi' = \bigotimes_w' \pi_w(\sigma_w)$ cuspidal on $GL_2(\mathbb{A}_E)$

say π' is a <u>base change lift</u> of π iff for $w|v$ σ_w' is the restriction of σ_v from WD_{F_v} to WD_{E_w}. This seems to be for <u>all</u> v inc. bad ones, +(infinite ones?)

If $\tau \in G(E/F)$ then τ acts on $GL_2(E)\backslash GL_2(\mathbb{A}_E)$ & gives an action on π' called $(\pi')^\tau$

$(\pi')^\tau = \bigotimes_w' \pi_w(\sigma_w'^\tau)$; $(\sigma_w')^\tau : WD_{E_w} \xrightarrow{\tau} WD_{E_{\tau w}} \xrightarrow{\sigma_{\tau w}'} GL_2(\mathbb{C})$

In ptic, $\pi' = \pi(\sigma) \Rightarrow (\pi')^\tau = \pi(\sigma^\tau)$ where $\sigma^\tau = \sigma_\circ$ (conjugation by τ)

π' is Galois-inv't if $(\pi')^\tau = \pi'$ $\forall \tau \in Gal(E/F)$.

<u>Thm 3.3</u> (Base change for GL_2) E/F cyclic of prime degree.

(a) every cusp. rep on $GL_2(\mathbb{A}_F)$ has a base change lift to $GL_2(\mathbb{A}_E)$

(b) π' a cuspidal repⁿ on $GL_2(\mathbb{A}_E)$ then it's a base change lift $\Leftrightarrow$ it Galois-inv't.

(c) If π & π' on $GL_2(\mathbb{A}_F)$ have the same base change lift then $\pi' = \pi \otimes \omega$, ω a char on G(E/F)

(d) If π' is a lift of π, then $\omega_{\pi'} = \omega_\pi \circ N_{E/F}$

$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_\pi(a)$, $a \in \mathbb{A}_F^\times / F^\times$

" Can we give him 10 more minutes?" It's 12:08. "No!" shouts the crowd.
"5 more minutes?"

$\pi = \bigotimes_v' \pi_v(\sigma_v)$ - automorphic on $GL_2(\mathbb{A}_F)$

<u>Def:</u> Π is a <u>GL_3-lift</u> if $\Pi = \bigotimes_v' \pi(A^2(\sigma_v))$ for almost all v

<u>Thm 3.4</u> (i) Every cuspidal π has a GL_3-lift to automorphic Π

(ii) Π is cuspidal iff $\pi \neq \pi(\sigma)$, σ a monomial repⁿ in G_F □

There's 1 more thing he'll need. It's sthg Tony mentioned.

Criterion for equality of cuspidals

$\pi = \otimes_v' \pi_v$, $\pi' = \otimes_v' \pi_v'$ cuspidal reps on $Gl_3(\mathbb{A}_F)$

$L_v(s, \pi_v \times \pi_v')$ is the Rankin product - this is for v s.t. π & π' are unram.

Thm 3.5 $\forall v$ can define $L_v(s, \pi_v \times \pi_v')$ - coincides with ↑ in unramified case,

s.t. $L(s, \pi \times \pi') = \prod_v L_v(s, \pi_v \times \pi_v')$ converges in RH plane &

(i) $L(s, \pi \times \pi')$ has mero continuation + functional eqn

Then if π & π' are unitary, $Re\, s \geq 1$

(ii) $L(s, \pi \times \pi')$ has pole at $s=1$ iff $\pi' \cong \tilde{\pi}$

(iii) $\forall v$, $L_v(s, \pi_v \times \pi_v') \neq 0$ @ $s=1$

(iv) $\forall v$, $L_v(s, \pi_v \times \pi_v')$ is pole-free for $Re(s) \geq 1$

ecture 3
Sat 27th Feb '03
3:30pm

§4 Artin's conjecture for tetrahedral rep^ns

$\sigma : G_F \rightarrow GL_2(\mathbb{C})$ an irred rep^n, $\tilde{\sigma}(G_F) \cong A_4$

$\pi(\sigma) = \otimes_v' \pi_v(\sigma_v)$.

We want to show $\pi(\sigma)$ is cuspidal.

Step 1 Construction of $\pi_{ps}(\sigma)$

We have E/F, $[E:F] = 3$, $\tilde{\sigma}(G_E) \cong V_4 \trianglelefteq A_4$

If $\Sigma = \sigma|_{G_E}$ we have that Σ is monomial.

This means all irred reps are induced fom chars of subgps.
1-d

[Rb (1997) I had always assumed Σ was reducible! But it aint! (I dont think)

By the converse thm, $\pi(\Sigma)$ is cuspidal on $GL_2(\mathbb{A}_E)$.

$\Sigma^\tau \cong \Sigma$, $\tau \in G(E/F) \Rightarrow \pi(\Sigma)^\tau = \pi(\Sigma^\tau) = \pi(\Sigma)$

By base change thm $\exists \pi$ cuspidal rep^n on $Gl_2(\mathbb{A}_F)$

s.t. $\pi(\Sigma)$ is a base change of π.

$\pi = \otimes_v' \pi(\sigma_v')$, $\underset{\text{Galois type}}{\sigma_v'}$ restricts to Σ_w for $w|v$, as restricting from W_{F_v} to W_{E_w}

There's some tricky reason stb. why we dont use WD_{F_v}. $N=0$ or sthg.

Replace π by $\pi\otimes\omega$, ω any char. of $G(E/F)$

Want ω s.t. $\underset{\overset{\|}{\omega_\pi \omega^2}}{\omega_{\pi\otimes\omega}} = \omega_{\pi(\sigma)} = \det(\sigma)$ ⊛

$\omega_\pi \circ N_{E/F} = \omega_{\pi(\Sigma)} = \det \Sigma = \det\sigma . N_{E/F}$

ω_π & $\det\sigma$ agree on $N_{E/F}C_E$. $C_F / N_{E/F}C_E \cong G(E/F) \cong C_3$

Can find ω on $G(E/F)$ s.t. ⊛, & it's unique.

$\underset{\overset{\|}{\otimes_v \pi(\sigma_v')}}{\pi_{ps}(\sigma)} = \pi\otimes\omega$ for that ω

Need $\sigma_v' = \sigma_v$ for almost all v

Claim $A^2(\sigma_v') = A^2(\sigma_v)$ for almost all v

Step 2 Claim $\Rightarrow \sigma_v = \sigma_v'$ for almost all v

If v splits in E/F, $\sigma_v' = \Sigma_w = \sigma_v$ ✓

If v remains prime in E/F, we may as well restrict to when σ_v, σ_v' are unramified & s.t. the claim is true.

$$\sigma_v(Fr_v) \sim \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}, \quad \sigma_v'(Fr_v) = \begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix}$$

As $Fr_v^3 = Fr_w \in W_{E_w}$ we have $\underset{\begin{pmatrix} a_v^3 & 0 \\ 0 & b_v^3 \end{pmatrix}}{\sigma_v(Fr_v^3)} = \underset{\begin{pmatrix} c_v^3 & 0 \\ 0 & d_v^3 \end{pmatrix}}{\sigma_v'(Fr_v^3)}$

∴ (WLOG) $c_v = \zeta a_v$

$d_v = \zeta' d_v$, ζ, ζ' cube roots of 1.

Also,
$\prod_v \det(\sigma_v') = \prod_v \omega_{\pi_v(\sigma_v')} = \omega_{\pi_{ps}(\sigma)} = \omega_{\pi(\sigma)} = \det\sigma = \prod_v \det\sigma_v$

$\Rightarrow \det\sigma_v' = \det\sigma_v \ \forall v$.

$\Rightarrow \zeta\zeta' = 1$ ie $\zeta' = \zeta^2$.

That's about all we can milk out here.

Recall $A^2 : GL_2(\mathbb{C}) \to GL_3(\mathbb{C})$; $\ker A^2 =$ scalars

$A^2(\sigma_v) = A^2(\sigma_v')$, $\begin{pmatrix} \zeta a_v & 0 \\ 0 & \zeta^2 b_v \end{pmatrix} \sim \begin{pmatrix} c_v & 0 \\ 0 & d_v \end{pmatrix} \sim \begin{pmatrix} \lambda a_v & 0 \\ 0 & \lambda b_v \end{pmatrix}$ for some λ.

So 2 cases:

(i) $\zeta a_v = \lambda a_v$, $\zeta^2 b_v = \lambda b_v \Rightarrow \lambda = \zeta = \zeta^2 \Rightarrow \lambda = 1$ ✓ $\Rightarrow \sigma_v = \sigma_v'$

(ii) $\left.\begin{matrix} \zeta a_v = \lambda b_v \\ \zeta^2 b_v = \lambda a_v \end{matrix}\right\} \Rightarrow \lambda^2 = 1$ & $b_v = \lambda \zeta a_v$

$\lambda = 1$ ✓ $\sigma_v = \sigma_v'$

$\lambda = -1$, $\zeta \neq 1$

$\sigma_v(Fr_v) \sim \begin{pmatrix} a_v & 0 \\ 0 & a_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \underset{\uparrow}{\lambda\zeta} \end{pmatrix}$ prim. 6th root of 1

$A^2(\sigma_v)(Fr_v) = \begin{pmatrix} \lambda\zeta & 0 & 0 \\ 0 & \lambda^{-1}\zeta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (w.r.t. basis $\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$ basis of $sl_2(\mathbb{C})$

↑ order 6 ; but $Im(A^2(\sigma_v)) \subseteq Im(A^2(\sigma)) = \underset{\substack{\uparrow \\ \text{no elts of} \\ \text{order 6.}}}{A_4}$ #.

So it suffices to prove the claim.

<u>Pf of claim</u> By Lemma 2.2, $A^2(\sigma)$ is monomial.

$\Rightarrow$ converse thm ; $\pi^* = \otimes_v' \pi_v(A^2(\sigma_v))$ is cuspidal on $GL_3(\mathbb{A}_F)$

$\Pi_{ps}(\sigma) = \pi(\sigma')$; $\sigma' : G_F \to GL_2(\mathbb{C}) \Rightarrow \sigma' = \sigma$.

$\Big[\sigma|_{G_E} = \sigma'|_{G_E} \Rightarrow \sigma' = \sigma \otimes \omega$, ω a <u>char of $G(E/F)$</u>

$\sigma' \subseteq Ind(Res_{G_E} \sigma') = Ind(Res\, \sigma_{F/E}) = \bigoplus_{\substack{\omega \text{ char.} \\ \text{of } G(E/F)}} \sigma \otimes \omega$;

$\det \sigma' = \omega_{\Pi_{ps}}(\sigma) = \det \sigma \Rightarrow \omega^2 = 1 \Rightarrow \omega = 1 \Big]$

σ isn't monomial $\underset{\text{Thm 3.4}}{\Longrightarrow} \exists\, GL_3$-lift Π (cuspidal) $= \otimes_v \pi_v(A^2(\sigma_v'))$ for almost all v.

$A^2(\sigma_v') = A^2(\sigma_v)$ follows from $\Pi = \pi^*$; $L(s, \pi^* \times \tilde{\pi}^*)$ & $L(s, \Pi \times \tilde{\pi}^*)$

For almost all v, $L(s,(\pi^*)_v \times (\tilde{\pi}^*)_v) = L(s, A^2(\sigma_v) \otimes A^2(\tilde{\sigma}_v))$ (by def$^{\text{n}}$)

& $L(s, \Pi_v \times (\tilde{\pi}^*)_v) = L(s, A^2(\sigma_v') \otimes A^2(\tilde{\sigma}_v))$ (")

v split: $\sigma_v = \sigma_v'$ – 2 local L-factors correspond

v remains prime, w|v ; $A^2(\sigma) = \mathrm{Ind}_{G_E}^{G_F}(\chi) \Rightarrow A^2(\sigma_v) = \mathrm{Ind}_{W_{E_v}}^{W_{F_v}}(\chi_v)$

$$A^4(\sigma_v) \otimes A^2(\tilde{\sigma}_v) = A^4(\sigma_v) \otimes \mathrm{Ind}_{W_{E_w}}^{W_{F_v}}(\chi_v^{-1}) \underset{\substack{\text{Frobenius}\\ \text{reciprocity}}}{\equiv} \mathrm{Ind}_{W_{E_w}}^{W_{F_v}}(A^4(\sigma_w)|_{W_{E_w}} \otimes \chi_v^{-1})$$

~~illegible~~ ~~remark~~

$$= \mathrm{Ind}(A^4(\sigma_v')|_{W_{E_v}} \otimes \chi_v^{-1}) = A^2(\sigma_v') \otimes A^2(\tilde{\sigma}_v')$$

So $L(s, \pi_v^* \otimes \tilde{\pi}_v^*) = L(s, \Pi_v \otimes \tilde{\pi}_v^*)$ for almost all v, $v \notin S$, S a finite set.

$$L(s, \Pi \times \tilde{\pi}^*) = \prod_{v \in S} \frac{L(s, \Pi_v \otimes \tilde{\pi}_v^*)}{L(s, \pi_v^* \otimes \tilde{\pi}_v^*)} L(s, \pi^* \otimes \tilde{\pi}^*)$$

Thm 3.5 $\Rightarrow L(s, \pi^* \otimes \tilde{\pi}^*)$ has a pole at 1.

$\Rightarrow$ finite product over S is finite & non-zero @ 1

$\Rightarrow$ RHS has a pole @ 1 $\Rightarrow$ LHS has a pole @ 1

$\Rightarrow$ (Thm 3.5) $\Pi \cong \pi^*$.

That's the pf of the claim.

$\pi_{ps}(\sigma) = \otimes_v' \pi_v(\sigma_v')$, $\pi(\sigma) = \otimes_v' \pi_v(\sigma_v)$

cuspidal

$\sigma_v' = \sigma_v$ for almost all v including $v|\infty$

$\exists$ finite set S of finite places where they could disagree

<u>Lemma</u> (J–L lemma 12.5) If $v_0 \in S$ then $\exists$ GC on C_F s.t. cond(ω) is highly divisible by all places in $S \setminus \{v_0\}$

& ω is unramified at v_0 [illegible] ($\omega(\pi_{v_0})$ is arbitrary) $\square$

Then $\underset{\underset{\otimes'_v \pi_v(\sigma'_v \otimes \omega)}{\|}}{\pi_{ps} \otimes \omega}$ & $\overset{\overset{\otimes'_v \pi_v(\sigma_v \otimes \omega_v)}{/\!/}}{\pi(\sigma) \otimes \omega}$ still agree locally $\forall v \notin S$

& both satisfy f'l eqns of the form

$$L(-, s) = \underset{\underset{A \cdot A_1^s}{\text{cst}}}{\underset{\uparrow}{\varepsilon(\sigma, s)}} L(-, 1-s)$$

∴ quotient of the 2 L-series for the twisted things is a finite product over quotients of local Euler factors for $v \in S$ & satisfy a functional eqn like the one above.

If ω is sufficiently ramified - ~~$v \neq v_0$~~ $v \in S$, $v \neq v_0$, then the local Euler products of both twisted L-series $= 1$, & $v = v_0$ it's changed by mult $\omega(\pi_{v_0})$ on 1 side & $\omega^{-1}(\pi_{v_0})$ on the contragredient side

$$\Rightarrow \text{at } v_0,\ L_{v_0}(\pi(\sigma) \otimes \omega) = L_{v_0}(\pi_{ps}(\sigma) \otimes \omega)$$

$$\therefore L_{v_0}(\pi(\sigma)) = L_{v_0}(\pi_{ps}(\sigma))$$

∵ 2 dim' local Weil rep's are determined by $L_v(s)$

$$\Rightarrow \pi_v(\sigma_v) = \pi_v(\sigma'_v) \Leftrightarrow \sigma_v = \sigma'_v$$

So in fact $\pi_{ps}(\sigma) = \pi(\sigma)$ is cuspidal in the 2-dim' case.

So we've not only shown that the L-f's are entire, but that the rep^n is auto. cuspidal.

∎

Eisenstein Series

Jean-Pierre Labesse

Lecture 1
Sat 27th Feb '93
9:30 am

Labesse wants to chat about lots of things. Maybe he'll talk about how to do (local results of yesterday) $\rightsquigarrow$ (global stuff)

This morning however he will talk a bit about GL_2.
In some sense GL_2 is easier, eg we have ~~cusp~~ Fourier expansion; in $D^\times$ case we only have Whittaker models. J-L is 'important. A lot of the ideas behind the pf we have met already.

Jean-Pierre did not follow Johns lectures very well because he missed the vast majority of them. He may say stuff that John said already.

Say $G = GL(2)$, F global

$\mathcal{A}(GL(2), F)$ = the space of auto. forms.

Auto reps are the irred subquotients of $\mathcal{H}$ in this space.

Set ~~$\mathcal{A}$~~ $\mathcal{A}_{cusp}(GL(2), F) = f$ s.t. $\int_{N(F)\backslash N(\mathbb{A})} f(ng)\,dn = 0 \quad \forall g.$

$\mathcal{A}_{cusp} = \bigoplus$ admiss irred rep with multiplicity 1.

The complement of this space can be described by the space generated by the Eisenstein series.

Define $f_N = \int_{N(F)\backslash N(\mathbb{A})} f(n*)$

If $\varphi \in C_c^\infty({}_{N(\mathbb{A})P(F)}\backslash G(\mathbb{A}))$ where P = "parabolique" = $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

f we can form $\langle \varphi, f_N \rangle_P$; $f_N \in C^\infty(N(\mathbb{A})P(F)\backslash G(\mathbb{A}))$

This converges.

The physicists are so important now that cx conjn has moved.

$$\langle \varphi, f_N \rangle_P = \int_{N(\mathbb{A})P(F)\backslash G(\mathbb{A})} f_N(x)\,\overline{\varphi(x)}\,dx = \langle E_\varphi, f \rangle_G .$$

where $E_\varphi(x) = \sum_{P(F)\backslash G(F)} \varphi(\gamma x)$

$$\& \langle E_\varphi, f \rangle_G = \int_{G(F)\backslash G(\mathbb{A})} \overline{E_\varphi(x)}\, f(x)\,dx$$

f is a cusp form $\Leftrightarrow f \perp \{E_\varphi \mid \varphi \in C_c^\infty(N(\mathbb{A})P(F)\backslash G(\mathbb{A}))\}$

Then $L^2(G(F)\backslash G(\mathbb{A})) = L^2_{cusp} \oplus L^2_{\underset{\sim}{P}}$

L^2 space genby E_φ I think.

Now $\langle E_\varphi, E_\psi\rangle_G = \langle \varphi, (E_\psi)_N\rangle_{\underset{\sim}{P}}$

& $(E_\psi)_N(x) = \int_{N(F)\backslash N(\mathbb{A})} E_\psi(nx) = \int \sum_{P(F)\backslash G(F)} \psi(\gamma m x)\,dx$

$P(F)\backslash G(F) = 1 \amalg wN(F)$ $\left(w = \begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ (Bruhat decomp)

$$(E_\psi)_N(x) = \int_{N(F)\backslash N(\mathbb{A})} \Big[\psi(x) + \sum_{\eta\in N(F)} \psi(w\eta n x)\Big]\,dx$$

$\therefore$ $(E_\psi)_N(x) = \psi(x) + (M\psi)(x)$

where $(M\psi)(x) = \int_{N(\mathbb{A})} \psi(wnx)\,dx$

This kind of object may well be factorizable into a product of local objects, if you fancy studying it.

So

$$\underset{\underset{L^2(G(F)\backslash G(\mathbb{A}))}{\uparrow}}{\langle E_\varphi, E_\psi\rangle_G} = \langle \varphi | \underset{\underset{L^2(N(\mathbb{A})P(F)\backslash G(\mathbb{A}))}{\uparrow}}{1+M} | \psi\rangle_{\underset{\sim}{P}}$$ ($= \langle\varphi, (1+M)\psi\rangle$ for all you non-physicists out there.)

M spoils things a bit. Shame. We must take our spectral analysis further.

"We are close to being Poinc. Series" though - $P(\mathbb{A})/N(\mathbb{A})P(F) \simeq M(\mathbb{A})/M(F)$

Define $\psi(x,\chi) = \int_{M(F)\backslash M(\mathbb{A})} \psi(mx)\,\chi(m)^{-1}\underset{\delta(m)^{-1/2}}{\underbrace{\ }}dx$ (cgt as ψ cptly supported in some sense ??)

Then $\psi(mx,\chi) = \delta(m)^{1/2}\chi(m)\psi(x,\chi)$

$\psi(nmx,\chi) = \delta(m)^{1/2}\chi(m)\psi(x,\chi)$ Its a global version of Poinc series

We have Fourier inversion too.

Write $E.\psi$ for E_ψ.

$$E.\psi = E\Big(\int_{\widehat{M(F)\backslash M(\mathbb{A})}} \psi(\cdot,\chi)\, d\mu(\chi)$$

$$E\,\psi(\cdot,\chi) = E_{\psi_\bullet(\cdot,\chi)}$$

$$E_{\psi(\cdot,\chi)}(x) = E(x,\psi,\chi) = \sum_{P(F)\backslash G(F)} \psi(\gamma x,\chi)$$

STOP this is not nec cgt.

Only apply this above formula "for good χ".

$$\text{Now } E_k(z) = \sum_{\substack{\text{"}P(F)\backslash G(F)\text{"} \\ =\Gamma_N\backslash\Gamma}} J(\gamma, gi)^{-k} \Big(/(\det)^? \text{ if he's on GL} \Big)$$

This stuff is cgt when $|\chi| = \delta^{s/2}$, $|\chi(m)| = \delta(m)^{s/2}$, $s\in\mathbb{C}$

cgt if $\mathrm{Re}(s)>1$.

$$\text{Let's try doing } \sum \delta(\gamma x)^{\frac{s+1}{2}} = \delta(x)^{\frac{s+1}{2}} + \underbrace{\sum_{\eta} \delta(w\eta x)^{\frac{s+1}{2}}}_{\text{must study this bit.}}$$

(here $\delta(nmk)=\delta(m)$ & δ is extended thus)

This is the same as studying $\displaystyle\int_{N(\mathbb{A})} \delta(wnx)^{\frac{s+1}{2}}\,dx$

G acts on $\mathbb{A}^2$ on the right.

$\mathbb{A}^2$ has some kind of $\|\cdot\|$ fn: $\|(\alpha,\beta)\| = \prod |(\alpha_v,\beta_v)|$

$\|vk\| = \|v\|$

$e_2 = (0,1)$. $e_2 n = e_2$. $\left\|e_2\begin{pmatrix} M_1 & 0 \\ 0 & M_2\end{pmatrix}\right\| = \|M_2\|$ & $\delta\begin{pmatrix} M & 0 \\ 0 & M_2\end{pmatrix} = \left|\frac{M_1}{M_2}\right|$

$\|e_2\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\| = \|m_2\|$ so $\delta(m) = |\det(m)| \Big/ \|e_2 m\|^2$

$\delta(wn) = \|e_2 wn\|^{-2} = \|e_1 n\|^{-2} = \|(1,u)\|^{-2}$, $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$

$\delta(wn)^{\frac{s+1}{2}} = \|(1,u)\|^{-(s+1)}$

$\int_{N(\mathbb{A})} \delta(wn)^{\frac{s+1}{2}} = \int_{\mathbb{A}} du \Big/ \|(1,u)\|^{s+1}$ & number theorists can do this explicitly - it's a product of local integrals

At ∞ get hand $\int \frac{du}{\sqrt{1+u^2}^{\,s+1}}$.

It turns out to be $\dfrac{L(s, \mathbb{1})}{L(s+1, \mathbb{1})}$ (exercise)

For $Re(s) > 1$ it's abscgt.

There's a pole @ $s=1$.

We need this for JL

$E(x, \psi, \chi)$; $|\chi| = \delta^{s/2}$ ...

Princ series $(|\cdot|^{\frac{s}{2}}, |\cdot|^{-s/2})$ locally everywhere (maybe twist by χ)

The Eisenstein series is nothing but an intertwining operator between induced rep of $P \to G$ & auto forms or sthg.

$\otimes \rho(|\cdot|^{s/2}, |\cdot|^{-s/2})$ we've got hunks of local triv rep's floating round

$(s-1)E(x, \psi, \delta^{s/2})$ in some sense reflects this

Time is too short to explain this further.

$$E(x, \psi, \chi)_N = \int_{N(F)\backslash N(\mathbb{A})} \sum_{P(F)\backslash G(F)} \psi(\gamma n x, \chi) = \psi(x, \chi) + (M\psi(\cdot, \chi))(x)$$

$$= \int_{N(\mathbb{A})} \psi(wnx, \chi)\, dn$$

$wn = n_1 m_1 k_1$

$$\int_{N(\mathbb{A})} \underbrace{\underset{\delta^{s/2}}{\chi(m_1(wn))}\, \underbrace{\delta(m_1(wn))^{1/2}}_{=\delta(wn)}}_{\text{we've made this computation already.}} \psi(k_1(wn)\, x \,\text{[illegible]}, \chi)\, dn$$

, $\chi = \delta^{s/2}\chi_0$

We get $L(s,\chi_0)/L(s+1,\chi_0)$. We've ignored k_1. So if, say, $\chi = 1$ & ψ is K-invt we get equality

So the convergence of $M\psi$ is controlled by the convergence of L-f^ns

Intertwining op. for par. subgps... rep^n.. L-group... general concept of L-function on L-groups. Langlands realised this & defined L-f^ns on L-groups, in his lecture

"Euler products" (Yellow lecture notes, 1966)

· He guesses he should stop here.

ecture 2
Sat 27th Feb '93
2:00pm

There are many things on the board. It appears that he himself has summarized the previous lecture.

N I have no idea what order the boards go in.

① Functional eqn $E(x,\psi,\chi) = E(x, M\psi, \tilde{\chi})$ (Holo in χ or sthg)

In ptic, $E(x,\psi,\chi_0\delta^{s/2}) = E(x, M\psi, \tilde{\chi}_0\delta^{-s/2})$

& if $\psi(xk) = \psi(x)$, $E(x,\psi,\chi_0\delta^{s/2}) = \dfrac{L(s,\chi_0)}{L(s+1,\chi_0)}\, E(x,\psi,\tilde{\chi}_0\delta^{-s/2})$

② At noontime you solved the exercise

$$\int_{F_v} \|(1,u)\|^{-(s+1)}\, du = \int_{\mathcal{O}} du + \sum_{n=1}^{\infty} \int_{\mathcal{O}^\times} |\varpi^{-n}\alpha|^{-(s+1)} q^n\, d\alpha$$

$$= 1 + (1-1/q)\sum_{n=1}^{\infty} q^{-n} q^{-(s+1)n}$$

$$= 1 + (1-1/q)\sum_{1}^{\infty} q^{-sn} = 1 + \frac{q^{-s}(1-1/q)}{1-q^{-s}}$$

③

$$= \frac{1-q^{-s}+(q^{-s}-q^{-(s+1)})}{1-q^{-s}} = \frac{(1-q^{-s})^{-1}}{(1-q^{-(s+1)})^{-1}} \quad \square$$

M intertwines I_χ with $I_{\tilde{\chi}}$

E intertwines "auto. forms" on $N(\mathbb{A})P(F)\backslash G(\mathbb{A})$ with auto forms on $G(F)\backslash G(\mathbb{A})$

[illegible]

④ $E(x, \psi, \chi) = \sum \psi(\gamma x, \chi)$

$$E(x, \psi, \chi)_N = \psi(x, \chi) + (M\psi)(x, \tilde{\chi})$$

Pf $\int \psi(wnx, \chi)\, dn = \iint \psi(m.wnx)\, \delta(m)^{-1/2} \chi(m)^{-1}\, dm\, dn$

$$= \int \psi(wn_1 \tilde{m} x)\, \delta(m)^{1/2} \chi(m)\, dm\, dn_1$$

$$= (M\psi)(x, \tilde{\chi}) \qquad (\delta(\tilde{m}) = \delta(m)^{-1})$$

⑤ Particular case: $\psi(xk) = \psi(x)$ & $\psi(zx) = \psi(x)$ $\forall z \in$ centre

& if $\chi = \delta^{s/2} \chi_0$, then

$$(M\psi)(x, \chi^{-1}) = \frac{L(s, \dot{\chi}_0)}{L(s+1, \dot{\chi}_0)} \psi(x, \chi^{-1})$$

where $\dot{\chi}(\alpha) = \chi\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \chi_0\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} |\alpha|^s = \chi_0(\alpha) |\alpha|^s$

⑥ $L(s, \dot{\chi}) = L(1-s, \dot{\chi}^{-1})$

$$\frac{L(s, \dot{\chi})\, L(-s, \dot{\chi}^{-1})}{L(1+s, \dot{\chi})\, L(1-s, \dot{\chi}^{-1})} = 1. \qquad M^2 = 1 \;(?)$$

That's the end of what was on the board.

<u>Truncation operator</u> (Langlands, Arthur)

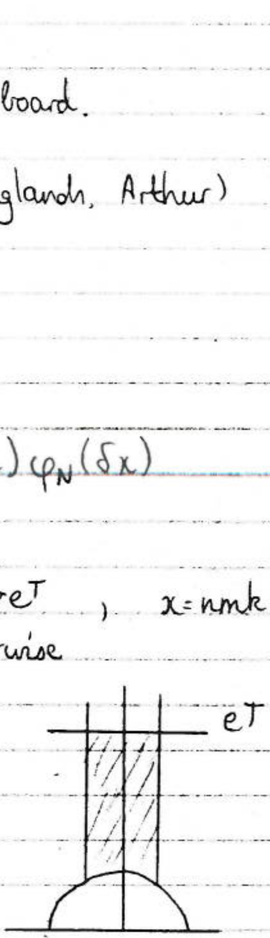
$$\Lambda^T \varphi = \varphi - E\, \hat{\tau}^T (\varphi_N)$$

$\varphi \in \mathcal{A}(GL_2, F)$

$$(\Lambda^T \varphi)(x) = \varphi(x) - \sum_{\delta \in P(F) \backslash G(F)} \hat{\tau}^T(\delta x)\, \varphi_N(\delta x)$$

$$\hat{\tau}^T(x) = \begin{cases} 1 & \delta(m) > e^T \\ 0 & \text{otherwise} \end{cases}, \quad x = nmk$$

$$\Lambda^T \varphi = \sum_{\underset{\sim}{P}} (-1)^{a_P} E_P^{\sigma} \hat{\tau}_{\underset{\sim}{P}}^T \varphi_{N_P}$$



Hopefully this makes it all very clear.

Only 1 cusp in the adelic case.

Now $(\Lambda^T)^2 = \Lambda^T$, $\Lambda^T = (\Lambda^T)^*$ so it's an orthogonal projector

$\varphi_N = 0 \quad \Lambda\varphi = \varphi$

φ automorphic, $\Lambda^T\varphi$ is rapidly decreasing.

$\Lambda^T E$ is square integrable

(Exercise - compute the scaler product $\int_{G(F)\backslash G(A)} \Lambda^T E(x, \psi, \chi)\, \Lambda^T E(x, \varphi, \mu)\, dx$

It only involves ψ, φ & $\bar{M}$.

If $K_f(x,y)$ is the kernel $K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$

$f \in C_c^\infty(G(A))$

Quotient not cpct. K_f not Hilbert Schmidt.

The truncated operator $\Lambda^T K$ is Hilbert-Schmidt.

$$\left| (K\Lambda^T\varphi)(x) \right| \le C\|\varphi\|_{L^2}, \quad \varphi \in L^2(\cdots)$$

$$\int \sum_\gamma f(x^{-1}\gamma y)\, (\Lambda^T\varphi)(y)\, dy$$

$x = n_1 a_1 k_1$
$y = n_2 a_2 k_2$

K cpct so drop k_i s

$$\int \sum_\gamma f(a_1^{-1} n_1^{-1} \gamma\, n_2 a_2)\, (\Lambda^T\varphi)(y)\, dy$$

$\gamma = \begin{pmatrix} * & * \\ c(\gamma) & * \end{pmatrix}$

$c(\gamma) \neq 0$

A priori it's divergent, a priori

Use the fact that we're truncated.

Apply Poisson sum, Fourier, we're slowly decreasing

↓

$\left[f(a_1^{-1} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} a_2) - \int f(a_1^{-1} n a_2)\, dn \right]$ Make your mayorsation.

Define $\underbrace{J^T}(f) = \int (\Lambda^T K)(x,x)\,dx$ (T is some big real, Alain reckons)

T large enough (w.r.t. ??? of f)

-this is a poly in T.

This is $J(f)$. This is what the trace formula is all about.

Eisenstein series ⇝ (after many difficulties) trace formula. Spectral sequence. It's the trace of nothing.

$f = \otimes f_v$

If at 2 places v_1, v_2 $\left\{ \begin{array}{l} O_\gamma(f_{v_i}) = 0 \\ \text{whenever } \gamma = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \end{array} \right.$

then the trace formula for $PGL_2 = G$

$$J(f) = \sum_{\gamma \text{ elliptic}} \text{vol}\left(G_\gamma(F) \backslash G_\gamma(\mathbb{A})\right) O_\gamma(f) = \sum_{\pi \subseteq L^2_{\text{disc}}} \text{tr}\, R(f)$$

centraliser/? is isotropic or sthg

Now we can quickly finish Jacquet-Langlands.

$D^\times(F) Z(\mathbb{A}) \backslash D^\times(\mathbb{A})$ $\quad f' = \otimes f'_v$

$GL(2,F) Z(\mathbb{A}) \backslash GL(2,\mathbb{A})$ $\quad f = \otimes f_v$

Compare 2 traces

$v \notin S \Rightarrow D_v$ split & take $f'_v = f_v$

$\uparrow$ bad places

v bad: $D_v^\times \not\hookrightarrow GL_2(F_v)$

but $\{\text{conj classes of } D_v^\times\} \hookrightarrow \{\text{conj classes in } GL_2(F_v)\}$

Richard Taylor has done exactly now what we need - matching stuff