PROBA2019
Problem Set 1

Coments, Hints & Some Solutions

1. (Algebras/σ-Algebras) Which of the following is an algebra or $\sigma-$algebra of sets:

(a) Family of open sets in $\mathbb{R}^n$ with Euclidean metric;

Not an algebra, does not contains complements which are closed sets.

(b) Family of open sets in $\mathbb{C}$ with discrete topology;

Yes, this family contains all subsets of $\mathbb{C}$ *and so it is the* σ*-algebra* $2^{\mathbb{C}}$.

(c) Family of open and closed sets in $\mathbb{R}$ with respect to $|\cdot - \cdot|$;

Not an algebra, it does not contains for example a set $(a, b] = (a, b) \cup \{b\}$

(d) Intersection of all σ-algebras containing left closed intervals in $\mathbb{R}$ with respect to $|\cdot - \cdot|$;

Intersection of σ*-algebras is a* σ*-algebra.* (*Since a* σ*-algebra is closed with respect to countable unions, it contains* $\cup_{n\in\mathbb{N}}[a - \frac{b-a}{n+1}, b) = (a, b)$*, and hence it contains all open set. Thus it coincides with the Borel* σ*-algebra.*)

(e) Family $\mathfrak{F}$ of subsets of an infinite set Ω consisting of finite sets or sets having a finite complement and the empty set and its complement.

It is not a σ*-algebra.* Since Ω is by our assumption infinite, it contains a sequence $(q_n \in \Omega)_{n\in\mathbb{N}}$. Consider a sequence of sets $A_n \equiv \{q_{2k+1}\}_{k=1,\dots,n}$, $n \in \mathbb{N}$. We have $\cup_{n\in\mathbb{N}} A_n = \{q_{2k+1}\}_{k\in\mathbb{N}}$ which is an infinite set with infinite complement, and so does not belong to the family in the question.

2. (Probability spaces) Which of the following is a probability space and which is not:

(a) $([0, 2], \Sigma_{\text{Leb}} \cap [0, 2], \lambda)$ with λ denoting the Lebesgue measure on $\Sigma_{\text{Leb}} \cap [0, 2]$.

No, as measure is not normalised to one.

(b) $(\mathbb{N}, 2^{\mathbb{N}}, \nu)$ with $\nu(\{n\}) \equiv 2^{-n}$

Yes, in this case for any $A \in 2^{\mathbb{N}}$ *one has by definition* $\nu(A) \equiv \sum_{k\in A} 2^{-k}$*, with sum equal to one if* $A = \mathbb{N}$.

(c) $(\mathbb{N}, \mathfrak{F}, \nu)$ with $\mathfrak{F}$as in (1e) and $\nu(A) \equiv \sum_{\{n\in A\}} 2^{-n}$ if A is finite and $\nu(A) = 2$ if A^c is finite.

ν *is not a sigma additive measure. Since for any finite* A *we have* $\sum_{\{n\in A\}} 2^{-n} < 1$*, for* $A_n \subset A_{n+1}$*with* $A \equiv \cup_{n\in\mathbb{N}} A_n$ *infinite (and so* $\nu(A) = 2$*), the property* $\lim_{n\in\mathbb{N}} \nu(A_n) = \nu(A)$ *fails.*

3. (Random Variables)

(a) Show that a sum and a product of random variables is a random variable.

See (b) below.

(b) Show that simple functions are random variables.

First of all one notes that a characteristic function χ_A*of a measurabel set* A *is a random variable, as we have*

$\chi_A^{-1}((-\infty, a)) = \emptyset$ if $a \in (-\infty, 0)$
$\chi_A^{-1}((-\infty, a)) = A^c$ if $a \in (-\infty, 1)$

$\chi_A^{-1}((-\infty, a)) = \Omega$ if $a > 1$

Next we note that for any measurable sets A and B, also $A \cup B$is measurable, and

$\chi_A + \chi_B = \chi_{A\cup B\setminus A\cap B} + 2\chi_{A\cap B}$*, with* $\chi_A + \chi_B = \chi_{A\cup B}$ *for disjoint sets.*

Hence any simple function can be represented as a sum of characteristic functions of measurable sets with different coefficients $f = \sum_{k=1,\ldots,n} c_n \chi_{A_n}$. If necessary rearranging the sum so that $c_k < c_{k+1}$, we get

$f^{-1}(-\infty, a) = \cup\{A_k : c_{_k} < a\}$ *which is measurable if each A_k is.*

The same applies to the sums of simple functions.

Next we note that using additionally $\chi_A \cdot \chi_B = \chi_{A\cap B}$ we can represent a product of two simple functions as a sum of charactersistic functions with different coefficients. Hence it follows that the product of simple functions is a random variable.

Since any random variable can be approximated by simple functions and the pointwise limit of random variables are random variables, we conclude that the sum and the product of random variables are the random variables.

(c) For a probability space (Ω, Σ, μ), such that $\Sigma \neq 2^\Omega$, and a set $V \notin \Sigma$, prove or disprove that the following is a r.v.

(i) χ_V ; (ii) $\chi_{V\cap A}$ for $A \in \Sigma$.

The first is explained before. The second can be true if the intersection is a zero set (and the measure is complete, i.e. is considered on σ-algebra containing all zero sets).

(d) Prove or disprove that every non-negative bounded random variable can be approximated by a monotone non-decreasing sequence of non-negative simple functions.

Choose a sequence $\varepsilon_n \equiv 2^{-n}$ and define

$f_n \equiv \sum_{0\geq k\leq 2^{2n}-1} k\varepsilon_n L\chi(\{k\varepsilon_n L \leq f < (k+1)\varepsilon_n L\})$

where L is choosen so that $f \leq L$.

4. (Distribution function)

(a) Find a distribution function of Bernoulli random variable.

A random variable f on a probability space $(\Omega, \Sigma, \mu)$is called Bernoulli r.v. iff

$\mu\{f = \alpha\} = p \in (0, 1)$ and $\mu\{f = \beta\} = q \equiv 1 - p$

for some $\alpha < \beta$. Then we have

$F_f(x) = 0$*if $x < \alpha$; p if $\alpha \leq x < \beta$; and* 1*if $\beta \leq x$.*

(b) Prove or disprove that "devil staircase" associated to the Tertiary Cantor set is a distribution function. Prove or disprove that the corresponding measure constructed via Lebesgue-Caratheodory construction is singular with respect to the Lebesgue measure.

The Cantor function is a continuous nondecreasing function F_C which is constant on each connected component of the complement of the Cantor set in $[0, 1]$. By definition of a measure of an interval, we have $\nu_C((a, b)) = F_C(b) - F_C(a) = 0$for any interval (a, b) contained in a connected component of the complement of the Cantor set. Since the Lebesgue measure of all interval in this complement is equal to one and the measure of their union according to ν_C is zero, the measures are singular.

5. (Variance and Entropy) For a product measure $\mu \equiv \nu_1 \otimes \nu_2$, prove or disprove

(a) $\mu(f - \mu f)^2 \le \nu_2(\nu_1(f - \nu_1 f)^2) + \nu_1(\nu_2(f - \nu_2 f)^2)$

For square integrable function f we can apply Fubini theorem to get

$\mu(f - \mu f)^2 = \nu_2(\nu_1(f - \mu f)^2) = \nu_2(\nu_1(f - \nu_1(f) + \nu_1(f) - \mu f)^2)$

$= \nu_2(\nu_1(f - \nu_1(f))^2) + \nu_2(\nu_1(f) - \mu f)^2$

and for the last term we have

$\nu_2(\nu_1(f) - \mu f)^2 = \nu_2(\nu_1(f - \nu_2 f))^2 \le \nu_2(\nu_1(f - \nu_2 f)^2) = \nu_1(\nu_2(f - \nu_2 f)^2)$

(b) $\mu\left(f^2 \log \frac{f^2}{\mu f^2}\right) = \nu_2\left(\nu_1\left(f^2 \log \frac{f^2}{\nu_1 f^2}\right)\right) + \nu_2\left(\left(\nu_1 f^2 \log \frac{\nu_1 f^2}{\nu_2(\nu_1 f^2)}\right)\right)$

One uses

$\mu\left(f^2 \log \frac{f^2}{\mu f^2}\right) = \mu\left(f^2 \log \frac{f^2 \cdot \nu_1 f^2}{\nu_1 f^2 \cdot \mu f^2}\right) = \mu\left(f^2 \log \frac{f^2}{\nu_1 f^2}\right) + \mu\left(f^2 \log \frac{\nu_1 f^2}{\mu f^2}\right)$

$= \nu_2\left(\nu_1\left(f^2 \log \frac{f^2}{\nu_1 f^2}\right)\right) + \nu_2\left(\nu_1 f^2 \log \frac{\nu_1 f^2}{\mu f^2}\right)$

6. (Inequalities)

(a) Prove that for a probability measure ν on a finite set one has

$\left(\sum_i (\nu(f_i))^2\right)^{\frac{1}{2}} \le \nu\left(\sum_i f_i^2\right)^{\frac{1}{2}}$

We have, using triangle inequality for the norm

$\left(\sum_i (\nu(f_i))^2\right)^{\frac{1}{2}} = \left(\sum_i \left(\sum_\omega \nu(\omega)(f_i(\omega))\right)^2\right)^{\frac{1}{2}} \le \sum_\omega \left(\sum_i (\nu(\omega)(f_i(\omega)))^2\right)^{\frac{1}{2}} = \sum_\omega \nu(\omega)\left(\sum_i (f_i(\omega))^2\right)^{\frac{1}{2}}$

(b) Generalise the above to any norm.

Similar as above using the triangle inequality and homogeneity of the norm.

(c) Prove Poincare inequality for a product of Bernoulli measures.

Use 5a and mathematical induction.

(d) Prove Log-Sobolev inequality for the Bernoulli measure.

(e) Prove Log-Sobolev inequality for a product of Bernoulli measures.

Use (5b) and the following inequality

$\left|\nabla(\nu f^2)^{\frac{1}{2}}\right|^2 \le \nu|\nabla f|^2$

which follows from triangle inequality for the norms.

7. (Independence)

(a) (Starting from simple functions) prove that for non-negative real valued random variables f_i, $i = 1, \ldots, n$, on (Ω, Σ, ν) which are independent and integrable, one has

$$\int \prod_{i=1}^n f_i d\nu = \prod_{i=1}^n \int f_i d\nu$$

First one shows this property for simple functions, which is a simple use of independendece together with rearrangement of sums of nonnegative terms. Then we can chose a monotone sequence of simpe functions for each f_i and use the monotone convergence theorem.

(b) Using (7a) prove similar relation for a product of integrable independent r.v.s.

First we decompose each f_i into positive and negative parts which for different indices are independent. In this way we represent each integral as a sum of terms as considered in (7a).

(c) Given distribution functions $F_i$of real valued independent r.v.s X_i , $i = 1, \ldots, n$, defined on the same probability space, find a distribution of the sum of these r.v.s.

Using mathematical induction, it is sufficient to consider the case $n = 2$. By definition we have

$F_{X+Y}(z) \equiv \mu\{X + Y \leq z\} = \int \chi_{\{X+Y \leq z\}} d\mu$

Because of independence of X and Y, we have $\mu\{X \in A, Y \in B\} = \mu\{X \in A\}\mu\{Y \in B\}$. Hence we have the following representation of the integral

$\int \chi_{\{X+Y \leq z\}} d\mu = \int \chi_{\{x+y \leq z\}} dF_X(x) dF_Y(y) = \int F_X(z - y) dF_Y(y) \equiv F_X * F_Y(z)$