SOLUTIONS 8. 8.3.2012

Q1 (Brownian bridge). As $B_0(t) := B_t - tB_1$, $B_0$ is Gaussian (it is obtained from the Gaussian process $B$, BM, by forming linear combinations, and linear combinations of multinormals are multinormal). It is continuous as BM is; it starts from 0 as BM does, and finishes at 0 from its definition.

(i) The covariance function is

$$
cov(B_0(s), B_0(t)) = E[(B_0(s)B_0(t))] = E[(B_s - sB_1)(B_t - tB_1)] = E[B_s B_t - tB_n B_1 - sB_1 B_1 + stB_1^2] = \min(s, t) - st - st + st = \min(s, t) - st \quad (0 \leq s, t \leq 1)
$$

(as BM has covariance $\min(s, t)$ and $\min(s, 1) = s$, etc.).

(ii) Since BM is

$$
B_t = \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) X_n
$$

with $\Delta_n$ the Schauder functions and $X_n$ independent $N(0, 1)$: recall $\Delta_0(t) = t$, $\Delta_n(1) = 0$ for $n \geq 1$, and $B_1 \sim N(0, 1)$. So putting $t = 1$ above gives $B_1 = X_0$. So $B_0$ is the sum of the remaining terms:

$$
B_0(t) = \sum_{n=1}^{\infty} \lambda_n \Delta_n(t) X_n.
$$

Q2. As in lectures, $X_t$ (defined by time-inversion for $t > 0$) has the same covariance as BM. So, away from the origin, $X$ is Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So $X$ is continuous. So we can define it at the origin by continuity. So $X$ is Brownian motion everywhere – $X$ is BM.

Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \rightarrow 0$ as $t \rightarrow 0$. This says that

$$
tB(1/t) \rightarrow 0 \quad (t \rightarrow 0),
$$

which is

$$
B(t)/t \rightarrow 0 \quad (t \rightarrow \infty),
$$

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as required.

Q3. Measurability of BM follows from the Schauder expansion. The partial sums are measurable in \((t, \omega)\) (recall the \(\Delta_n(t)\) are continuous, and \(X_n = X_n(\omega)\)), and limits of measurable functions are measurable.

Q4. By LIL, \(\limsup_{t \to \infty} B_t = +\infty\) a.s., and similarly (or by symmetry), \(\liminf_{t \to \infty} B_t = -\infty\) a.s. So by continuity of BM, there must be arbitrarily large zeros of BM: the zero-set \(Z\) is unbounded, a.s. Then time-inversion (as in Q2) shows that there are zeros \(t_n \downarrow 0\), a.s. – the zero at the starting-point \(t = 0\) is followed by infinitely many zeros at positive times. Using the Strong Markov Property: any zero of BM must be a limit-point of zeros from the right. So any zero is a limit of zeros other than itself: \(Z\) is closed (by continuity of BM), and has no isolated points: \(Z\) is a perfect set.

With \(\lambda\) Lebesgue measure, we now evaluate \((\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\})\) in two ways, by Fubini’s Theorem (which we can use, because \(B_t(\omega) = B(t, \omega)\) is measurable). First,

\[
(\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\}) = \int_0^\infty P(B_t = 0)dt = \int_0^\infty 0dt = 0.
\]

Next,

\[
(\lambda \times P)(\{(t, \omega) : B_t(\omega) = 0\}) = \int_\Omega \lambda(\{t : B_t(\omega) = 0\})dP = E[\lambda(\Omega)].
\]

Combining, \(E[\lambda(\Omega)] = 0\). But as \(\lambda(\Omega) \geq 0\), this says \(\lambda(\Omega) = 0\) a.s.: \(Z\) is a.s. \(\text{Lebesgue-null}\).

As \(B_t\) is nowhere differentiable (given), \(B_t\) cannot vanish throughout any interval \(I\) (or it would have derivative 0 there). So by continuity, any interval \(I\) contains a subinterval \(J\) on which \(B_t\) is non-zero, i.e. \(J\) does not meet \(Z\). So \(Z\) is nowhere dense.

Q5 (Scheffé’s Lemma). \(|f_B f_n - f_B f| = |f_B(f_n - f)| \leq f_B |f_n - f|\). Taking sups over \(B\) proves the inequality. Next, with \(a \wedge b := \min(a, b)\), \(|f_n - f| = f_n + f - 2f_n \wedge f\) (check). Integrate: \(\int f_n = 1\), \(\int f = 1\) as these are densities. As \(0 \leq f_n \wedge f \leq f\), integrable, dominated convergence gives \(\int f_n \wedge f \to \int f = 1\). So the integral of RHS \(\to 1+1-2 = 0\). So the integral of LHS \(\to 0\) also. //