Abstract

We trace the interplay between the Hardy-Littlewood school of analysis, prominent in the UK in the first half of the last century, and probability theory, barely known in the UK at that time.\footnote{MSC: 01, 60}

1. Introduction

This piece is intended as a sequel to the splendid paper by Diaconis, *G. H. Hardy and probability* [Dia], published in *Bull. LMS* in 2002. There, Diaconis took the view that, despite Hardy’s public disdain for applied mathematics, including probability as he then saw it (and as it largely was then), his work has many probabilistic overtones. Littlewood always had a more positive view of both.

We follow Diaconis’ ordering of subject-matter: probabilistic number theory (§2), summability and Tauberian theory (§3) and Hardy spaces (§4), before turning to the principal individuals involved.

2. Probabilistic number theory

Recall the prime divisor functions: \( \omega(n) := \# \) distinct prime divisors \( n \); \( \Omega(n) := \# \) prime divisors \( n \) (counted with multiplicity) – additive arithmetic functions, as they satisfy \( f(mn) = f(m) + f(n) \) if \( (m, n) = 1 \). For background, see e.g. Tenenbaum [Ten, III.3, III.4], Montgomery and Vaughan [MonV, §§1.3, 2.4, 7.4].

Edmund Landau (1877-1938) proved in 1900 [Lan1] that if \( \pi_k(x) \) is the number of \( n \leq x \) with \( k \) distinct prime factors \( (k = 1, 2, \ldots) \),

\[
\pi_k(x) \sim \frac{x}{(k - 1)!} \frac{(\log \log x)^{k-1}}{\log x}.
\]

As each \( n \) has at least one prime factor, it is better to work with \( k + 1 \) rather than \( k \). Writing

\[
\lambda := \log \log x
\]
(so $\lambda \to \infty$ as $x \to \infty$ – though extremely slowly):

$$\frac{1}{x} \pi_{k+1}(x) \sim \frac{(\log \log x)^k}{k! \log x} = \frac{e^{-\lambda} \lambda^k}{k!} \quad (k = 0, 1, 2, \ldots) \quad (\lambda, x \to \infty).$$

Now $\{e^{-\lambda}\lambda^k/k! : k = 0, 1, 2, \ldots\}$ forms the Poisson distribution $P(\lambda)$ of Probability Theory, with parameter $\lambda$ (mean $\lambda$, variance $\lambda$), though Landau made no mention of Poisson or of probability (the extensive 54-page bibliography of his Handbuch of 1909 [Lan2] does not cite Poisson). So, although Landau did not do so, one may rephrase his result above as follows: the proportion of integers $\leq x$ with $k + 1$ distinct prime factors is asymptotically that of a Poisson distribution $P(\lambda)$ with parameter $\lambda := \log \log x$. This first hint of probability theory in analytic number theory is a precursor of probabilistic number theory, our subject here.

Regarding the prime divisor functions, the following result was proved by Hardy and Ramanujan in 1917 ([HarRa]; [HarW, Th. 430]).

(i) $\sum_{n \leq x} \omega(n) = x \log \log x + C_1 x + O(1/\log x),$

(ii) $\sum_{n \leq x} \Omega(n) = x \log \log x + C_2 x + O(1/\log x),$

where the $C_i$ are specified constants. This result (without the constants or error terms) is summarised by Hardy and Wright as saying that the average order of both $\omega(n)$ and $\Omega(n)$ is $\log \log n$. They also treat normal order, and show that $\omega(n), \Omega(n)$ have normal order $\log \log n$ [HarW Th. 431]. See also Tenenbaum [Ten, §3.4], where this result is derived from the Turáň-Kubilius inequality [Ten §3.2], and Turáň’s quantitative form of 1934 [Tur] is also given. Equip the set $\mathbb{N}_N := \{1, 2, \ldots, N\}$ with the uniform distribution $P_N$ (mass $1/N$ on each member). If $\xi(n) \to \infty,$

$$P_N\{n : |\omega(n) - \log \log n| > \xi(n) \sqrt{\log \log n}\} < 1/\xi^2(N),$$

and similarly for $\Omega(n)$. Tenenbaum remarks that this result may be viewed nowadays as the birth of probabilistic number theory.

If one has the Poisson distribution in mind (which presumably Hardy and Ramanujan did not, any more than Landau did), one sees the $\log \log x$ term on the right as the mean and variance of the Poisson distribution $P(\lambda)$ in Landau’s result above. What this suggests, of course, is a law of large numbers (LLN). And where one has an LLN, one may seek a central limit theorem (CLT) . . . Since the Poisson law $P(\lambda)$ is the $n$-fold convolution of $P(\lambda/n)$ for each $n$, the ordinary CLT shows that for large $\lambda$ $P(\lambda)$ is approximately normal $N(\lambda, \lambda)$ with mean and variance $\lambda$. 

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The CLT for this case followed in 1939, as the Erdős-Kac central limit theorem (completed in 1939, during a seminar given by Kac: Erdős was in the audience, and completed the proof by using sieve methods), the other candidate for the birth of probabilistic number theory. Write as usual

\[ \Phi(x) := \int_{-\infty}^{x} \phi(u) \, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \, du. \]

**Theorem (Erdős-Kac CLT, 1939).**

\[ P_N(n : \omega(n) \leq \log \log N + x\sqrt{\log \log N}) \to \Phi(x) \quad (N \to \infty), \]

and similarly for \( \Omega \).

Where one has a central limit theorem, one looks for a Berry-Esseen rate-of-convergence result (following A. C. Berry in 1941 and C.-G. Esseen in 1942). This was found by Rényi and Turán [RenT] in 1958, following earlier work by Turán in 1934 and 1936 on error estimates in the Hardy-Ramanujan theorem. For an integrated treatment of the Erdős-Kac and Rényi-Turán theorems, see [Ten III.4.4]. This uses the Selberg-Delange method (Selberg in 1954, Delange in 1959 and 1971, following work of Sathe in 1953 and 1954; [Ten II.5]). This extends the Landau result – estimates of \( \pi_k(x) \) and its analogue with repetitions allowed – from fixed \( k \) to \( k = O(\log \log x) \) [Ten II.6.1].

The Poisson \( P(\lambda) \) and normal laws occur together here, as above (though in discrete situations such as Landau’s result the Poisson is perhaps more natural). Poisson and normal approximation can both be handled by the Chen-Stein method [BarHJ] (see §3 below); this method has recently been applied to give a short proof of the Erdős-Kac theorem by Harper [Harp].

Diaconis tellingly says [Dia, 387]: ‘My interests in Hardy and probability started when Erdős said: ‘You know, if Hardy had known anything of probability, he certainly would have proved the law of the iterated logarithm. He was an amazing analyst, and only missed the result because he didn’t have any feeling for probability’ (which could also be said of Landau). This also delayed the start of probabilistic number theory, but the subject is now flourishing; see e.g. [Kub], [Ten], [Ell].

Diaconis [Dia, §§2, 3] covers much of the material here, including the

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2Hardy and Littlewood obtained a partial result in 1914, following Hausdorff in 1913. The complete result was obtained by Khinchin in 1924.
Turán-Kubilius inequality, and Hildebrand’s use of it and sieve methods to prove the Prime Number Theorem [Hil]; as he says, these results played a major role in his mathematical development.

This fascinating interplay between probabilistic methods on the one hand, and the distribution of the primes (as ‘God-given’ and deterministic as anything could be) on the other is neatly summarised by Kac in the chapter heading of Ch. 4 in his delightful book [Kac1]: ‘Primes play a game of chance’. We shall call this Kac’s dictum. Another way of putting this is due to the analytic number theorist R. C. Vaughan (1945-): ‘It’s obvious that the primes are randomly distributed – we just don’t know what that means yet’ – Vaughan’s dictum.

3. Summability and Tauberian theory

The connections between probability and summability were also influential on Diaconis’ early work [Dia, §4].

The most basic summability method is the Cesàro method of order 1, $C_1$, which maps a sequence $(s_n)$ into its sequence of arithmetic means. This preserves convergence, and limits, when present, but may well produce convergence when not. The second-order method $C_2$ is obtained by applying $C_1$ twice, etc.; the family $C_\alpha$ for $\alpha > 0$ is obtained by replacing all factorials by Gamma functions. See e.g. [Har3, V, VI] for background and details.

The next summability method one usually meets is the Abel method $A$, with geometrically decreasing weights. See [Har3, VII] for details, and connections between the Cesàro and Abel methods.

Next, one has the Borel method, with Poisson weights $e^{-x}x^k/k!$ (and one takes limits as $x \to \infty$), and the Euler methods $E_p$ ($0 < p < 1$), with weights those of the binomial distribution $B(n,p)$, namely $\binom{n}{k}p^k(1-p)^{n-k}$, and takes limits as $n \to \infty$. For details, see [Har3, VIII, IX].

In classical summability theory, the interest of such methods is largely in the possibility they offer to make divergent sequences converge. In particular, in complex analysis they may be used to extend the domain of analyticity of a power series, and this is largely the view taken in [Har3].

Laws of large numbers (LLNs).

In probability theory, however, one has another view – that of laws of

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My wife’s instant response to this:

(Cecilie) Bingham’s dictum: Primes play a game of chance – we just don’t know the rules yet.
large numbers. If \( X, X_1, \ldots, X_n, \ldots \) are independent and identically distributed (iid) random variables, then

\[
\frac{1}{n} \sum_{k=1}^{n} X_k \to c \quad (n \to \infty), \quad \text{i.e.} \quad X_n \to c \quad (C_1)
\]

for some constant \( c \) if and only if

\[
E[|X|] < \infty, \quad \text{and then} \quad E[X] = c.
\]

This is Kolmogorov’s strong law of large numbers (SLLN) of 1933, one of the key results in his pioneering Grundbegriffe [Kol1]. Incidentally, this result completes a large number of preliminary results of this type, going back to Bernoulli’s theorem (Jakob Bernoulli (1654-1705), in his posthumous Ars Conjectandi (AC) of 1713 – weak LLN for Bernoulli trials). It is the final mathematical form of the folklore idea of a ”Law of Averages”, and underlies the interpretation (it is no more than that) of probability as a limiting frequency. (One can weaken independence, and identical distribution, and work on more general spaces, but we keep to the real iid case here for simplicity.)

Probability and summability.

Remarkably, Kolmogorov’s result holds verbatim if the summability method \( C_1 \) in it is replaced by \( C_\alpha \) for each (and so all) \( \alpha \geq 1 \), and by the Abel method \( A \). This result is due to T.-L. Lai in 1974 [Lai]. This gives a sense in which the Abel-Cesàro family of summability methods is the natural family in probability theory in the \( L_1 \) case – when the mean or first moment \( E[X] \) exists.

There is a complement to this. Y.-S. Chow [Cho] showed in 1973 that

\[
\sum_{k=1}^{\infty} \frac{e^{-x}x^k}{k!} X_k \to c \quad (x \to \infty), \quad \text{i.e.} \quad X_n \to c \quad (B)
\]

for some constant \( c \) if and only if

\[
E[|X|^2] < \infty, \quad \text{and then} \quad E[X] = c.
\]

The same result holds if the Borel method \( B \) is replaced by the Euler method \( E_p \) for some (and so for all) \( p \in (0,1) \). This makes precise a sense in which the Euler-Borel family of summability methods is the natural family in probability in the \( L_2 \) case – when the variance or second moment exists.

It turns out that Kolmogorov’s SLLN is reflected in a discontinuity of
functional form in the result for $C_\alpha$ as $\alpha$ passes through 1 (Bingham and Tenenbaum [BinT], cf. [Bin7]):

**Theorem (LLN for $C_\alpha$).**

(i) For $0 < \alpha \leq 1$,

$$X_n \to \mu \ (C_\alpha) \iff E[|X|^{1/\alpha}] < \infty \ & E[X] = \mu.$$ 

(ii) For $\alpha \geq 1$,

$$X_n \to \mu \ (C_\alpha) \iff E[|X|] < \infty \ & E[X] = \mu.$$ 

Case (ii) is Lai’s result above. For case (i), the case $\alpha \in (\frac{1}{2}, 1)$ is due to G. G. Lorentz in 1955 [Lor], for $\alpha \in (0, \frac{1}{2})$ to Chow and Lai in 1973 [ChoL]; the case $\alpha = \frac{1}{2}$ is in Déniel and Derriennic [DenD] in 1988.

Riesz means.

Chow’s ‘delayed sums’ [Cho] (‘moving averages’ in [Bin7]) involved averages of the form $\frac{1}{c_n} \sum_{n \leq k < n + c_n} X_k$. It turns out that these already had a long pedigree in classical summability theory, as *typical means*, or Riesz means. These were introduced by Marcel Riesz (1886-1969) in 1909, as they are ideally suited to analytic continuation of Dirichlet series, as ubiquitous in analytic number theory as power series are in complex analysis. They are used for this purpose in the book by Hardy and Riesz [HarRi]; there is a monograph account in [ChaM].

The theorem above deals with the powers, or $L_p$-spaces, but Riesz means enable one to tie general moments accurately to the length of the interval along which one takes the moving average. Take a function $\lambda(.) \uparrow \infty$, and write $\lambda_n = \lambda(n)$. For $s_n = \sum_{0}^{n} a_k$, write

$$s_n \to s \quad R(\lambda, 1)$$

(or just $R(\lambda)$ if, as here, one uses only order $k = 1$) for

$$\frac{1}{x} \int_{0}^{x} \{ \sum_{n: \lambda_n \leq y} a_n \} dy \to s, \quad \text{i.e.} \quad \frac{1}{\lambda(x)} \int_{0}^{x} s(y) d\lambda(y) \to s \quad (x \to \infty)$$

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This book was published in 1915; Riesz was Hungarian. The title page contains a memorable dedication ending *Auctores hostes idemque amici*: The authors, enemies and yet friends.
(cf. Karamata [Kar]). It turns out that for $p \geq 1$

$$R_p := R(\exp\{\int_1^n x^{-1/p}dx\}, 1)$$

gives a summability method equivalent to the method $M_p$ obtained from
moving averages over intervals $[n, n + cn^{1/p}]$ ([BinT], [Bin7]), and that each
of $R_p$, $M_p$ is interchangeable with $C_{1/p}$ in Case (i) of the theorem above.

Self-neglecting functions.

More generally, call a (continuous) function $\phi$ self-neglecting, or Beurling
slowly varying, $\phi \in SN$, if

$$\phi(x + t\phi(x))/\phi(x) \to 1 \quad (x \to \infty) \forall t \in \mathbb{R}$$

(see [BinGT, §2.11], [BinO1], [BinO2] for some of the extensive background
on such functions). For $\phi(.) \uparrow \infty$, write

$$\Phi(x) := \int_1^x dt/\phi(t), \quad \lambda(x) := \exp\{\Phi(x)\}, \quad \psi := \phi^{-}. \quad \text{Then [BinG, Th. 1] moving averages over intervals of lengths } t\phi(x) \text{ converge to } s (s_n \to s (M(\phi)) \text{ iff } s_n \to s R(\lambda). \text{ In the probabilistic context, one}

needs the condition that } \psi := \phi^{-} \in BI \text{ (is of bounded increase: } [BinG, \quad \text{§2.1.2]). Then } X_n \to \mu \text{ (M(} \phi) \text{)} \text{ iff } E[\phi(X)] < \infty \text{ and } E[X] = \mu \text{ [BinG, Th. 3]. The condition of bounded increase is needed to preserve LLN-}
type behaviour, in which the details of the distribution are lost in the limit,
leaving only the mean. If } \psi \text{ grows too fast (e.g., exponentially), the entire}
distribution survives the passage to the limit (Erdős-Rényi law, or almost-
sure non-invariance principle: see e.g. [BinG] for details and references).

Random-walk and circle methods.

My own entry into this field was triggered by writing the Mathematical
Reviews to two papers, by Diaconis and Stein [DiaS] and by Schmaal, Stam
and de Vries [Sch]. For the Euler and Borel methods, their weights are
probabilities (of Poisson or binomial distributions). For matrix methods of
summation $A = (a_{nk})$, call $A$ a random-walk method if

$$a_{nk} = P(S_n = k)$$

for some random walk $S_n = X_1 + \ldots + X_n$, with $X_n$ iid. Quite a lot can be
said about these methods; see e.g. Korevaar [Kor] VI, [Bin1-7].
If one uses normal (Gaussian) approximation in the weights above, one obtains a (discrete version of) a Valiron method. These, together with Taylor methods and Meyer-König methods, are included in Meyer-König’s family of circle methods or Kreisverfahren [MeyK]. Some of this material is included in [Har3, VIII], though not all: Hardy died in 1947 and both appeared in 1949. The motivation here is analytic continuation of power series (hence ‘circle’, for the circle of convergence). See [Bin3] and [BinT] for more detail here.

Gap theorems.

The Borel method involves a continuous passage to the limit, the Euler (and random-walk) methods a discrete passage to the limit. It turns out that there are subtle differences. The Borel and Euler methods have gap or high-indices theorems [Har3, §7.13]: if \( s_n = \sum a_k \) with the \( a_n \) vanishing off some suitably sparse subsequence \( (n_k) \), then no order condition is needed in the relevant Tauberian theorem. The ‘Euler gap theorem’ for the Euler method \( (n_{k+1} - n_k \geq h\sqrt{n_k} \text{ for some } h > 0) \) is due to Meyer-König and Zeller in 1956 [MeyK-Z1] (cf. [MeyK-Z3]); the corresponding ‘Borel gap’ theorem is due to Gaier in 1965 [Gai] (following partial results by Erdős and others). That there is no (pure) gap theorem for the discrete Borel method was shown by Meyer-König and Zeller in 1960 [MeyK-Z2] by functional-analytic methods. For more detail and further references, see [Bin6, §3.1].

The Chen-Stein method.

A powerful way of approximating by the normal distribution was introduced by Charles Stein in 1970 (unpublished), and extended to Poisson approximation by Chen in 1976 [Che]. A monograph on the Chen-Stein method is given by Barbour, Holst and Janson [BarHJ]. This is clearly highly relevant to the probabilistic number theory of §2; see Harper [Harp].

Law of the iterated logarithm (LIL) and of the single logarithm (LSL).

We have discussed LIL above, in connection with Erdős’ comment to Diaconis about Hardy. It concerns averages, and so is a strong limit theorem for \( C_1 \), holding in \( L_1 \). There are analogues for \( C_{1/\alpha} \), holding in \( L_\alpha \), for \( \alpha > 1 \), but with only a single rather than an iterated logarithm; see e.g. [Bin5,6,7].

The logarithmic method.

Above, one has summability methods requiring more than the Cesàro method (or in probabilistic terms, one deals with a refinement of the Kolmogorov strong law). The classical logarithmic method is a prime example of where one requires less (or deals with coarsenings of the Kolmogorov strong law). This is useful in a number of areas, particularly probability theory and analytic number theory ([Har3, §§4.16, 5.16]; Bingham and Gashi [BinGa];
4. Hardy spaces.

Hardy spaces grew out of a series of papers by Hardy on functions analytic on the unit disk \( \mathbb{D} \), and the boundary values (on the unit circle \( \mathbb{T} \)) that they have under suitable growth restrictions. The first monograph treatment, by Duren in 1970 [Dur], cites three papers by Hardy and eight by Hardy and Littlewood, over the period 1913-1941 (distributed over Volumes II, III and IV of Hardy’s Collected Papers). Hardy spaces \( H_p \) were named and studied by F. Riesz in 1923. The focus originally was on individual functions, but with the development of functional analysis (a milestone here being Banach’s 1932 book [Ban]), spaces of functions began to be studied for their own sake — by A. E. Taylor and by S. S. Walters in 1950, by W. Rudin and by R. P. Boas in 1955, and others. In addition to [Dur], other useful books are by Hoffman in 1962 [Hof], Garnett in 1981 [Gar], and Rosenblum and Rovnyak in 1985 [RosR]. The general area of spaces of holomorphic functions is nowadays called holomorphic spaces for short; see e.g. the survey by Sarason in 1998 [Sar] and the book containing it, by Axler et al. [Axl].

Hardy spaces were studied (with the notation \( H_p \) and with references to Hardy, but without the name Hardy spaces) by Marcinkiewicz and Zygmund in 1940 [MarZ]. For background on the links between the Polish school and probability, see e.g. Holgate [Hol], [Bin9], [Bin11]. For modern textbook accounts, see e.g. Durrett [Durr, Ch. 6,7], Bass [Bas, Ch. IV].

Another probabilistic strand grew largely out of the work of Gabor Szegő (1895-1985), in papers from the period 1915-21 (so a little after Hardy’s work began) and later of 1952. An important development was Kolmogorov’s major paper of 1941 on stationary sequences in Hilbert space [Kol2]. This paper introduced the Kolmogorov isomorphism theorem (KIT), which (for a stationary stochastic process \( X = (X_n) \) in discrete time) enables one to pass at will between the time domain, in which one focusses explicitly on time \( n \), and the frequency domain, in which one focusses on the spectral measure \( \mu \) on \( \mathbb{T} \). The next year, Cramér [Cra2] showed that \( X \) is the sequence of Fourier coefficients of a random measure \( Y \) on \( \mathbb{T} \), \( Y \) and \( \mu \) being linked by

\[
E[(dY(\theta))^2] = d\mu(\theta).
\]

This shows clearly the link with Itô calculus of 1944, where for Brownian motion \( B = (B_t) \) one has \( E[(dB_t)^2] = dt \). For textbook expositions making the links here clear, see e.g. Janson [Jan] and Kallenberg [Kal].
In 1948, Beurling [Beu] continued the use of Hilbert-space methods to study stationary discrete-time stochastic processes, and showed that the Lebesgue decomposition $\mu(d\theta) = w(\theta)d\theta + \mu_s(d\theta)$ of $\mu$ has an interpretation in terms of invariant subspaces of Hilbert space (under the shift, $n \mapsto n + 1$), the decomposition of $\mu$ corresponding to the role of the remote past. The nice situation is that where the remote past is forgotten (the tail $\sigma$-field at $-\infty$ is trivial). For this, one needs Szegö’s condition ($\log w \in L_1$). This visibly involves entropy, and is in fact related to the Gibbs variational principle of statistical mechanics. One also needs $\mu_s = 0$. There is a great deal more to be said, but for brevity we refer here to the writer’s recent surveys on this subject, [Bin12] (one dimension), [Bin13] (higher dimensions), [Bin14] (statistics), to [BinIK], [KasB], [BinM], to the books on orthogonal polynomials on the unit circle and Szegö’s theorem by Simon [Sim1,2], and the books by Nikolskii [Nik1,2] on the shift operator and spectral function theory.

5. Hardy.

Hardy was a pure mathematician through and through, famous (or notorious, depending on one’s point of view) for his lack of interest in applied mathematics; he left a fine account of his views of mathematics in the autobiographical *A mathematician’s apology* [Har2]. It is not quite true that Hardy showed no interest in probability: he wrote on (what is now known as) the Hardy-Weinberg law in Mendelian genetics in 1908 ([Har1]; the result was obtained independently by Weinberg [Wei], also in 1908; [Fel, V.5]). Indeed, J. M. Hammersley (1920-2004)\(^5\) was fond of saying that it was for the Hardy-Weinberg law that Hardy would ultimately be principally remembered. Hardy appears in §2 in his work of 1917 with Ramanujan, in the middle of the transitional period between the introduction of measure theory by Lebesgue in 1901 and its successful harnessing to the service of probability theory by Kolmogorov in 1933 (see e.g. [Bin9] for an account of this interesting transitional period). But Williams’ description of probability theory pre-measure theory as a shambles [Wil, §1.3] is fair, and it is understandable that Hardy chose not to divert his energies from his many interests in pure mathematics to study probability theory.

Hardy began publication in 1899, before Lebesgue’s work. His series *Notes on some points in the integral calculus*, I-LXIV (NIC in Hardy’s works)\(^5\)Hammersley taught the present writer when he was an undergraduate at Trinity College, Oxford, 1963-66)
covered 1901-29, beginning with the Riemann integral and later using the Lebesgue integral when needed. Hardy’s views on the Lebesgue integral are available directly, e.g. in [HarLP, §6.1] of 1934. This passage is quoted by Burkhill in the Preface to his Cambridge Tract [Bur] of 1951, the first standard work in English on the Lebesgue integral (though Burkhill there acknowledges his debt to the treatment [Tit, X-XII] in the 1932 book by Titchmarsh, a pupil of Hardy); here Burkhill (writing in 1949, after Hardy’s death in 1947) writes ‘I wish to record that one of my many debts to G. H. Hardy lay in his encouragement to write this tract’. But Hardy’s views on the Lebesgue integral are also available vividly indirectly, through his influence on Wiener ($\S7$ below).

Despite his openness to the Lebesgue integral, Hardy seems to have taken little interest in functional analysis. It fell to Frank Smithies (1912-2002), whose Cambridge PhD of 1936 on integral equations was written under Hardy’s supervision [Smi], to introduce the then still new functional analysis to the UK in the late 1930s (from France – Laurent Schwartz and co-workers)¹.

As noted in $\S4$, Hardy spaces are a prime example of the intermingling of complex analysis and functional analysis – or, as they are sometimes called, ‘hard’ (real or complex) and ‘soft’ (abstract or functional) analysis. Hardy is often quoted as making comments in which his preference for the first is clear.


J. E. Littlewood (1885-1977) was the younger partner (to G. H. Hardy (1877-1947) in the Hardy-Littlewood collaboration (95 papers, from 1912 to 1948) – the longest and most famous in mathematical history. Both wrote excellent and readable autobiographical books, [Har2] and [Lit1]. Their styles were complementary: Hardy normally did the initiating and writing-up; Littlewood bore the brunt of the hard problem-solving.

Volume I of Littlewood’s Collected Papers [Lit2] contains his collaboration with R. E. A. C. Paley (1907-33): three papers, ‘Theorems on Fourier series and power series’, a short one of 1931 and two long ones of 1937 after Paley’s death (recall that the book by Paley and Wiener [PalW] was also posthumous). The resulting Littlewood-Paley theory has connections with martingales, and is important in both analysis and probability. See e.g. Stein

¹Smithies lectured to the present writer on functional analysis while he was a research student at Cambridge, 1966-69.
Volume II of [Lit2] (§5, Probabilistic analysis, p.1578) contains five papers with his pupil A. C. (Cyril) Offord (1906-2000) on real roots of random polynomials. This subject was continued, by Kac [Kac2, I], by Offord’s pupil J. E. A. Dunnage, and by Dunnage’s pupil K. Farahmand.

7. Wiener.

The American Norbert Wiener (1894-1964) was a wonderfully broad and prolific mathematician. In addition to a number of excellent mathematics book, and four volumes of collected papers (1976-1985), Wiener wrote a two-volume autobiography, [Wie2,3]. In [Wie2, XIV] (Emancipation: Cambridge, June 1913 – April, 1914) Wiener gives (as well as his observations on the Cambridge and the England of that time) an account of what a revelation Hardy’s lectures (on the Lebesgue integral) were. In [Wie3, Ch. 7] (An unofficial Cambridge don, 1931-2), he describes his dealings with a much older Hardy and with Littlewood; this led to his writing his book [Wie1] on the Fourier integral.7 He also comments on how Hardy teased him, saying that he was really a pure mathematician at heart, and only did applicable work to curry favour with his engineering colleagues at MIT. In [Wie3, Ch. 8] (Back home, 1932-33) Wiener describes his short but productive collaboration with Paley, before Paley’s untimely death in a skiing accident and the posthumous book [PalW].

8. Pitt.

In 1937-38 H. R. Pitt (1914-2005; Sir Harry Pitt from 1978), a pupil of Hardy, visited MIT and worked with Wiener, mainly on Tauberian theory. This year had a transformative effect on Pitt, and led to Pitt’s form of Wiener’s (Tauberian) theorem (or Wiener-Pitt theorem). For details, see [BinH].

Pitt was primarily an analyst, best known for his work on Tauberian theorems, but he retained the interest in probability theory that he acquired from Wiener during his annus mirabilis of 1937-8. Pitt later supervised C. W. J. Granger (1934-2009; Nobel Prize in Economics, 2003; Sir Clive Granger, 2005). Granger’s work has been extremely influential in econometrics and

7This book, incidentally, makes the case for teaching the Fourier integral via the Lebesgue integral so convincingly that, despite the extra difficulties that learning measure theory and the measure-theoretic integral presents, I find it both saddening and surprising that it is so often not taught to UK undergraduates.
time series; see e.g. [Bin12, §1].


The Swedish mathematician Harald Cramér (1893-1985) took his PhD at Stockholm in 1917 on Dirichlet series, under Marcel Riesz, who had excellent links with Hardy ([HarRi]; cf. [Bin14]). Cramér worked originally on analytic number theory (see [Cra5, I]), but moved to probability and statistics as he had a family to support and took up a job as an actuary. From his autobiographical account [Cra4, §3.1, §3.2]: he visited Cambridge in 1920, where he worked under Hardy and met Wiener [Wie2, 64]. ‘During a visit to England in 1927, I saw my old teacher and friend G. H. Hardy. When I told him that I had become interested in probability theory, he said that there was no mathematically satisfactory book in English on this subject, and encouraged me to write one.’ This task actually took a decade. Cramér was much influenced by Kolmogorov’s Grundbegriffe of 1933. His book finally appeared as a Cambridge Tract in 1937 [Cra1]. It was the first clear treatment of measure-theoretic probability theory in English, and it owes its existence to Hardy’s encouragement. Cramér followed this up in 1946 with [Cra3], the first book to synthesise both the measure-theoretic probability of Kolmogorov and the pioneering work of Fisher in statistics.

10. Daniell.

P. J. Daniell (1889-1946) was Professor of Mathematics at Sheffield from 1923 (and the last person to be Senior Wrangler in Cambridge, in 1909). He is remembered for the Daniell integral (work of 1918), and the Daniell-Kolmogorov theorem, the foundational existence theorem for a stochastic process (Daniell, 1919; Kolmogorov, Grundbegriffe, 1933). A fascinating account of Daniell’s life, times and work is given by Aldrich [Ald]. His paper has many points of contact with this and with [Dia], and is strongly recommended to all readers of this piece.

11. Others.

Perhaps the other two people most relevant to British probability before WWII were Sir Harold Jeffreys (1891-1989) and R. A. (Sir Ronald) Fisher (1890-1962). Jeffreys was primarily a geophysicist. He was the first to realise that the Earth’s core is molten iron\(^8\). He wrote a book Theory of probability

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\(^8\)(but was a scientific conservative: he was a vehement opponent of Continental Drift)
[Jef], largely devoted to Bayesian statistics (on which see e.g. [Bin10]). Fisher was both the greatest statistician ever and a geneticist of the first rank. His theory of ‘fiducial probability’ is often described as Fisher’s great mistake. Both had a traditional background in applied mathematics, and (perhaps as a result) both actively disliked measure-theoretic probability. Both were in Cambridge for much of their careers, but in different areas (Fisher was Professor of Genetics, Jeffreys worked in mathematics, then geophysics, and was then Professor of Astronomy), and did not get on.

Most of the key figures in UK probability (and statistics) after WWII were much affected by the War itself. Turing led the attack on Enigma at Bletchley Park; Good was Turing’s statistical assistant. D. G. Kendall was an analyst at Oxford before the War, with leanings towards astronomy; he learned statistics from Bartlett for his war work (on rockets, with Bartlett), and became the leading UK probabilist after the war, first in Oxford (where he was a colleague of J. M. Hammersley, who was in the Royal Artillery in WWII), and then in Cambridge (see e.g. [Bin8]). Daniell’s health was badly damaged by his war work. Barnard worked under Bartlett on the then new area of Operational Research.

For an account of the vexed passage from Lebesgue’s work on measure theory to Kolmogorov’s Grundbegriffe on probability theory, with particular reference to the Russian, Polish, French, German, Italian and American schools, see [Bin9]. The contributions of the UK were mainly statistical and philosophical rather than mathematical (e.g. Keynes and Ramsey, in addition to the names above).

For an entertaining glimpse into probability in Cambridge, 1957-67, see Kingman [Kin]: it was a shambles. Fortunately, UK probability is in better health nowadays.

12. Conclusion.

The Hardy-Littlewood school of analysis in general, and G. H. Hardy in particular, emerge from all this not as probabilists, but as analysts aware of and sympathetic to probability. In particular, Hardy’s benign and creative influence on Wiener and Cramér, and on Pitt and others within the UK, deserves our thanks.

This piece is offered as a sequel and complement to Diaconis’ G. H. Hardy

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9Both men worked on remanent magnetism – study of the Earth’s history as locked in by the magnetisation of rocks. This has clear implications for Continental Drift.
and probability?? [Dia]. I hope that all readers will read (or re-read) this first.

References

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Mathematics Department, Imperial College, London SW7 2AZ. n.bingham@ic.ac.uk