M5F22 Math. Finance MSc: EXAMINATION SOLUTIONS 2016-17

Q1. *Reinsurance and limited liability.*
(i) *Limited liability.*

Lloyd's of London pre-dates limited liability (which developed in the mid-19th C.). The Lloyd's participants, or *names*, had unlimited liability, and were liable for the full extent of losses, irrespective of their investment or their assets. This changed, following the Lloyd's scandal of the 1990s.

Insurance is now done (and most was before the Lloyd's scandal) by *limited liability companies*. So for these, the possibility or *ruin* is crucial. Not only would this wipe out the company, its assets and expertise, the jobs of its employees etc., but it would leave policy-holders without cover. **[5]**

(ii) *Reinsurance.* Because a run of large claims could bankrupt an insurance company, companies seek to lay off large risks – to reinsure – insure themselves – with larger, specialist reinsurance companies.

The question arises as to where reinsurance companies re-reinsure themselves ... This raises the modern form of Juvenal's question: Quis custodiet ipsos custodes – Who guards the guards? Who reinsures the reinsurers? **[5]**

(iii) *Regulation.* It is in the interest of some industries to agree to cover each other's liabilities in the event of a bankruptcy – e.g., *travel firms*. If a travel firm goes bust, leaving large numbers of people stranded abroad, or unable to travel on a foreign holiday booked and paid for, this would destroy public confidence in the whole industry – *unless* other firms, by prior agreement, step in to cover. This works well – a form of *self-regulation* (like the press).

As motor insurance is compulsory by law, motor insurance companies are regulated by the state, giving some protection against bankruptcy. **[5]**
[Basel I, II and III are relevant to regulation generally, less to insurance.]

(iv) *Lender of last resort.* When a big concern is facing bankruptcy, the knock-on effects for the nation's economy may be so severe that it may be in the national interest to intervene. This is done by the *lender of last resort* – the *central bank* (Bank of England (BoE) in the UK, Federal Reserve (Fed) in the US, European Central Bank (ECB) in the European Union (EU), etc.), acting on behalf of the state (or e.g. EU). This raises questions as to the relationship between the central bank and the national government: how *independent* of government is the central bank, and so how free of *political* pressures? **[5]**

[Mainly seen – lectures]

Q2. (i) *Volatility*. The Black-Scholes formula involves the parameter σ (where σ^2 is the *variance of the stock per unit time*), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information - how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches: [**2**]
(ii) *Historic volatility*: here we use Time Series methods to estimate σ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is, and given enough data we can estimate σ in this way. [**3**]
Implied volatility: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone). [**3**]
Volatility surface. If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model. Volatility graphed against price S, or strike K, typically shows a *volatility smile* (or even smirk). Graphed against S and K in 3 dimensions, we get the *volatility surface*. [**3**]
(iii) Volatility dependence is given by *vega* $:= \partial c/\partial\sigma$ for calls, $\partial p/\partial\sigma$ for puts. From the Black-Scholes formula (which gives the price explicitly as a function of σ), one can check by calculus that $\partial c/\partial\sigma > 0$, and similarly for puts (or, use the result for calls and put-call parity). *Options like volatility*. The more uncertain things are (the higher the volatility), the more valuable protection against adversity becomes (the higher the option price). [**3**]
(iv) The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account periods when the markets are open and closed [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This suggests that *trading itself is one of the main causes of volatility*. [**3**]

The introduction of a small transaction tax would have the effect of decreasing trading. This would increase market stability: trading is one of the causes of volatility; options like volatility. So trading tends to cause an increase in trading in options, and so on. Ultimately this tends to induce market instability. So conversely, market stability would benefit from a reduction in trading volumes caused by a transaction tax. [**3**]
[Mainly seen – lectures]

Q3. *American options.*
(i) The discounting rate per unit time is $1+\rho$. With 'up' and 'down' factors $1+u$, $1+d$ and 'up' and 'down' probabilities q, $1-q$, the discounted price process is a martingale iff $(1+u)q+(1+d)(1-q)=1+\rho$:

$$uq+d(1-q)=\rho; \qquad (u-d)q=\rho-d: \qquad q=\frac{\rho-d}{u-d}. \qquad [\mathbf{2}]$$

(ii) To price the American put in this (CRR) binomial-tree model:
1. Draw a binary tree showing the initial stock value S and with the right number, N, of time-intervals. **[2]**
2. Fill in the stock prices: after one time interval, these are Su (upper) and Sd (lower); after two, Su^2, Sud and Sd^2; after i time-intervals, $Su^j d^{i-j}$ at the node with j 'up' steps and $i-j$ 'down' steps. **[2]**
3. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j}=\max[K-Su^j d^{N-j},0]$) from the option at the terminal nodes (where the values of the European and American options coincide). **[2]**
4. Work back down the tree one time-step. Fill in (a) the 'European' value at the penultimate nodes as the discounted values of the terminal values, under the risk-neutral (P^*, Q) measure – 'q times upper right plus $1-q$ times lower right'; (b) the 'intrinsic' (early-exercise) value; (c) the American put value as the higher of these. **[2]**
5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'. **[2]**
6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the set where the early-exercise value is the higher, the rest the 'continuation region'. **[2]**
(iii) *Connection with the Snell envelope.*
Let $Z=(Z_n)_{n=0}^N$ be the payoff on exercising at time n. To price Z_n, by U_n say, so as to avoid arbitrage: we work backwards in time. Recursively:

$$U_N := Z_N, \qquad U_{n-1} := \max(Z_{n-1}, \frac{1}{1+\rho}E^*[U_n|\mathcal{F}_{n-1}]), \qquad [\mathbf{2}]$$

the first alternative corresponding to early exercise, the second to the discounted expectation under P^* (or Q), as usual. With discounting,

$$\tilde{U}_N=\tilde{Z}_N, \qquad \tilde{U}_{n-1}=\max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}]): \qquad [\mathbf{2}]$$

$(\tilde{U}_n)$ is the *Snell envelope* of the discounted payoff process $(\tilde{Z}_n)$. **[2]**
[Seen – lectures]

Q4 *Optional stopping; optimal stopping.*
(i) *Optional Stopping Theorem (OST*

The OS) states that for a stopping time T and a supermartingale $X = (X_n)$, if one of the following conditions holds:
(a) T is bounded;
(b) X is bounded;
(c) $E[T] < \infty$ and $X_n - X_{n-1})$ is bounded –
then X_T is integrable, and $E[X_T] \leq E[X_0$.

If here X is a martingale, the $E[X_T] = E[X_0]$. **[4]**

(ii) The OST fails in the gambling context where one either plays 'the martingale' (bet on heads till one is first ahead, then quit; double stakes whenever one loses), or just betting (heads, say) till one is first ahead and then quit. In each case, if T is our stopping time, $T < \infty$ a.s., so eventual gain of one is certain. So $S_T = 1$, but $S_0 = 0$. Neither strategy is viable in practice: no bound can be put on losses before eventual gain, so one needs unlimited capital (or credit)(and unlimited time without doubling stakes). This shows that boundedness (or integrability) restrictions are needed for practical trading (or gambling!) strategies. **[4]**

(iii) With a uniformly integrable (UI) martingale (mg), $X = (X_n)$, all the randomness is concentrated in the final value – or *closing value*, X_∞ (infinite time-horizon) or X_T (finite time-horizon T). Then the mg converges to this closing value, a.s. and in L_1:

$$E[X_\infty|\mathcal{F}_n] \to X_\infty \text{ or } E[X_T|\mathcal{F}_n] \to X_T \quad a.s. \text{ and in } L_1.$$

This applies in the Risk-Neutral Valuation Formula, with $T < \infty$, X_T the payoff $h(S_T)$, h the payoff function and S_T the terminal stock price

$$V_t = e^{-r(T-t)} E^*[h(S_T)|\mathcal{F}_t]. \qquad \textbf{[5]}$$

Optimal stopping.
(iv) Here the aim is to optimise one's expected payoff, given only current information. A typical application is to American puts, where one uses the *least supermartingale majorant* of (smallest supermg dominating) the discounted payoff process $\tilde{Z}$. This is the discounted Snell envelope $\tilde{U}$ of $\tilde{Z}$. **[4]**

(v) With infinite time-horizon, one again meets the least supermartingale majorant. (One doesn't need a finite expiry-time to work back from, unlike the Snell envelope.) The optimal stopping problem now involves a *free boundary-value problem.* **[3]**

[Seen – lectures]

Q5. (i) *The exponential martingale for Brownian motion.*
The MGF of $X \sim N(\mu, \sigma^2)$ is $E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, $(*)$, given. For $B = (B_t)$ Brownian motion (BM), write

$$M_t := \exp\{\theta B_t - \frac{1}{2}\theta^2 t\}.$$

Then with $\mathcal{F} = (\mathcal{F}_t)$ the Brownian filtration, for $s \leq t$,

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E[\exp\{\theta B_t - \frac{1}{2}\theta^2 t\}|\mathcal{F}_s] \\ &= E[\exp\{\theta(B_s + (B_t - B_s)) - \frac{1}{2}\theta^2 s - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s] \\ &= \exp\{\theta B_s - \frac{1}{2}\theta^2 s\}.E[\exp\{\theta(B_t - B_s)) - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s], \end{aligned}$$

taking out what is known. The first term on the right is M_s. The conditioning in the second term can be omitted, by independent increments of BM. But $B_t - B_s \sim N(0, t-s)$, which has MGF

$$E[\exp\{\theta(B_t - B_s)\}] = \exp\{\frac{1}{2}\theta^2(t-s)\}$$

(by $(*)$, with $\mu \mapsto 0$, $\theta^2 \mapsto t-s$, $t \mapsto \theta$). So the second term on RHS 1:

$$E[M_t|\mathcal{F}_s] = M_s.$$

So M is a martingale. // **[8]**

(ii) By the normal MGF (given), $M_Y(t) = E[e^{tY}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$. Taking $t = 1$, $M_Y(1) = E[e^Y] = \exp\{\mu + \frac{1}{2}\sigma^2\}$. As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}. \qquad [\mathbf{2}]$$

(iii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations

$$dS = S(\mu dt + \sigma dB), \qquad (GBM)$$

with B Brownian motion. This has solution (quote – Itô's Lemma)

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normal, so S_t is lognormal. **[5]**
(iv) In Girsanov's theorem, we have a process

$$L_t := \exp\{\int_0^t \mu_s dB_s - \frac{1}{2}\int_0^t \mu_s^2 ds\} \qquad (0 \leq t \leq T),$$

with (μ_s) adapted (with $\int_0^T \mu_s^2 ds < \infty$), and $L = (L_t)$ a martingale. By (i), this martingale condition is satisfied for μ_t constant, identically equal to μ, interpreted as the interest rate – of the risky stock, which Girsanov's theorem transforms by change of measure to r, the riskless interest rate. So (i) enables us to apply Girsanov's theorem, and so obtain the Black-Scholes formula in continuous time. **[5]**
[(i), (iii), (iv), seen, Lectures; (ii), seen – Problems]

Q6. *Poisson process; compound Poisson process.*
(i) The *Poisson process* $N = (N_t)$ of *rate* λ has stationary independent increments, and N_t is Poisson with parameter λt (so mean and variance λt). The *compound Poisson process* $CP(\lambda, F)$ is the process $S = (S_t)$, where (X_n) are independent copies with law F, independent of $N = (N_t)$, with $S_t := \sum_{n \leq N_t} X_n$. **[2, 2]**
(ii) The characteristic function (CF) of $CP(\lambda, F)$ follows from

$$\psi(u) = E[e^{iuS_t}] = E[\exp\{iu(X_1 + \ldots + X_{N_t})\}]$$
$$= \sum_n E[\exp\{iu(X_1 + \ldots + X_{N_t})\} | N_t = n].P(N_t = n)$$
$$= \sum_n e^{-\lambda t} \lambda^n t^n / n!.E[\exp\{iu(X_1 + \ldots + X_n)\}]$$
$$= \sum_n e^{-\lambda t} \lambda^n t^n / n!.(E[\exp\{iuX_1\}])^n$$
$$= \sum_n e^{-\lambda t} \lambda^n t^n / n!.\phi(u)^n = \exp\{-\lambda t(1 - \phi(u))\}. \qquad [\mathbf{4}]$$

(iii) Given N_t, $S_t = X_1 + \ldots + X_{N_t}$ has mean $N_t EX = N_t \mu$ and variance $N_t \; var \; X = N_t \sigma^2$. As N_t is Poisson with parameter λt, N_t has mean λt and variance λt. So by the Conditional Mean Formula,

$$E[S_t] = E[E[S_t | N_t]] = E[N_t \mu] = \lambda t \mu. \qquad [\mathbf{2}]$$

By the Conditional Variance Formula,

$$\begin{aligned} var \; S_t &= E[var(S_t | N_t)] + var \; E[S_t | N_t] \\ &= E[N_t var \; X] + var([N_t \; E[X]) \\ &= E[N_t].var \; X + var \; N_t.(EX)^2 \\ &= \lambda t(E[X^2] - (E[X])^2) + \lambda t.(E[X])^2 \end{aligned}$$
$$= \lambda t E[X^2] = \lambda t(\sigma^2 + \mu^2). \qquad [\mathbf{5}]$$

(iv) As the convolution of two Poisson distributions $P(\lambda)$ and $P(\mu)$ is Poisson $P(\lambda + \mu)$, a Poisson distribution with large parameter is the convolution of many (Poisson) distributions, each with finite mean and variance. So by the Central Limit Theorem, it is approximately normal. So by (ii), for λt large,

$$Z := (S_t - \lambda t \mu)/\sqrt{\lambda t E[X^2]} \sim N(0, 1) : \quad S_t \sim \lambda t \mu + Z\sqrt{\lambda t E[X^2]},$$

giving a normal approximation to the total-claims distribution. **[5]**
[Seen – lectures]