M3F22 SOLUTIONS 8. 8.12.2017

Q1: *Lognormal distribution.* By the normal MGF,

$$M_Y(t) = E[e^{tY}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Taking $t = 1$,

$$M_Y(1) = E[e^Y] = \exp\{\mu + \frac{1}{2}\sigma^2\}.$$

As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}.$$

(ii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations

$$dS = S(\mu dt + \sigma dB), \qquad (GBM)$$

with B Brownian motion. This has solution (from Itô's lemma, VI.6 – as in VII.1)

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normally distributed, so S_t is lognormal.

Q2 *Brownian covariance.* For $s \leq t$,

$$B_t = B_s + (B_t - B_s), \qquad B_s B_t = B_s^2 + B_s(B_t - B_s).$$

Take expectations: on the left we get $cov(B_s, B_t)$. The first term on the right is, as $E[B_s] = 0$, $var(B_s) = s$. As BM has independent increments, $B_t - B_s$ is independent of B_s, so

$$E[B_s(B_t - B_s)] = E[B_s].E[B_t - B_s] = 0.0 = 0.$$

Combining, $cov(B_s, B_t) = s$ for $s \leq t$. Similarly, for $t \leq s$ we get t. Combining, $cov(B_s, B_t) = \min(s, t)$.

Q3 *Brownian scaling*. With $B_c(t) := B(c^2 t)/c$,

$$cov(B_c(s), B_c(t)) = E[B(c^2 s)/c.B(c^2 t)/c] = c^{-2}\min(c^2 s, c^2 t) = \min(s,t) = cov(B_s, B_t).$$

So B_c has the same mean 0 and covariance $\min(s,t)$ as BM. It is also (from its definition) continuous, Gaussian, stationary independent increments etc. So it has all the defining properties of BM. So it *is* BM.

So BM is a *fractal*: it reproduces itself if time and space are scaled together in this way. This is why if we "zoom in and blow up" a Brownian path, it still looks like a Brownian path – however often we do this. By contrast, if we zoom in and blow up a smooth function, it starts to look straight (because it has a tangent).

Specialising to the zero set Z of BM B, this too is a fractal because B is.

Q4 *Time-inversion*. Like BM, X is continuous (where it is defined – away from 0) and Gaussian. Its covariance is

$$cov(X_s, X_t) = cov(sB(1/s), tB(1/t)) = st\,cov(B(1/s), B(1/t))$$

$$= st\min(1/s, 1/t) = \min(t,s) = \min(s,t).$$

So as X has the same covariance as BM, X *is* BM. But BM is continuous everywhere, not just away from 0. So X is continuous at 0 too, and has $X(0) = 0$ as BM does. So

$$X_t \to 0 \quad (t \to 0): \qquad tB(1/t) \to 0 \quad (t \to 0): \qquad B(t)/t \to 0 \quad (t \to \infty).$$

Q5. We calculate $\int B(u)dB(u)$. We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} B(0) = 0 & \text{if} \quad 0 \le u \le t/n, \\ B(t/n) & \text{if} \quad t/n < u \le 2t/n, \\ \vdots & \vdots \\ B((n-1)t/n) & \text{if} \quad (n-1)t/n < u \le t. \end{cases}$$

By definition,

$$\int_0^t B(u)dB(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} B(kt/n)(B((k+1)t/n) - B(kt/n)).$$

Replacing $B(kt/n)$ by $\frac{1}{2}(B((k+1)t/n)+B(kt/n))-\frac{1}{2}(B((k+1)t/n)-Bkt/n))$, the RHS is

$$\sum \frac{1}{2}(B((k+1)t/n) + B(kt/n)).(B((k+1)t/n) - B(kt/n))$$

$$-\sum \frac{1}{2}(B((k+1)t/n) - B(kt/n)).(B((k+1)t/n) - B(kt/n)).$$

The first sum is $\sum \frac{1}{2}(B((k+1)t/n)^2 - B(kt/n)^2)$, which telescopes (as a sum of differences) to $\frac{1}{2}B(t)^2$ $(B(0) = 0)$. The second sum is $\frac{1}{2}\sum(B(k+1)t/n) - B(kt/n))^2$, an approximation to the quadratic variation of B on $[0, t]$, which tends to $\frac{1}{2}t$ by Lévy's theorem on the QV. Combining,

$$\int_0^t B(u)dB(u) = \frac{1}{2}B(t)^2 - \frac{1}{2}t.$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of B.

NHB