

## M3F22 SOLUTIONS 4. 10.11.2017

Q1. Since  $f$  is clearly non-negative, to show that  $f$  is a (probability density) function (in two dimensions), it suffices to show that  $f$  integrates to 1:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1, \quad \text{or} \quad \int \int f = 1.$$

Write

$$f_1(x) := \int_{-\infty}^{\infty} f(x, y) dy, \quad f_2(y) := \int_{-\infty}^{\infty} f(x, y) dx.$$

Then to show  $\int \int f = 1$ , we need to show  $\int_{-\infty}^{\infty} f_1(x) dx = 1$  (or  $\int_{-\infty}^{\infty} f_2(y) dy = 1$ ). Then  $f_1, f_2$  are densities, in *one* dimension. If  $f(x, y) = f_{X,Y}(x, y)$  is the *joint* density of *two* random variables  $X, Y$ , then  $f_1(x)$  is the density  $f_X(x)$  of  $X$ ,  $f_2(y)$  the density  $f_Y(y)$  of  $Y$  ( $f_1, f_2$ , or  $f_X, f_Y$ , are called the *marginal* densities of the *joint* density  $f$ , or  $f_{X,Y}$ ).

To perform the integrations, we have to *complete the square*:

$$(1 - \rho^2)Q \equiv \left[ \left( \frac{y - \mu_2}{\sigma_2} \right) - \rho \left( \frac{x - \mu_1}{\sigma_1} \right) \right]^2 + (1 - \rho^2) \left( \frac{x - \mu_1}{\sigma_1} \right)^2$$

(reducing the number of occurrences of  $y$  to 1, as we intend to integrate out  $y$  first). Then (taking the terms free of  $y$  out through the  $y$ -integral): for

$$c_x := \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1),$$

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x - \mu_1)^2/\sigma_1^2)}{\sigma_1 \sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_2 \sqrt{2\pi} \sqrt{1 - \rho^2}} \exp\left(\frac{-\frac{1}{2}(y - c_x)^2}{\sigma_2^2(1 - \rho^2)}\right) dy. \quad (*)$$

The integral is 1 ('normal density'). So

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x - \mu_1)^2/\sigma_1^2)}{\sigma_1 \sqrt{2\pi}},$$

which integrates to 1 ('normal density'), proving

**Fact 1.**  $f(x, y)$  is a joint density function (two-dimensional), with marginal density functions  $f_1(x), f_2(y)$  (one-dimensional). So we can write

$$f(x, y) = f_{X,Y}(x, y), \quad f_1(x) = f_X(x), \quad f_2(y) = f_Y(y).$$

**Fact 2.**  $X, Y$  are normal:  $X$  is  $N(\mu_1, \sigma_1^2)$ ,  $Y$  is  $N(\mu_2, \sigma_2^2)$ . For, we showed  $f_1 = f_X$  to be the  $N(\mu_1, \sigma_1^2)$  density above, and similarly for  $Y$  by symmetry.

**Fact 3.**  $EX = \mu_1, EY = \mu_2, \text{var}X = \sigma_1^2, \text{var}Y = \sigma_2^2$ .

This identifies four out of the five parameters: two means  $\mu_i$ , two variances  $\sigma_i^2$ . Next, recall conditional densities [L9]:

$$f_{Y|X}(y|x) := f_{X,Y}(x,y)/f_X(x) = f_{X,Y}(x,y) / \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy.$$

Returning to the bivariate normal:

**Fact 4.** The conditional distribution of  $y$  given  $X = x$  is  $N(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$ .

*Proof.* Go back to completing the square (or, return to (\*) with  $\int$  and  $dy$  deleted):

$$f(x, y) = \frac{\exp(-\frac{1}{2}(x - \mu_1)^2/\sigma_1^2) \cdot \exp(-\frac{1}{2}(y - c_x)^2/(\sigma_2^2(1 - \rho^2)))}{\sigma_1\sqrt{2\pi} \cdot \sigma_2\sqrt{2\pi}\sqrt{1 - \rho^2}}.$$

The first factor is  $f_1(x)$ , by Fact 1. So,  $f_{Y|X}(y|x) = f(x, y)/f_1(x)$  is the second factor:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1 - \rho^2}} \exp\left(\frac{-(y - c_x)^2}{2\sigma_2^2(1 - \rho^2)}\right),$$

where  $c_x$  is the linear function of  $x$  given below (\*). //

This not only completes the proof of Fact 4 but gives Facts 5 and 6:

**Fact 5.** The conditional mean  $E(Y|X = x)$  is *linear* in  $x$ :

$$E(Y|X = x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

**Fact 6.** The conditional variance of  $Y$  given  $X = x$  is

$$\text{var}(Y|X = x) = \sigma_2^2(1 - \rho^2).$$

**Fact 7.** The correlation coefficient of  $X, Y$  is  $\rho$ .

*Proof.*

$$\rho(X, Y) := E\left[\left(\frac{X - \mu_1}{\sigma_1}\right)\left(\frac{Y - \mu_2}{\sigma_2}\right)\right] = \int \int \left(\frac{x - \mu_1}{\sigma_1}\right)\left(\frac{y - \mu_2}{\sigma_2}\right) f(x, y) dx dy.$$

Substitute for  $f(x, y) = c \exp(-\frac{1}{2}Q)$ , and make the change of variables  $u := (x - \mu_1)/\sigma_1$ ,  $v := (y - \mu_2)/\sigma_2$ :

$$\rho(X, Y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int \int uv \exp\left(\frac{-[u^2 - 2\rho uv + v^2]}{2(1-\rho^2)}\right) dudv.$$

Completing the square,  $[u^2 - 2\rho uv + v^2] = (v - \rho u)^2 + (1 - \rho^2)u^2$ . So

$$\rho(X, Y) = \frac{1}{\sqrt{2\pi}} \int u \exp\left(-\frac{u^2}{2}\right) du \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int v \exp\left(-\frac{(v - \rho u)^2}{2(1-\rho^2)}\right) dv.$$

Replace  $v$  in the inner integral by  $(v - \rho u) + \rho u$ , and calculate the two resulting integrals separately. The first is zero ('normal mean', or symmetry), the second is  $\rho u$  ('normal density'). So ('normal variance')

$$\rho(X, Y) = \frac{1}{\sqrt{2\pi}} \cdot \rho \int u^2 \exp\left(-\frac{u^2}{2}\right) du = \rho.$$

This completes the identification of all five parameters in the bivariate normal distribution: two means  $\mu_i$ , two variances  $\sigma_i^2$ , one correlation  $\rho$ .

**Fact 8.** The bivariate normal law has *elliptical contours*.

For, the contours are  $Q(x, y) = \text{const}$ , which (Galton) are ellipses.

*Moment Generating Function (MGF)*. Recall (see e.g. Haigh (2002), 102-6)  $M(t)$ , or  $M_X(t)$ ,  $:= E(e^{tX})$ . For  $X$  normal  $N(\mu, \sigma^2)$ ,

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}} \int e^{tx} \exp(-\frac{1}{2}(x - \mu)^2/\sigma^2) dx.$$

Change variable to  $u := (x - \mu)/\sigma$ :

$$M(t) = \frac{1}{\sqrt{2\pi}} \int \exp(\mu t + \sigma u t - \frac{1}{2}u^2) du.$$

Completing the square,

$$M(t) = e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int \exp(-\frac{1}{2}(u - \sigma t)^2) du \cdot e^{\frac{1}{2}\sigma^2 t^2},$$

or  $M_X(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$  (recognising that the central term on the right is  $1 -$  'normal density'). So  $M_{X-\mu}(t) = \exp(\frac{1}{2}\sigma^2 t^2)$ . Then (check)  $\mu = EX =$

$M'_X(0)$ ,  $\text{var}X = E[(X - \mu)^2] = M''_{X-\mu}(0)$ .

Similarly in the bivariate case: the MGF is  $M_{X,Y}(t_1, t_2) := E \exp(t_1X + t_2Y)$ . In the bivariate normal case:

$$\begin{aligned} M(t_1, t_2) &= E(\exp(t_1X + t_2Y)) = \int \int \exp(t_1x + t_2y)f(x, y)dx dy \\ &= \int \exp(t_1x)f_1(x)dx \int \exp(t_2y)f(y|x)dy. \end{aligned}$$

The inner integral is the MGF of  $Y|X = x$ , which is  $N(c_x, \sigma_2^2, (1 - \rho^2))$ , so is  $\exp(c_x t_2 + \frac{1}{2}\sigma_2^2(1 - \rho^2)t_2^2)$ . By Fact 5  $c_x t_2 = [\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)]t_2$ , so

$$M(t_1, t_2) = \exp(t_2\mu_2 - t_2\frac{\sigma_2}{\sigma_1}\mu_1 + \frac{1}{2}\sigma_2^2(1 - \rho^2)t_2^2) \int \exp([t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}]x)f_1(x)dx.$$

Since  $f_1(x)$  is  $N(\mu_1, \sigma_1^2)$ , the inner integral is a normal MGF, which is thus  $\exp(\mu_1[t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}] + \frac{1}{2}\sigma_1^2[. . .]^2)$ . Combining the two terms and simplifying:

**Fact 9.** The joint MGF is

$$M_{X,Y}(t_1, t_2) = M(t_1, t_2) = \exp(\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]).$$

**Fact 10.**  $X, Y$  are independent if and only if  $\rho = 0$ .

Proof. For densities:  $X, Y$  are independent iff the joint density  $f_{X,Y}(x, y)$  *factorises* as the *product* of the marginal densities  $f_X(x).f_Y(y)$  (see e.g. Haigh (2002), Cor. 4.17).

For MGFs:  $X, Y$  are independent iff the joint MGF  $M_{X,Y}(t_1, t_2)$  *factorises* as the *product* of the marginal MGFs  $M_X(t_1).M_Y(t_2)$ . From Fact 9, this occurs iff  $\rho = 0$ . Similarly with CFs, if we prefer to work with them. //

*Note.* We can re-write Fact 5 above as

$$E[Y|X] = \mu_2 + \frac{\rho\sigma_1}{\sigma_2}(X - \mu_1).$$

So as  $E[X] = \mu_1$ , this illustrates the Conditional Mean Formula (II.4 Property 6, L10):

$$E[E[Y|X]] = \mu_2 + \frac{\rho\sigma_1}{\sigma_2}(E[X] - \mu_1) = \mu_2 = E[Y].$$

Similarly, Fact 6 illustrates the Conditional Variance Formula.

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