m3f22soln10

M3F22 SOLUTIONS 10. 15.12.2017

Q1. *Exponential distributions and renewal.*
(i) Recall the exponential law $E(\lambda)$: density and distribution function

$$f(x) = \lambda e^{-\lambda x}, \qquad F(x) = 1 - e^{-\lambda x} \quad (x > 0).$$

The Laplace transform of f (Laplace-Stieltjes transform, or LST, of F) is

$$\begin{aligned} \hat{F}(s) &= \int_{[0,\infty)} e^{-\lambda s} dF(x) \\ &= \int_{[0,\infty)} \lambda e^{-\lambda s}.e^{-\lambda x} dx \\ &= \lambda/(\lambda + s). \end{aligned}$$

The LST of the nth convolution F^{*n} of F is the nth power of this. Summing over n: for the renewal function

$$U(x) = \sum_{n=0}^{\infty} F^{*n}(x),$$

its LST is

$$\hat{U}(s) = \sum_{n=0}^{\infty} (\lambda/(\lambda + s))^n = \frac{1}{\big(1 - \lambda/(\lambda + s)\big)} = \frac{1}{s/(\lambda + s)} = (\lambda + s)/s :$$

$$\hat{U}(s) = 1 + \lambda/s.$$

Now with δ_0 the Dirac measure at 0 (probability measure with all its mass 1 at the origin 0), its LST is 1. The LST of Lebesgue measure on $(0, \infty)$ (the measure with mass x on $(0, x)$) is, putting $u := sx$,

$$\int_0^\infty e^{-sx} dx = \int_)^\infty e^{-u} du/s = 1/s.$$

Combining, for $F = E(\lambda)$,

$$U(x) = \delta_0(x) + \lambda x \qquad (x \geq 0).$$

Interpretation. The first term 1 here just says that, by definition, there is always a renewal at time 0 (we always start with a new item). After that, because the hazard rate for $E(\lambda)$ is the constant λ, the expected number of renewals in $(0, x)$ is λx.
(ii) Recall that the mean μ of $E(\lambda)$ is, putting $u := \lambda x$ as above,

$$\mu = \int_0^\infty x.f(x)dx = \int_0^\infty x.\lambda e^{-\lambda x}dx = \int_0^\infty ue^{-u}du/\lambda = 1/\lambda$$

(the integral is $\Gamma(2) = 1! = 1$, or check by integration by parts). So the Renewal Theorem holds:

$$E(t) = E[N(t)] = 1 + t\lambda = 1 + t/\mu \sim t/\mu \quad (t \to \infty).$$

Blackwell's renewal theorem holds here with equality, as

$$U(x + h) - U(x) = h\lambda = h/\mu.$$

The Key Renewal Theorem holds, as

$$Z(t) = (z * U)(t) = \int_0^t z(u).u(t-u)du = \int_0^t z(u).\lambda du$$

$$= \int_0^t z(u)du/\mu \to \int_0^\infty z(u)du/\mu \quad (t \to \infty).$$

Q2. *Gamma distributions and Renewal.*
(i) *Density.*

$$\int f = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda x}.\lambda^\alpha x^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u}.u^{\alpha-1}du = 1,$$

putting $u := \lambda x$ and using the definition of the Gamma function.
(ii) *Mean.*

$$\begin{aligned}
\mu &= \int xf(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x.e^{-\lambda x}.\lambda^\alpha x^{\alpha-1}dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty (u/\lambda).e^{-u}.u^{\alpha-1}du = \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty e^{-u}.u^\alpha du \\
&= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \\
&= \alpha/\lambda.
\end{aligned}$$

(iii) *LST*: $\hat{f}(s)$.

$$\begin{aligned}
\hat{f}(s) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-sx}.e^{-\lambda x}.\lambda^\alpha x^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-(\lambda+s)x}.\lambda^\alpha x^{\alpha-1}dx \\
&= = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-u}\Big(\frac{\lambda}{\lambda+s}\Big)^\alpha u^{\alpha-1}du \quad (u := (\lambda+s)x) \\
&= \Big(\frac{\lambda}{\lambda+s}\Big)^\alpha.
\end{aligned}$$

(iv) *LST*: $\hat{U}(s)$.

$$\begin{aligned}
\hat{U}(s) &= \frac{1}{1-\hat{f}(s)} = \frac{1}{1-\Big(\frac{\lambda}{\lambda+s}\Big)^\alpha} \\
&= \frac{(\lambda+s)^\alpha}{(\lambda+s)^\alpha-\lambda^\alpha} = \frac{(\lambda+s)^\alpha}{\lambda^\alpha\Big[\Big((1+\frac{s}{\lambda}\Big)^\alpha-1\Big]}.
\end{aligned}$$

(v) As $s \downarrow 0$,

$$(\lambda+s)^\alpha/\lambda^\alpha \to 1, \qquad \Big[\Big((1+\frac{s}{\lambda}\Big)^\alpha - 1\Big] \sim \sigma.s/\lambda = \mu s,$$

by the (generalised) Binomial Theorem (Newton: see M3H). So

$$\hat{U}(s) \sim \frac{1}{\mu s} \qquad (s \downarrow 0).$$

(vi) So by HLK with $\rho = 1$,

$$U(x) \sim x/\mu \qquad (x \to \infty),$$

giving the Renewal Theorem in this case.

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