m3f22soln1.tex

SOLUTIONS 1. 14.10.2016

Q1 (*Random sums; Wald's identity*).

$$\begin{aligned} R(s) &:= E[s^Y] = E[s^{X_1+\cdots+X_N}] \\ &= \sum_{n=0}^{\infty} E[s^{X_1+\cdots+X_N}|N=n]P(N=n) \\ &= \sum q_n E[s^{X_1+\cdots+X_n}] \\ &= \sum q_n (P(s))^n \\ &= Q(P(s)) \end{aligned}$$

(this uses the Conditional Mean Formula (Ch. III), also known as the Law of Total Expectation). Differentiating, $R'(s) = E[Ys^Y]$, $E[Y] = R'(1)$, and similarly
$E[N] = Q'(1), E[X] = P'(1)$.

$$R'(s) = Q'(P(s)).P'(s):$$

$$R'(1) = Q'(1)P'(1): \qquad E[Y] = E[N].E[X].$$

This is (a special case of) *Wald's identity* (Abraham Wald (1902-1950) in 1944).

Q2 (*Compound Poisson: CF, mean and variance*).
(i) The characteristic function (CF) follows from

$$\begin{aligned} \psi(t) = E[e^{itY}] &= E[\exp\{it(X_1+\ldots+X_N)\}] \\ &= \sum_n E[\exp\{it(X_1+\ldots+X_N)\}|N=n].P(N=n) \\ &= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1+\ldots+X_n)\}] \\ &= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n = \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\ &= \exp\{-\lambda(1-\phi(t))\}. \end{aligned}$$

(ii) If X has CF $\phi(t) = E[e^{iXt}]$: differentiating,

$$\phi'(t) = E[iXe^{iXt}] : \qquad \phi'(0) = iE[X]; \qquad E[X] = -i\phi'(0);$$

$$\phi''(t) = E[-X^2 e^{iXt}] : \qquad \phi''(0) = -E[X^2]; \qquad E[X^2] = -\phi''(0).$$

Differentiate the CF ψ of Y:

$$\psi'(t) = \psi(t).\lambda\phi'(t); \qquad \psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

By above, ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives

$$E[Y] = \lambda\mu.$$

Thus the mean of the random sum $Y := X_1 + \cdots + X_N$ is the product of the means of X (short for a typical X_i) and N:

$$E[Y] := E[X_1 + \cdots + X_N] = E[X].E[N].$$

Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2 = \lambda(\mu^2 + \sigma^2).$$

Q3. If $S = (S_t)$ is $CP(\lambda, F)$, $S_t := X_1 + \cdots + X_{N(t)}$, $N(t) \sim P(\lambda t)$. So this follows from Q2 on replacing λ by λt.

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