m3f33prob8

M3F22 PROBLEMS 8. 1.12.2017

Q1 (*The lognormal distribution*). The *lognormal distribuktion* $LN(\mu, \sigma^2)$ is defined as the distribution of $X := e^Y$, where $Y \sim N(\mu, \sigma^2)$. v
(i) Show that

$$E[X] = \exp\{\mu + \frac{1}{2}\sigma^2\}.$$

(ii) Explain why stock prices in the Black-Scholes model are lognormal.

Q2 *Brownian covariance*. The *covariance* of random variables X, Y is

$$cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Show that for $B = (B_t)$ Brownian motion (BM), its covariance is

$$cov(B_s, B_t) = min(s, t).$$

We quote that for a Gaussian process (one all of whose finite-dimensional distributions are Gaussian, such as BM), the process is characterised by its mean function and covariance function (so mean 0 and covariance $\min(s, t)$ characterise BM).

Q3 *Brownian scaling*. With $c > 0$ and B Brownian motion, show that B_c, where

$$B_c(t) := B(c^2 t)/c,$$

has the same covariance function $\min(s, t)$ as Brownian motion B. Deduce that (as B_c is also continuous and Gaussian) that B_c *is* Brownian motion. It is formed from B by *Brownian scaling*.

Deduce that B is *self-similar*: it reproduces itself it time and space are both scaled as above. We call such a self-similar process a *fractal*.

If Z is the zero-set of B and Z_c that of B_c, deduce that Z, Z_c are fractals.
Note. 1. Those with experience of computer graphics will recall 'zooming in and blowing up' – selecting a portion of a graphic (of particular interest), and blowing it up to full screen. The essence of a fractal is that it *looks just the same* under this process – even if we iterate it. By contrast, a reasonably smooth function (differentiable, say) looks *quite different* – it starts to look

straight, *because it has a tangent.*

2. You all know calculus, and you may not have met fractals before (or at least, not often). You might suspect on this basis that 'a typical function' is smooth (as in calculus), and that fractals are rare and pathological: the first examples of (what we now call) fractals were constructed to be continuous and nowhere differentiable (try drawing one!). On the contrary: in a way that can be made precise (Baire category), *a typical continuous function is nowhere differentiable.*

Q4 *Time inversion.* For B BM, and

$$X_t := tB(1/t) \qquad (t \neq 0),$$

$X = (X_t)$ is also BM.

Deduce or prove otherwise that for B BM

$$B(t)/t \to 0 \qquad (t \to \infty).$$

Q5. By writing

$$\int_0^t B(u)dB(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} B(kt/n)(B((k+1)t/n) - B(kt/n))$$

$$= \sum \frac{1}{2}(B((k+1)t/n) + B(kt/n)).(B((k+1)t/n) - B(kt/n))$$

$$-\sum \frac{1}{2}(B((k+1)t/n) - B(kt/n)).(B((k+1)t/n) - B(kt/n)),$$

or otherwise, show that

$$\int_0^t B(u)dB(u) = \frac{1}{2}B(t)^2 - \frac{1}{2}t.$$

Comment on the difference between this Itô calculus result and ordinary (Newton-Leibniz) calculus.

NHB