m3f33prob1

M3F22 PROBLEMS 10. 15.11.2017

Q1. *Exponential distributions and renewal.*
(i) For the exponential law $E(\lambda)$ with parameter $\lambda > 0$ (density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, 0 otherwise), show that the renewal function is

$$U(x) = 1 + \lambda x.$$

Interpret the two terms in this result.
(ii) Show that this agrees with the renewal theorem, Blackwell's renewal theorem and the key renewal theorem in the $E(\lambda)$ case.

Q1. *Gamma distributions and Renewal.*
For $\alpha, \lambda > 0$, the Gamma distribution $\Gamma(\alpha, \lambda)$ is defined by

$$f(x) := \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} \quad (x > 0).$$

Show tha:
(i) this is a density;
(ii) the mean is $\mu = \alpha/\lambda$;
(iii) the LST of f is

$$\hat{f}(s) = \Big(\frac{\lambda}{\lambda + s}\Big)^\alpha;$$

(iv) the LST of the renewal function $U(x)$ is

$$\hat{U}(s) = \frac{(\lambda + s)^\alpha}{\lambda^\alpha[(1 + s/\lambda)^\alpha - 1]};$$

(v)

$$\hat{U}(s) \sim \frac{1}{\mu s} \qquad (s \downarrow 0).$$

(vi) Deduce that the Renewal Theorem holds here:

$$U(x) \sim x/\mu \qquad (x \to \infty).$$

You may quote (Hardy-Littlewood-Karamata Tauberian theorem, HLK) that for $\rho \geq 0$,

$$U(x) \sim cx^\rho/\Gamma(1 + \rho) \quad (x \to \infty) \;\Leftrightarrow\; \hat{U}(s) \sim c/s^\rho \quad (s \downarrow 0).$$

NHB