M3F22 EXAMINATION 2017-18

Q1. (i) For a probability distribution F on $(0, \infty)$, define the *lack of memory property*. Find the functional equation satisfied by the tail $\overline{F}(x)$ of $F(x)$ with this property.
(ii) Solve this functional equation, quoting any results you need.
(iii) Explain the relevance of $E(\lambda)$ (the exponential distribution with parameter $\lambda > 0$) to the Poisson process with parameter (or rate) $\lambda > 0$, and the modelling of insurance claims.
(iv) Comment on the limitations of this model.

Q2. (i) Prove the converse part of the No-Arbitrage Theorem: that if an equivalent martingale measure P^* exists, then there is no arbitrage.
(ii) Why is the direct part more difficult?
(iii) Describe the use of the No-Arbitrage Theorem in pricing assets such as options.
(iv) To what extent do arbitrage opportunities exist in real markets?
(v) How would market be affected by arbitrage opportunities on any sizeable scale?

Q3. The *Theta*, Θ, of an option is defined as the time-derivative of its value.
(i) Given the Black-Scholes formula for the price c_t of European calls,

$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

with S_t the stock price at time $t \in [0, T]$, K the strike price, r the riskless interest rate, σ the volatility and

$$d_1 := [\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}, \qquad d_2 := d_1 - \sigma\sqrt{T-t}:$$

(a) find Θ and show that $\Theta < 0$;
(b) interpret this.
(ii) Given the corresponding Black-Scholes formula for the price p_t of European puts,

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1),$$

(a) find Θ, and show that this time Θ can change sign.
(b) Describe the conditions under which Θ will be positive, and interpret this.

You may quote that $Ke^{-r(T-t)}\phi(d_2) = S\phi(d_1)$.

Q4. (i) In the Cramér-Lundberg risk model, define the *safety loading* $\rho > 0$ in terms of the premium rate $c > 0$, the claim rate $\lambda > 0$ and the mean claim size $\mu \in (0, \infty)$.
(ii) State without proof the *key renewal theorem*.
(iii) Define the *Lundberg coefficient* $r > 0$ (assumed to exist).
(iv) Define the *Esscher transform* $F \mapsto G$ of the claim-size distribution F.
(v) Given the integral equation for the ruin probability $\psi(u)$ with initial capital u:

$$\psi(u) = \frac{1}{(1+\rho)} \int_u^\infty \frac{(1-F(x))}{\mu} dx + \frac{1}{(1+\rho)} \cdot \int_0^u \psi(u-x) \frac{(1-F(x))}{\mu} dx, \quad (*)$$

obtain the *Cramér estimate of ruin* $\psi(u) \sim Ce^{-ru}$ as $u \to \infty$, with C a constant.
(vi) Comment briefly on where the assumptions here may be unrealistic.

Q5 (Mastery question). (i) Define *geometric Brownian motion* (GBM) with parameters μ, σ, and find the stochastic differential equation it satisfies.
(ii) Give the financial interpretation of this.
(iii) Find the quadratic variation of GBM.
(iv) Find the expected quadratic variation of GBM.

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