m3f22cwsoln(1718)

SMF ASSESSED COURSEWORK: SOLUTIONS. 4.12.2017

(i) With initial stock-price S, the martingale probability p^* satisfies

$$S = Sup^* + Sd(1-p^*); \qquad 1 = \frac{6}{5}p^* + \frac{5}{6}(1-p^*) :$$

$$1 = \frac{5}{6} + p^*(\frac{6}{5} - \frac{5}{6}) = \frac{5}{6} + p^* \cdot \frac{11}{30} : \qquad \frac{1}{6} = p^* \cdot \frac{11}{30} : \qquad p^* = \frac{5}{11}.$$

(ii) In an obvious notation, write the nodes as:
Time 0: 0;
Time 1: $1u$, $1d$;
Time 2: $2uu$, $2ud$ ($= 2du$), $2dd$;
Time 3: $3uuu$, $3u$, $3d$, $3ddd$.

The corresponding stock prices are:
Time 0: 60;
Time 1: $1u$, 72; $1d$, 50.
Time 2: $2uu$, $72 \times 6/5 = 86.4$; $2ud = 2du$, 60; $2dd$, $50 \times 5/6 = 250/6 = 41.666...$
Time 3: $3uuu$, $86.4 \times 6/5 = 103.68$; $3u$ ($= 1u$), 72; $3d$ ($= 1d$), 50;
$3ddd$, $250/6 \times 5/6 = 1250/36 = 34.7222$.

With strike 50:
The payoffs (values) at time 3 are: $3uuu$, 53.68; $3u$, 22; $3d$, $3dd$, 0.
The values at the three Time 2 nodes are "$p^* \times$ upper RH neighbour + $(1-p^*)\times$ lower RH neighbour", giving (with $p^* = 5/11$ and the values above):
2uu:

$$\frac{5}{11} \times 53.68 + \frac{6}{11} \times 22 = 24.4 + 12 = 36.4;$$

2ud = 2du:

$$\frac{5}{11} \times 22 + \frac{6}{11} \times 0 = 5 \times 2 = 10;$$

2dd: 0.
The values at the two Time 1 nodes are:
1u:

$$\frac{5}{11} \times 36.4 + \frac{6}{11} \times 10 = \frac{182 + 60}{11} = \frac{241}{11} = 22;$$

1d:
$$\frac{5}{11} \times 10 + \frac{6}{11} \times 0 = \frac{50}{11} = 4.5454....$$
The value at the Time 0 node (the root) is the value of the option:

$$0: \qquad C = \frac{5}{11} \times 22 + \frac{6}{11} \times \frac{50}{11} = 10 + \frac{300}{121} = 10 + 2.4793 = 12.4793.$$

Check/Alternative method. By the discrete Black-Scholes formula, the value of the option is

$$C = \Big(\frac{5}{11}\Big)^3 \times 53.68 + 3.\Big(\frac{5}{11}\Big)^2 \Big(\frac{6}{11}\Big) \times 22 = \frac{125 \times 53.68 + 75 \times 132}{1,331}$$

$$= \frac{6,710 + 9,900}{1,331} = \frac{16,610}{1,331} = \frac{1,510}{121} = 12.4793.$$

Note. The fractions here are exact. But they cannot be quoted as prices. Prices are given in terms of *money*; no currency is denominated in submultiples of 11 (still less 11^2 or 11^3). So conversion to decimal form is needed, obviously to *at least* two decimal places (pounds and pence, etc.). But give *four*. The extra two decimal places (*basis points* in the City) are money!

(iii) Early exercise of the American option is never better than continuing, by Merton's theorem. So the early-exercise region is empty, and the continuation region is the whole tree. But early exercise may be worse; we proceed to find out how much worse, and where. ('Worse' here is from a financial point of view. Recall from lectures that American calls may well be exercised early for economic reasons.)

The American option needs to be valued at each node, not just the root. Its value is the higher of
(a) the (European, or intrinsic) values above, and
(b) early-exercise value – excess of stock price over strike if positive, 0 if not.
At expiry (too late to exercise early), (a) and (b) agree.
They also agree at the Time 2 nodes (both 36.4 at 2uu, 10 at 2ud, 0 at 2dd).
They agree at node 1u (both 22).
They differ at node 1d ((a) is 4.5454..., (b) is 0), and at node 0 ((a) is 12.4794, (b) is 10), with (a) higher – so continuation is better – in both cases. So: early exercise is never better, but is only worse at nodes 0 and 1d.

Note that the calculation above gives a numerical example of Merton's theorem in action.

(iv) With strike 70, the payoffs at time 3 are: 3uuu, 33.68; 3u, 2; 0 else. European (intrinsic) values:
2uu: 16.4 (down 20 from 36.4, as both values being averaged are down 20).
2ud = 2du:

$$\frac{5}{11} \times 2 + \frac{6}{11} \times 0 = \frac{10}{11} = 0.9090...;$$

2dd: 0.
1u:

$$\frac{5}{11} \times 16.4 + \frac{6}{11} \times \frac{10}{11} = \frac{82 \times 11 + 60}{11} = \frac{962}{121} = 7.9504;$$

1d:

$$\frac{5}{11} \times \frac{10}{11} + \frac{6}{11} \times 0 = \frac{50}{121} = 0.4132....$$

0: The value of the option at time 0 is

$$\frac{5}{11} \cdot \frac{962}{121} + \frac{6}{11} \cdot \frac{50}{121} = \frac{4,810 + 300}{1,331} = \frac{5,110}{1,331} = 3.8392,$$

or

$$\frac{5}{11} \times 7.9504 + \frac{6}{11} \times 0.4132 = \frac{39.752 + 2.4792}{11} = \frac{42.2312}{11} = 3.8392.$$

Early-exercise values: 2uu, 16.4; 2ud, 2dd, 0; 1u, 2; 1d, 0; node 0, 0.
Taking the maximum: again, early exercise is never better (Merton's theorem again), and worse at 0 and 1d, *but is now worse at nodes 1u and 2ud also.*
Check/Alternative method. By the discrete Black-Scholes formula, the option value is

$$C = \Big(\frac{5}{11}\Big)^3 \times 33.68 + 3.\Big(\frac{5}{11}\Big)^2\Big(\frac{6}{11}\Big).2 = \frac{125 \times 33.68 + 75 \times 12}{11^3}$$
$$= \frac{4,210 + 900}{1,331} = \frac{5,110}{1,331} = 3.8392.$$

(v) Interpretation: with a higher strike, the situation is worse for the option holder than before (so of course he got the option more cheaply). One should expect that early exercise will now be worse at *more* nodes than before (if anything): continuing provides more opportunities to escape from an unfavourable situation. In a bad situation, "give chance a chance to work in one's favour": give the cards a chance to break for you, as bridge-players say. NHB