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## Chapter II: ECONOMIC AND FINANCIAL BACKGROUND

### 1. Time Value of Money; discounting

Recall first the definition of *simple interest*, at $100x\%$ p.a. With one unit (a pound, say) invested for one year, at the end of the year we have:
$1+x$ with interest calculated yearly;
$(1+\frac{x}{2})^2$ with interest calculated twice-yearly;
$(1+\frac{x}{n})^n$ with interest calculated (or *compounded*) $n$ times per year (12 for monthly, 365 for daily, etc.).

We would like the compounding to be done as often as possible (so as to be able to exploit the essence of compound interest – getting interest on the interest). To do this, we use

$$(1+\frac{x}{n})^n \uparrow e^x \qquad (n\to\infty) \qquad (x>0):$$

*exponential growth is the limit of compound interest as the interest is compounded continuously.*
*Note.* 1. Exponential growth (or decay) occurs widely in nature, as well as in economics/finance! – e.g.
*Growth of biological populations.* A species expanding into a new environment with no limit on resources can and does grow exponentially (rabbits in Australia, bacteria overwhelming a weakened organism, etc.).
*Radioactive decay.* The amount of material decays exponentially (the rate is measured by the *half-life* – the time it takes for half of what is left to decay).
2. Exponential growth is extremely rapid, and hence is *destabilizing.* This is one of the underlying causes behind financial crises (such as the Credit Crunch, or Crash of 2008). The wonder is that the financial world is as stable as it is, most of the time.
3. Possibly for this reason, there were religious prohibitions on lending money at interest – 'usury', as the (Christian, Catholic) Church called it in the Middle Ages (there are still such reservations in parts of the Islamic world). But:
4. Without the inducement of interest, it is difficult to justify taking the risk of lending money. Without lending to finance investment and business activity, credit dries up, business activity slows down and the economy shrinks (as recently).

*Discounting.*

With economic activity driven by money lent and borrowed at interest, one must distinguish between prices in nominal terms and prices in real terms, whenever any length of time has elapsed. Otherwise, one is not comparing like with like. (This gives a good way to distinguish 'good' from 'bad' politicians, with an election pending!) We do this by *discounting*. With $r$ the current short-term interest rate (short rate), one discounts over a time-interval of length $t$ by a factor $e^{-rt}$. In much of the mathematics below, we will *discount everything*, and if $S_t$ denotes a price at the present time $t$, its real value at a later time $T$ in present (time $t$) prices is $e^{-r(T-t)}S_t$. This is done for accounting purposes in Net Present Value (NPV) calculations. These are also used as a tool in assessing whether or not an investment should be made. We will not pursue this in detail here: part of the message of Real Options (VII.5) is that the NPV approach to investment decisions is misleading.

For most purposes, we use as interest rate the rate for *riskless* lending/borrowing. Of course, this overlooks a lot of complications! For example:

1. *Borrowing v. lending.* Traditionally, banks made their money on the difference between the interest they paid to depositors on their savings accounts, and the higher rate they charged borrowers, whether private customers on overdrafts or businesses on borrowing for investment. However, banks not only lend, they borrow, from each other (LIBOR = London Inter-Bank Offer Rate, etc.).[1] The difference between such borrowing and lending rates introduces *friction* into the markets. One often (e.g., in the standard Black-Scholes theory below) neglects this, and works with an idealized market.
2. *Risky v. riskless.* Major government debt (Treasury bonds in the US, gilts in the UK) is traditionally regarded as riskless. However, governments do occasionally default on their debts (Mexico and Russia, within fairly recent memory). Banks also occasionally default (though government institutions – the Fed in US, the Bank of England in UK – may intervene as 'lender of last resort' to rescue a troubled institution – e.g. Northern Rock, UK, 2008).
3. *Fixed rate or variable.* Interest rates vary, and one may need to reflect this by using a function $r(t,T)$ rather than a constant $r$.
4. *Stochastic or deterministic.* It may be that the variability $r(t,T)$ is better

[1] In the recent LIBOR scandal – known inevitably as the Lie-bor scandal – it emerged that individual traders in the participating banks (which only included the large and reputable ones) were systematically distorting the quotes at which they purported to lend or borrow. This is a blatant form of market manipulation; it has led to large fines already, and to heavy (and expensive) reputational damage to the banks.

modelled as random, or *stochastic* – see Ch. IV below, and VII.5.
5. *Term structure of interest rates.* The mathematics of this – the *term structure of interest rates* – is very interesting and important. But it is (a lot) harder than most of the mathematics we shall be doing here.
6. *Stock markets v. money markets (bond markets).* There are two principal kinds of financial markets, stock markets and money markets. In stock markets, what are traded – bought and sold – are stocks and shares – part-ownership in a company, both its physical assets, its intangible assets (goodwill, brand name etc.) and its future earnings, which may be released to shareholders as *dividends*. Investors may buy for capital appreciation, dividends or both. The prototype of the relevant mathematics is the *Black-Scholes theory*, of which more below in Ch. V and VII. On the other hand, in money markets, money is being borrowed (usually by governments, or large companies) over time, to be repaid later at specified rates of interest and/or with specified payments or *coupons*.

The borrower binds himself to repay with interest, and the agreement to do so is called a bond, hence *bond markets* (Shylock, *Merchant of Venice*: I will have my bond!). Bonds issued by leading governments are regarded as safe, or 'gilt-edged', hence the term *gilts* in the UK. Their US equivalents are called Treasury bonds, or T-bonds.

## 2. Economics and Finance; Utility

Economics is largely concerned with questions such as supply and demand (for everything: commodities, manufactures, capital, labour, etc.). Much of economics deals with how prices are determined.

Finance is concerned with the borrowing and lending of money needed to engage in economic activity. We will be dealing here with the more mathematical side of this. In (mathematical) finance, one takes prices largely as given, and deals with questions such as how to price options (real or otherwise). Thus Finance is a part of Economics – and a fairly small and specialized part at that.

Small economic agents are *price takers*. They have no power to influence prices, which they can either take or leave – but equally, do have the power to enter the market without thereby moving the market against them. By contrast, large economic agents are *price makers*. They do have the power to influence prices – but against this, are visible, and so are vulnerable, when forced to enter the market through weakness (example: the financial authorities of a major country, defending the value of its currency by buying it on the market).

*Trading.*

The price of common everyday items is accurately known at any given time. Anyone trying to sell at a higher price than the 'going rate' would tend to lose market share to cheaper competitors, and eventually have to reduce towards the going rate or go out of business. At the other extreme, items never bought and sold do not have a price – are literally priceless (Buckingham Palace, Westminster Abbey, the Houses of Parliament, ...).

In between the two, prices are known but not accurately – to within some interval. This is the *bid-ask spread* – the gap between the price at which a market participant will buy, and the (higher) price at which he will sell.

When large trades are made, prices jump. This is because the large trade affects the current balance between supply and demand, and the price is the level at which markets clear – that is, at which supply and demand balance. One can model such a market by means of a stochastic process (Ch. IV) with jumps (prototype: Poisson process).

With small trades, one can look at things at two different levels of detail. 'From a distance', prices seem to move continuously – so can be modelled by a stochastic process which is continuous (prototype: Brownian motion, VI.3). But imagine a trader spending a trading day tracking the price movement of a heavily traded stock under normal market conditions. From a distance, price movement looks continuous, but close up, prices move by lots of little jumps – the effects of the individual small trades, and how they briefly affect the current balance between supply and demand. Such movement of prices by 'lots of little jumps' is called *jitter*.

*Utility.*

A pound is worth much more to a poor man than to a rich man. For ordinary people, a 10% increase in income might well give an extra 10% of satisfaction. So for small amounts $x$ of money, we can think of utility as being the same as money. But to a billionaire, it would be hardly noticeable. To model this, one uses a *utility function*, $U(.)$, which measures how much *utility* – genuine use – money is to the economic agent in question: income $x$ gives utility $U(x)$. The effect above is called the *Law of diminishing returns* (or *diminishing utility* – $U$ is strictly increasing, but its graph bends below the line $y = x$, and indeed is typically bounded above. Often used here are the *Inada conditions* (Ken-Ichi Inada in 1963):

$U(0) = 0$; $U \in C^1$; $U \uparrow$; $U'' < 0$ (so $U$ concave); $U'(0+) = \infty$; $U'(\infty-) = 0$.

A guiding principle that is often used here is that each economic agent should seek to *maximize his expected utility.* This approach goes back to John Von Neumann and Oscar Morgenstern in 1947 (in their classic book *Theory of games and economic behaviour*, one of 'the books of the last century'), and earlier to F. P. Ramsey (1906-1930) in 1931 (posthumously).

*Loss.*

This is often looked at the other way round. One uses a *loss function* – which can usually be thought of as a negative of utility. One then seeks to *minimize one's expected loss.*

*Arbitrage.*

An *arbitrage* opportunity (see II.6) is the possibility of extracting riskless profit from the market. In an orderly market, this should not be possible – at least, to a first approximation. For, an arbitrage opportunity is 'free money'; arbitrageurs will take this, in unlimited quantities – until the person or institution being so exploited is driven from the market (bankrupt or otherwise). In view of this, we make the assumption that the market is *free of arbitrage* – is *arbitrage-free*, or has *no arbitrage*, NA.

*Idealized markets.*

Various assumptions are commonly made, in order to bring to bear the tools of mathematics on the broad field of economic/financial activity. All are useful, but valid to a first approximation only.

1. No arbitrage (NA).
2. No transaction costs or transaction taxes.
3. Same interest rates for borrowing and lending.
4. Unlimited liquidity (the ability to turn goods into money, and vice versa, at the currently quoted prices).
5. No limitations of scale.

Markets satisfying such assumptions will be called *perfect*, or *frictionless* – unrealistic in detail, but a useful first approximation in practice.

## 3. Brief history of Mathematical Finance

*Mathematical Finance I: Markowitz and CAPM.*

We deal with the history of put-call parity (II.7) below. It has ancient roots, but entered the textbooks around 1904. Louis Bachelier (1870-1946) first put mathematics to work on finance in his 1900 thesis *Théorie de la spéculation.*[2] Bachelier's thesis is also remarkable as he used *Brownian mo-*

[2] Mark Davis and Alison Etheridge: Louis Bachelier's *Theory of speculation*: The origins of modern finance, translated and with a commentary; foreword by Paul A. Samuelson.

*tion* as a model for the driving noise in the price of a risky asset. This was remarkable, as the relevant mathematics did not exist until 1923 (Wiener), and later (Itô, stochastic calculus, 1944).

Until 1952, finance was more an art than a science. This changed with the 1952 thesis of Harry Markowitz (1927–), which introduced modern *portfolio theory*. Markowitz gave us two key insights, both so 'obvious' that they are all around us now. There is no point in investing in the stock market, which is risky, when one can instead invest risklessly by putting money in the bank, unless one expects the (rate of) *return* on the stock, $\mu$, to be higher than the riskless return $r$. The riskiness of the stock is measured by a parameter, the *volatility* $\sigma$, which corresponds to the standard deviation (square root of the variance) in a model of the risky stock price as a stochastic process (Ch. IV), while $\mu$, $r$ correspond to *means*, for risky and riskless assets respectively. Markowitz's first key insight is: *think of risk and return together, not separately.* This leads to *mean-variance analysis.*

Next, the investor is free to choose which sector of the economy to invest in. He is investing in the face of uncertainty (or risk), and in each sector he chooses, prices may move against him. He should insure against this by holding a *balanced portfolio*, of assets from a number of different sectors, chosen so that they will tend to 'move against each other'. Then, 'what he loses on the swings he will gain on the roundabouts'. This tendency to move against each other is measured by *negative correlation* (the term comes from Statistics). Markowitz's second key insight is:
*diversify, by holding a balanced portfolio with lots of negative correlation.*

Markowitz's theory was developed during the 1960s, in the *capital asset pricing model* (CAPM – 'cap-emm'), of Sharpe, Lintner and Mossin (William Sharpe (1964), John Lintner (1965), Jan Mossin (1966); Jack Treynor (1961, 1962)). In CAPM, one looks at the excess of a particular stock over that of the market overall, and the risk (as measured by volatility), and seeks to obtain the maximum return for a given risk (or minimum risk for a given return), which will hold on the *efficient frontier.* The relevant mathematics involves Linear Regression in Statistics, and Linear Programming in Operational Research (OR).

*Mathematical Finance II: Black, Scholes and Merton.*

If one is contemplating buying a particular stock, intending to hold it for a year say, what one would love to know is the price in a year's time,

Princeton UP, 2006.

compared with the price today (one should discount this, as above). If the (discounted) price goes up, one will be glad in a year's time that one bought; if it goes down, one will be sorry.

Suppose one's Fairy Godmother appeared, and gave one a piece of paper, which said that if one bought now, then in a year's time if one was glad one had done so one did buy, but if one was sorry, one didn't. Such pieces of paper do exist, and are called *options* – see Ch. V, VII. Clearly such options are valuable: they may lead to a profit, but cannot lead to a loss.
*Question*: What is an option worth?

Note that unless one can *price* options, they will not be traded (at least in any quantity) – as with anything else.

Before 1973, the conventional wisdom was that this question had no answer: it *could have no answer*, because the answer would necessarily depend on the economic agent's attitude to risk (that is, on his utility function, or loss function – see above). It turns out that this view is incorrect. Subject to the above assumptions of an idealized market (NA, etc.), one *can* price options, according to the famous *Black-Scholes formula* of 1973 (Ch. V, VII – Fischer Black (1938-1995) and Myron Scholes (1941-)). They derived their formula by showing that the option price satisfied a partial differential equation (PDE), of parabolic type (a variant of the *heat equation*). In 1973 Robert Merton (1944-) gave a more direct approach. Meanwhile, 1973 was also the year when the first exchange for buying and selling options opened, the Chicago Board Options Exchange (CBOE).

To see why options can be priced, one only needs to know that the standard options are (under our idealized assumptions) *redundant* financial assets: an option is equivalent to an appropriate combination of cash and stock. Knowing how much cash, how much stock and the current stock price, one can thus calculate the current option price by simple arithmetic.

In 1981, it was shown (by J. M. Harrison and S. R. Pliska) that the right mathematical machinery to use in this area involves a particular type of stochastic process – *martingales* – and a particular type of calculus, for stochastic processes – *Itô calculus* (Kiyosi Itô (1915-2008)); see Ch. VII.

The subject of Mathematical Finance is by now well-established, and rapidly growing in popularity in universities, in UK, US and elsewhere. This is because of its relevance to the needs of the financial sector (or financial services industry) in the City of London (also Edinburgh) within UK, New York in USA, Tokyo in Japan, Frankfurt in Germany, etc. This sector needs technical people with good skills in mathematics, statistics, numerics etc., as well as economic insight and financial awareness, problem-solving skills and ability to work in a team, etc. Such people are variously called financial engineers, quantitative analysts ('quants') or 'rocket scientists'.

Academically, the subject falls broadly in the interface between Economics on the one hand and Mathematics on the other. In Economics, much of the subject, again broadly speaking, relates to *how prices are determined* – by the interplay between supply and demand, etc. By contrast, here in this course we will usually take prices as given. Our task is to study how, starting from the given prices, we can price other things related to them (options, and other financial derivatives – see below), and guard our operations against unpredictable hazards (hedge – again, see below).

In this sense, Finance as a subject appears as a small – specialised, highly mathematical – part of Economics (note that Finance here is not used quite in the traditional non-technical sense). *Risk* is the key danger – the key concept even – in finance; risk reflects uncertainty; uncertainty reflects chance or probability. So it was clear that Probability Theory, a branch of Mathematics related to Statistics, had to be relevant here. Quite how was shown in 1981 by J. M. (Michael) Harrison (a probabilist) and David Kreps (an economist), who simplified and generalized the Black-Scholes-Merton theory by using the language of Probability Theory and Stochastic Processes – in particular, *martingales* (and Itô calculus, again). These developments – and what followed – constituted the 'second revolution in mathematical finance'. This is the subject-matter of this course. (We can cover the mathematics of the developments outlined above. More recent developments are very important, but go beyond a first undergraduate course – see e.g. our MSc in MF.) On the mathematical side: you will learn a lot about stochastic processes, martingales and Itô calculus, and see them put to use on financial problems. On the practical side: the best proof of the relevance and usefulness of these ideas is the explosive growth in volumes of trades in financial derivatives over the last forty-odd years, and the corresponding explosive growth in employment opportunities (and salaries!) for those who understand what is going on.

*Black, Scholes and Merton*

As everywhere, triumph and disaster can always happen, and one has to use common sense. Triumph: Scholes and Merton were awarded the Nobel Prize for Economics in 1997 (Black died in 1995, and the prize cannot be awarded posthumously). Disaster: Scholes and Merton were on the board of the hedge fund Long Term Capital Management, which ignominiously collapsed with enormous losses in 1998. Pushing a good theory too far – beyond all sensible limits – is asking for trouble, even if one invented the theory and got the Nobel Prize for it, and if one asks for trouble, one can expect to get it.

**4. Markets and Options.**

*Markets.*

This course is about the mathematics of *financial markets.* Types:
*Stock markets* [New York, London, ...], dealing in stocks/shares/equities, etc.,
*Bond markets*, dealing in government bonds (gilts, ...),
*Currency or foreign exchange* ('forex') markets,
*Futures* and *options markets*, dealing in financial instruments derived from the above - *financial derivatives* such as *options* of various types.

*Options.*

Economic activity, and trading, involves *risk.* One may have to, or choose to, make a judgement involving committing funds ('taking a position') based on prediction of the future in the presence of uncertainty. With hindsight, one might or might not regret taking that position. An *option* is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. Whether or not the option will be exercised depends on (is contingent on) the uncertain future, so is also known as a *contingent claim.*

*Types of option.*

*Call* options give one the right (but not – without further comment now – the obligation) to *buy.* [Remember: to *buy* something, one *calls* for it.]
*Put* options give one the right to *sell.* [Remember: to *sell* something, one *puts* it on the market.]
*European* options give one the right to buy/sell *on* the specified date, the *expiry* date, when the option *expires* or matures.
*American* options give one the right to buy/sell *at any time* prior to or at expiry. Thus:

*European* options: exercise *at* expiry,
*American* options: exercise *by* expiry.

*Note.* The terms European, American (Asian, etc.) refer only to the type of option, and no longer bear any relation to the area in the name. Most options traded worldwide these days are American.
*History.* As discussed in §1, over-the-counter (OTC) options were long ago negotiated by a broker between a buyer and a seller. Then in 1973 (the year of the Black-Scholes formula, the central result of the course), the Chicago

Board Options Exchange (CBOE) began trading in options on some stocks. Since then, the growth of options has been explosive. Options are now traded on all the major world exchanges, in enormous volumes. Often, the market in derivatives is *much larger* than the market in the underlying assets – an important source of instability in financial markets.

The simplest call and put options are now so standard they are called *vanilla* options. Many kinds of options now exist, including so-called *exotic* options (Asian, barrier, etc.) – on which you need a separate course. For *real options* (also called *investment options*) – see L30.

*Terminology.*

The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the *exercise price* or *strike price*. We shall usually use $K$ for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

*Calls.*

Consider a European *call* option, with strike price $K$ and underlying worth $S_t$ at time $t$. If $S_T > K$, the option is *in the money*: the holder will/should *exercise* the option, obtaining an asset worth $S_T$ ($> K$) for $K$. He can immediately sell the asset for $S_T$, making a *profit* of $S_T - K$ ($> 0$).
If $S_T = K$, the option is said to be *at the money*.
If $S_T < K$, the option is *out of the money*, and should not be exercised. It is worthless, and is thrown away.

The *pay-off* from the option is thus

$$S_T - K \text{ if } S_T > K, \quad 0 \quad \text{otherwise},$$

which may be written more briefly as

$$max(S_T - K, 0) \text{ or } (S_T - K)_+$$

$(x_+ := max(x, 0), x_- := -min(x, 0); x = x_+ - x_-, |x| = x_+ + x_-; (-x)_+ = max(-x, 0) = -min(x, 0) = x_-)$.

*Puts.*

Similarly, the payoff from a *put* option is

$$K - S_T \text{ if } S_T \leq K, \quad 0 \text{ if } S_T > K, \quad \text{or} \quad (K - S_T)_+.$$

*Option pricing.*

The fundamental problem in the mathematics of options is that of *option*

*pricing*. The modern theory began with the *Black-Scholes formula* for pricing European options in 1973. We shall deal with the Black-Scholes theory, and cover the pricing of European options in full. We also discuss American options: these are harder, and lack explicit formulae such as the Black-Scholes formula; consequently, one needs to evaluate them numerically. The pricing of Asian options is even harder.
*Perfect Markets*. For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. First, there are no *transaction costs* (one can include transaction costs in the theory, but this is considerably harder). Similarly, we assume that interest rates for borrowing and for lending are the same (which is unrealistic, as banks make their money on the difference), and also that all traders have access to the same – perfect – information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading). We shall assume no restrictions on *liquidity* – that is, one can buy or sell unlimited quantities of stock at the currently quoted price. That is, our economic agents are *price takers* and not *price makers*. (This comes back to §1 on the relationship between Economics and Finance. In practice, big trades do move markets. Also, in a crisis, no-one wants to trade, and liquidity dries up – basically, this is what did for LTCM.) In practice, very small trades are not economic (the stockbroker may only deal in units of reasonable size, etc.). We shall ignore all these complications for the sake of simplicity.

## 5. Portfolios and Hedging.

*Portfolios.*

We consider an investor with capital to invest. The simplest model is that in which he has two (or more) choices: to invest in
(i) a *bank account* – assumed *riskless*, and yielding interest. For simplicity, we assume the interest rate is a constant $r > 0$ (usually called the *short rate* of interest: interest rates may be different outside $[0, T]$); thus $\$B$ invested at time $t$ grows to $\$Be^{r(T-t)}$ by time $t$;
(ii) one (or more) *risky assets* or stocks, whose value at time $t$ is $S_t$ (scalar or vector).
A *portfolio* is a division $(B_t, S_t)$ of the investor's capital between bank account and stock holdings at time $t$.
A *trading strategy* is a rule (suitably restricted – see Chapters V and VII) chosen by the investor to update his portfolio over time as new price information on the risky stock(s) comes in.

*Hedging.* Hedging is an attempt to reduce exposure to risk by adopting opposite positions – e.g., in holding both call and put options in the same underlying, or by adjusting a portfolio as above to cover possible losses.
*Why buy options?* Options have two main uses: speculation and hedging. In speculation, available funds ('hot money') are invested opportunistically in the hope of making a profit: the underlying itself is irrelevant to the investor (speculator), who is only interested in the potential for possible profit that trade involving it may present. Hedging, by contrast, is typically engaged in by companies who have to deal habitually in intrinsically risky assets such as foreign exchange next year, wheat/copper/oil next year, etc. This protects their economic base (trade in wheat/copper/oil, or manufacture of products using these as raw materials), and lets them focus their effort in their chosen area of trade or manufacture. But for speculators, it is the market (forex, commodities or whatever) itself which is their main focus.

Because the value of an option at expiry is so sensitive to price – it reflects movements in the price of the underlying in exaggerated form – the holding (or trading) of options and other derivatives presents greater opportunities for profit (and indeed, for loss) than trade in the underlying (this is why speculators buy options!). They are correspondingly more risky.

One of the main insights of the fundamental work of Black and Scholes was that one can (at least in the most basic model) *hedge* against meeting a contingent claim by *replicating* it: constructing a portfolio, adjusted or re-balanced as time unfolds and new price information comes in, whose pay-off *is* the amount of the contingent claim.

**6. Arbitrage**.

Economic agents go to the market for various reasons. One the one hand, companies may wish to insure, or *hedge*, against adverse price movements that might affect their core business. On the other hand, *speculators* may be uninterested in the specific economic background, but only interested in making a profit from some financial transaction. The relation between hedging ('good') and speculation ('bad') is to some extent symbiotic (one cannot lay off a risk unless someone else is prepared to take it on, and why should he unless he expects to make money by doing so). Nevertheless, one feels that it should not be possible to extract money from the market without genuinely engaging in it, by taking *risk*: all business activity is risky. Indeed, were it possible to do so, people would do so – in unlimited quantities, thus sucking money parasitically out of the market, using it as a 'money-pump'. This would undermine the stability and viability of the market in the long run – and in particular, make it impossible for the market to be in *equilibrium*.

The view we take of modelling markets as NA is not that arbitrage opportunities do not exist, but that if they do exist in any quantity, people will rush to exploit them, and thereby dissipate them – 'arbitrage them away'.

Financial markets involve both riskless (bank account) and risky (stocks, etc.) assets. For investors, the only point of taking risk is the chance of a greater profit than the riskless procedure of putting one's money in the bank (the mathematics of which – compound interest – does not require a degree or MSc course!). In general, the greater the risk, the greater the return required to make investment an attractive enough prospect to attract funds.

It is usually better to work, not in face-value or nominal terms, but in *discounted* terms, allowing for the exponential growth-rate $e^{rt}$ of risklessly invested money. So, profit and loss are generally reckoned against this discounted benchmark. So a market with arbitrage opportunities would be a disorderly market – too disorderly to model. The remarkable thing is the converse. It turns out that the minimal requirement of absence of arbitrage opportunities is enough to allow one to build a model of a financial market which – while admittedly idealised (frictionless market – no transaction costs, etc.) – is realistic enough both to provide real insight and to handle the mathematics necessary to price standard options (Black-Scholes theory). We shall see that arbitrage arguments suffice to determine prices – the *arbitrage pricing technique* (APT – S. A. Ross, 1976, 1978).

*Short-selling.*

Just as we can borrow money from the bank, in many markets, risky assets such as stocks may be treated in the same way. We may have a positive or zero holding – or a *negative holding* (notionally borrowing stock, which we will be obliged to repay – or repay its current value). In particular, we may be allowed to *sell stock we do not own.* This is called *short-selling*, and is perfectly legal (subject to appropriate regulation) in many markets. Think of short-selling as *borrowing.*

Not only is short-selling both routine and necessary in some contexts, such as foreign exchange and commodities futures, it simplifies the mathematics. So we assume, unless otherwise specified, no restriction on short-selling. By extension, we call a portfolio, or position, *short* in an asset if the holding of the asset is negative, *long* if the holding of the asset is positive. It turns out that in some important contexts – such as the Black-Scholes theory of European and American calls – short-selling can be avoided. In such cases, it is natural and sensible to do so: see Ch. VII.

**7. Put-Call Parity.**

Just as long and short positions are diametrical opposites, so are call and put options. We now use arbitrage to show how they are linked.

Suppose there is a risky asset, value $S$ (or $S_t$ at time $t$), with European call and put options on it, value $C, P$ (or $C_t, P_t$), with expiry time $T$ and strike-price $K$. Consider a portfolio which is long one asset, long one put and short one call; write $\Pi$ (or $\Pi_t$) for the value of this portfolio. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call)}.$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ & \text{for a call, } C, \\ \max(K-S,0) & \text{or} \quad (K-S)_+ = (S-K)_- & \text{for a put, } P. \end{cases}$$

So the value of the above portfolio at expiry is $K$: for, it is

$$S + 0 - (S-K) = K \quad \text{if } S \geq K, \qquad S + (K-S) - 0 = K \quad \text{if } K \geq S.$$

Alternatively, use $x = x_+ - x_-$ and $(-x)_+ = x_-$ with $x = S - K$.
This portfolio thus guarantees a payoff $K$ at time $T$. How much is it worth at time $t$?

Short answer (correct, and complete): $Ke^{-r(T-t)}$, because it is financially equivalent to cash $K$, so has the same time-$t$ value as cash $K$.

Longer answer (included as an example of arbitrage arguments). The riskless way to guarantee a payoff $K$ at time $T$ is to deposit $Ke^{-r(T-t)}$ in the bank at time $t$ and do nothing. If the portfolio is offered for sale at time $t$ too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time $T$ my portfolio yields $K$ (above), while my bank debt has grown to $K$. I clear my cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time $t$ at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio short* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time $T$, my bank deposit has grown to $K$, and I again clear my cash account –

use this to meet my obligation $K$ on the portfolio I sold short, again locking in my earlier riskless profit. So the rational price for the portfolio at time $t$ is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make a riskless profit) – which they will take! Thus
(i) The price (or value) of the portfolio at time $t$ is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

This link between the prices of the underlying asset $S$ and call and put options on it is called *put-call parity*.
(ii) The value of the portfolio $S+P-C$ is the *discounted value of the riskless equivalent*. This is a first glimpse at the central principle, or insight, of the entire subject of option pricing. But in general, we will have 'risk-neutral' in place of 'riskless'; see II.8 below, Ch. V and Ch. VII.
(iii) Arbitrage arguments, although apparently qualitative, have quantitative conclusions, and allow one to calculate precisely the rational price – or *arbitrage price* – of a portfolio. The put-call parity argument above is the simplest example – though typical – of the *arbitrage pricing technique* (APT).
(iv) The APT is due to S. A. Ross in 1976-78 (details in [BK], Preface). Put-call parity has a long history (see Wikipedia).
*Note.* 1. History shows both that arbitrage opportunities exist (or are sought) in the real world and that the exploiting of them is a delicate matter. The collapse of Baring's Bank in 1995 (the UK's oldest bank, and bankers to HMQ) was triggered by unauthorised dealings by one individual, who tried and failed to exploit a fine margin between the Singapore and Osaka Stock Exchanges. The leadership of Baring's Bank at that time thought that the trader involved had discovered a clever way to exploit price movements in either direction between Singapore and Osaka. This is obviously impossible on theoretical grounds, to anyone who knows any Physics. See Problems 2 Q1 (key phrases: perpetual motion machine; Maxwell's demon; Second Law of Thermodynamics; entropy).
2. Major finance houses have an *arbitrage desk*, where their *arbs* work.

**8. An Example: Single-Period Binary Model.**

We consider the following simple example, taken from
[CRR] COX, J. C., ROSS, S. A. & RUBINSTEIN, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263.
For definiteness, we use the language of foreign exchange. Our risky asset will be the current price in Swiss francs (SFR) of (say) 100 US \$, supposed

$X_0 = 150$ at time 0. Consider a call option with strike price $K = 150$ at time $T$. The simplest case is the binary model, with two outcomes: suppose the price $X_T$ of 100 \$ at time $T$ is (in SFR)

$$X_T = \begin{cases} 180 & \text{with probability } p \\ 90 & \text{with probability } 1-p. \end{cases}$$

The payoff $H$ of the option will be $30 = 180 - 150$ with probability $p$, 0 with probability $1-p$, so has expectation $EH = 30p$. This would seem to be the fair price for the option at $t = 0$, or allowing for an interest-rate $r$ and discounting, we get the value

$$V_0 = E\big(\frac{H}{1+r}\big) = \frac{30p}{1+r}.$$

Take for simplicity $p = \frac{1}{2}$ and $r = 0$ (no interest): the naive, or expectation, value of the option at time 0 is

$$V_0 = 15.$$

The *Black-Scholes value* of the option, however, is different. To derive it, we follow the Black-Scholes prescription (Ch. IV, VI):
(i) First replace $p$ by $p^*$ so that the price, properly discounted, *behaves like a fair game*:

$$X_0 = E^*\big(\frac{X_T}{1+r}\big).$$

That is,

$$150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90);$$

for $r = 0$ this gives $60 = 90p^*$ or $p^* = 2/3$.
(ii) Now compute the fair price of the expected value in this new model:

$$V_0 = E^*\big(\frac{H}{1+r}\big) = \frac{30p^*}{1+r};$$

for $r = 0$ this gives the Black-Scholes value as $V_0 = 20$.
*Justification*: it works! – as the arbitrage constructed below shows. For simplicity, take $r = 0$.
We *sell* the option at time 0, for a price $\pi(H)$, say. We then prepare for the resulting contingent claim on us at time $T$ by the option holder by using the

following strategy:

| | |
|---|---|
| Sell the option for $\pi(H)$ | $+\pi(H)$ |
| Buy \$33.33 at the present exchange rate of 1.50 | $-50$ |
| Borrow SFR 30 | $+30$ |
| Balance | $\pi(H) - 20.$ |

So our balance at time 0 is $\pi(H) - 20$. At time $T$, two cases are possible:
(i) The dollar has risen:

| | |
|---|---|
| Option is exercised (against us) | $-30$ |
| Sell dollars at 1.80 | $+60$ |
| Repay loan | $-30$ |
| Balance | 0. |

(ii) The dollar has fallen:

| | |
|---|---|
| Option is worthless | 0.00 |
| Sell dollars at 0.90 | $+30$ |
| Repay loan | $-30$ |
| Balance | 0. |

So the balance at time $T$ is zero in both cases. The balance $\pi(H) - 20$ at time 0 should thus also be zero, giving the Black-Scholes price $\pi(H) = 20$ as above. For, *any other price* gives an arbitrage opportunity. Argue as in put-call parity in §4: if the option is offered too cheaply, buy it; if it is offered too dearly, write it (the equivalent for options to 'sell it short' for stock). Thus any other price would offer an arbitrageur the opportunity to extract a riskless profit, by appropriately buying and selling (Swiss francs, US dollars and options) so as to exploit your mis-pricing.

The same argument with interest-rate $r$ also applies: divide everything through by $1 + r$.

*Note.* This argument, and result, are **independent** of $p$, the 'real' probability, and depend instead **only** on this 'fictitious' new probability, $p^*$ (which is called the *risk-neutral* or *risk-adjusted* probability.

The example above is highly instructive. First, it clearly represents the simplest possible non-trivial case: only two time-points (with one time-period between them, hence the 'single-period' of the title), and only two possible outcomes (hence the 'binary' of the title). Secondly, it shows that there is a theory hidden here, which gives us a definite prescription to follow (and some surprises, such as not involving the 'real' probability $p$ above). This prescription is simple to implement, and can be justified by explicitly constructing an arbitrage to exploit doing anything else [if the option is offered for sale too cheaply, buy it, if too dearly, write it]. This theory is the Black-Scholes theory, which we consider in detail in Chapters V and VII. The technical key to the Black-Scholes prescription is the introduction of $p^*$ and its associated expectation operator $E^*$. In technical language, this is the *equivalent martingale measure*. Now each of these three terms needs full introduction. We shall talk about measures in III.1 below, about equivalent measures in III.4, and martingales in IV.3 and VI.2. We stress: the Black-Scholes theory – that is, rational option pricing – cannot be done without all these concepts. This is why we need Chapter III on the necessary background on measure theory, and Chapters IV and VI on the necessary background on stochastic processes.

There are basically three options open to those teaching, and learning, how to price options etc.

1. One can avoid measure theory altogether (cf. [CR]). This is technically possible rigorously in the discrete-time setting of Ch. IV – though at greater length, because the key concepts cannot be addressed explicitly. It is also possible non-rigorously in continuous time; cf. [WHD], who base their approach on partial differential equations (PDE).
2. One can learn measure theory first – say, from the excellent book [W]. This, however, puts the subject beyond the reach of most people who need it and use it in practice – and beyond reach of most of this audience.
3. One can do as we shall do (and as [BK] and a number of other books do): state what we need from measure theory, and use its language, concepts, viewpoint and results, without proving anything. This makes good sense: the constructions and proofs of measure theory are quite hard (say, final year undergraduate or first-year postgraduate level for good mathematics students with a bent for analysis – quite a select group!). Using measure theory taking its results for granted, however, is quite easy, as we shall see.

## 9. Complements

*1. Types of risk.*

Institutions encounter risks of various types. These include:

*Market risk.*

This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper,

and/or things one is short on get dearer).
*Credit risk.*

This is the risk that counter-parties to one's financial transactions may default on their obligations.

When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. This raises issues of *moral hazard*, below.
*Operational risk.*

This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions (the failure of the Taurus clearing system on the London Stock Exchange was one example); fraudulent or unauthorised trading made possible by inadequate supervision; etc.
*Liquidity risk.*

This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up.
*Model risk.*

To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful.[3] Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models.
*2. Risk management.* The problems of 2007/8 on have made the importance of risk management obvious. For an excellent book-length treatment, see e.g. [MFE] A. J. McNEIL, R. FREY & P. EMBRECHTS: *Quantitative risk management: Concepts, techniques, tools.* Princeton UP, 2005.

We know from Markowitz that we should have a balanced portfolio, with lots of negative correlation. The danger is *large* losses. These are quantified by the *tails* of the distributions – the joint distribution of our portfolio. The

[3]George E. Box, 1919-. British statistician

point of diversifying is so that what we lose on the swings we gain on the roundabouts. Two comments:
(a) Whether this works for large losses depends on the tail properties of the joint distribution. It does *not* work if this is normal – as it is in the benchmark Black-Scholes model.
(b) When the whole market is falling – as in a financial crisis – none of the risk-management techniques useful under normal market conditions work.
*3. Moral hazard*. Before the limited liability company, if one defaulted, one was liable to the *whole* of the loss incurred by one's counter-party. This made trading very dangerous (the early traders were called merchant adventurers) – all the more as insurance had not developed by then.[4]

Limited liability was what made ordinary people willing to undertake the risks of trading, and so paved the way for the development of modern business, commerce, capitalism etc.

The moral hazard here is the possibility of gambling with other people's money (see John Kay's excellent new book, *Other people's money*, L0). If it works, fine. If not, walk away (writing off one's limited liability) and leave them to bear the loss. "Playing Russian roulette with someone else's head" is a rather brutal description of this.

Bankruptcy law varies from country to country, and is too complicated to pursue here. But one sees moral hazard where it concerns us in, e.g.:
(a) start-ups of hedge funds (or, dot-com companies);
(b) aggressive traders – who (for the sake of their bonuses) gamble with their careers – but with other people's money;
(c) credit rating – where the credit rating agencies had a financial incentive to pass as AAA some highly questionable financial asset, etc.
*4. Securitization*. This term covers the drive in recent years to seek out new financial markets by identifying risks that people might want to cover themselves against, and creating new financial derivatives that can be sold to address this perceived need. These derivatives too could be traded, etc. The upshot was an explosion of trade in increasingly artificial financial products, developed by the R&D departments of the financial institutions. By 2007/8, the leaders of these institutions did not understand these products – could not price them, and could not value their holdings of them (above).

[4]Lloyds of London predates limited liability. The Lloyds participants – "names" – had unlimited liability. Many were driven into personal bankruptcy in the Lloyds scandals of the 90s. See Google for the ghastly details.

One specific trigger of the US crash in 2007 was the explosive growth in sub-prime mortgages. These were granted to people who would not have qualified as financially sound enough to get a mortgage previously, but who wanted to buy their own house. This new and profitable market proved irresistible to US banks – leading to a great house-price bubble, which burst (as bubbles do) in 2007. The knock-on effects hit the UK in 2008 (Northern Rock, etc.). The real damage of this failure of the financial sector has been its devastating and ongoing consequences on the real economy.

*5. Macro-prudential issues.* As the above illustrates, financial matters are too important to be left to financiers. Proper regulation is vital.

*6. Forwards and futures.* Forwards are agreements between buyer and seller made now, but concerning delivery in the future. They are not traded. Futures are options on things that will come to market in the future (next year's grain crop, for example), and these are traded (extensively). There are good accounts in Hull's books, [H1], [H2].

*7. OTC and exchange-traded contracts.* OTC – "over-the-counter" – denotes a transaction made between an individual buyer and an individual seller. As options on standard transactions develop, these are assets themselves that can be traded in exchanges (e.g., the CBOE, which opened in 1973: II.3).

*8. Marking to market.* This is a system whereby the exchanges cover themselves and their clients against the risk of large losses. If one party to a trade is, on current market prices, exposed to a potentially heavy loss, a *margin call* will be required by the exchange. Margin calls actually trigger many financial failures (but limit the losses of the counter-parties).

*9. Forex.* Forex is an abbreviation for foreign exchange. International trade involves more than one currency' currencies move against each other. There is a vast market in derivatives to cover the risks involved.

*10. Swaps.* From Hull [H2] Ch. 5: "Swaps are private agreements between two companies to exchange cash flows in the future ... The first swap contracts were negotiated in 1981. Since then the market has grown very rapidly. ..." There are even options on swaps – *swaptions* – etc.

**10. Postscript to Ch. II: Systemic issues – "Big-picture stuff"** *(not examinable).*

Recall (Ch0) the three underlying truths about this course (indeed, this subject, at this level): *Anything important enough becomes political. Politics is not an exact science. Mathematics is an exact science.*

Thus, while this course is self-contained as "F22", it is not and cannot be self-contained if we broaden the picture: the big picture is, to some extent, unmathematisable – even in principle. We illustrate this with a few themes.

(i) *Finance and economics.* What ultimately counts – in terms of jobs, people's lives, growing food, making things that people want, etc. – is the *real economy.* Finance is the means (the supply of money) to an end (the production of food, goods, services, creation of jobs, etc.). But, damage to the financial system (the Slump of 1929 and the 1930s, the Crash of 2007/8/...) can be longer-lasting than economic damage. Think of the financial system as the *nervous system* of the economy. A nervous breakdown can disrupt life more than a broken leg, etc. – and have longer-lasting effects (cf. PTSD).

(ii) *Economics: Keynes, Hayek/Friedman, the postwar consensus, the neo-liberal consensus.* The great crisis of world economics 1900-50 was the Slump (or Depression), the misery caused by which was a major cause of WWII. Three factors helped to cure this: F. D. Roosevelt and the New Deal (US, 1930s); J. M. Keynes (books, 1930s); the stimulus to economies of WWII. Keynesian economics (governments should spend their way out of a slump by infrastructure projects etc.) formed the post-war consensus (1945-1979/80). Then came the neo-liberal consensus (politically, Thatcher and Reagan; intellectual underpinning from Hayek and Friedman) – free markets, legal restrictions on trade unions, etc. (and the collapse of communism). The Nobel-prize winning US economist Paul Krugman puts it thus: *big business doesn't like Keynesian economics, as it diminishes its bargaining power.* Do your own digging here, and form your own view. But note that in science, liking something or not has zero effect on whether it is right, or whether it works: a scientist should like what he sees, rather than see what he likes. But in economics this is less true, because big issues involve politics, and so perception, and so psychology. Ordinary members of the public do not even know the names of the economists mentioned above, still less what they did; this limits their ability to make informed decisions about matters of current political controversy, where they see things through the 'distorting prism' of media coverage, partisan political debate, etc. Three comments:

(a) The above underlines the difference in mission between the Imperial Col-

lege of Science, Technology and Medicine and (say) the London School of Economics and Political Science;
(b) Anyone intending to work in, say, the financial services industry should know *at least as much* about such things as a well-informed member of the public. This cannot be done overnight! One obvious way is to read a decent newspaper regularly over a period of time (or online equivalent).
(c) The profound importance of the issues here can be seen in, e.g., the damage to the life prospects of the young. Even discounting Greece here, youth unemployment has been over 50% in some European countries for long periods since the Crash (and much higher in disadvantaged parts of the community, with predictable consequences for social cohesion, crime etc.).
(iii) *The Euro.* By common consent, the euro was *political* in aim and in origin. It is *financial* and *economic* in substance and in content. There is a tension between these two. This is seen dramatically in the events concerning Greece (Grexit?) and the UK (Brexit). What is your view on all this?
(iv) *Asset-price bubbles.* The Greenspan years were one long asset-price bubble, which ended in tears. The Chinese economy now has another one ... .
(v) *Quantitative easing (QE).* Since the Crash, governments moved to rescue a banking system at risk of collapse by QE. The idea was to 'create electronic money' for banks to lend to businesses, to free up and kick-start the real economy, so that life could get back to normal. What tended to happen instead was that banks – also under pressure to rebuild their balance sheets – did so by holding onto this new money, rather than lending it as intended. At the same time, interest rates have been at historic lows for long periods – indeed, real (as distinct from nominal) interest rates have often been *negative*, which would have been dismissed as laughable a decade ago. But keeping interest rates at near-zero for long periods has itself had distorting effects – such as fuelling a new asset-price bubble, and so sewing the seeds for the next Crash. How should policy-makers proceed here?
(vi) *Banks: too big to fail? separate investment banking from retail banking?* As Keynes said: if you owe the bank a thousand pounds, it's your problem; if you own the bank a million pounds, it's the bank's problem. Seeming 'too big to fail' (without risking a collapse of confidence as with Lehman Brothers) helped the banks in their dealings with governments post-Crash. The Glass-Steagall Act was passed in the US in 1932 to separate investment (UK: merchant; "casino") banks from retail ("high-street") banks, but this was repealed in 1999. What is the right approach to regulation and legislation here?