M3A22 MATHEMATICAL FINANCE: MOCK EXAM SOLUTIONS 2014-15

Q1. (a) *Types of risk*. Institutions encounter risks of various types. Perhaps the biggest one starts at the top: how good is the board? If the board of directors, and particularly the chairman and CEO, do not have a good overview and good judgement, this alone can bring the institution down. [**2**]

Other specific types of risk include:

Market risk. This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer). [**3**]

Credit risk. This is the risk that counter-parties to one's financial transactions may default on their obligations. When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. [**3**]

Operational risk. This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions; fraudulent or unauthorised trading made possible by inadequate supervision; etc. [**3**]

Liquidity risk. This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up. [**3**]

Model risk. To handle real-world phenomena of any complexity, one needs to model them mathematically. Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models. [**3**]

(b) *Stress testing*. Financial regulators test the adequacy of the performance of a financial institution by subjecting it to *stress testing*: seeing how well its operations would perform under hypothetical but unfavourable market scenarios. This tests various aspects: their models, systems (how management and trading teams would react under pressure), capital reserves, etc. [**3**]

[Mainly seen – lectures]

Q2. *Martingale transforms.* Call a process $C = (C_n)_{n=1}^{\infty}$ *previsible* (or *predictable*) if

$$C_n \text{ is } \mathcal{F}_{n-1} - \text{measurable for all } n \geq 1. \qquad \textbf{[2]}$$

Think of a gambling game, or series of speculative investments, in discrete time. There is no play at time 0; there are plays at times $n = 1, 2, \cdots$, and

$$\Delta X_n := X_n - X_{n-1}$$

represents our net winnings per unit stake at play n. Think of C_n as your stake on play n (C_0 is not defined, as there is no play at time 0). Previsibility says that you have to decide how much to stake on play n based on the history *before* time n (i.e., up to and including play $n-1$). Your winnings on game n are $C_n \Delta X_n = C_n(X_n - X_{n-1})$. Your total (net) winnings up to time n are

$$Y_n = \sum_{k=1}^{n} C_k \Delta X_k = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

We define $Y = C \bullet X$ as the *martingale transform* of X by C:

$$Y = C \bullet X, \qquad Y_n = (C \bullet X)_n, \qquad \Delta Y_n = C_n \Delta X_n$$

($(C \bullet X)_0 = 0$ as $\sum_1^0$ is empty). **[5]**

Interpretation. In mathematical finance, X plays the role of a price process: discounted asset prices are martingales under the risk-neutral measure. C plays the role of our trading strategy (saying how much stock we hold at each time), and the mg transform $C \bullet X$ plays the role of our gains (or losses!) from trading. **[5]**

Theorem. If C is bounded and previsible and X is a martingale, $C \bullet X$ is a martingale null at zero.

Proof. With $Y = C \bullet X$ as above,

$$\begin{aligned} E[Y_n - Y_{n-1} | \mathcal{F}_{n-1}] &= E[C_n(X_n - X_{n-1}) | \mathcal{F}_{n-1}] \\ &= C_n E[(X_n - X_{n-1}) | \mathcal{F}_{n-1}] \end{aligned}$$

(as C_n is bounded, so integrable, and $\mathcal{F}_{n-1}$-measurable, so can be taken out)

$$= 0$$

as X is a martingale. // **[8]**

[Seen – lectures]

Q3. *Vega for calls.* With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$ the standard normal density and distribution functions, $\tau := T - t$ the time to expiry, the Black-Scholes call price is

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \qquad (BS)$$

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} :$$

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} :$$

$$\phi(d_2) = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*)$$

Differentiating (BS) partially w.r.t. σ gives

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial d_1/\partial\sigma - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial\sigma.$$

So by $(*)$,

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial(d_1 - d_2)/\partial\sigma = S\phi(d_1)\partial\sigma\sqrt{\tau})/\partial\sigma = S\phi(d_1)\sqrt{\tau} > 0.$$

[12]

Vega for puts.

The same argument gives $v := \partial P/\partial\sigma > 0$, starting with the Black-Scholes formula for puts. Equivalently, we can use put-call parity

$$S + P - C = Ke^{-r\tau} : \qquad \partial P/\partial\sigma = \partial C/\partial\sigma > 0. \qquad [\mathbf{3}]$$

Interpretation: "Options like volatility": the more uncertainty, i.e. the higher the volatility, the more the "insurance policy" of an option is worth. So vega is positive for positions *long* in the option – but negative for *short* positions. **[2]**

American calls. Vega is also positive for American calls. For, the Snell envelope U of a process Z, which passes from European to American option prices, is order-preserving: U increases as Z increases. Increase in σ increases both, so since 'European vega' is positive, so is 'American vega'. **[3]**

[Seen – lectures]

Q4. *Geometric Brownian Motion (GBM).* (a) Consider the Black-Scholes model, with dynamics given by the stochastic differential equation (SDE)

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t. \qquad (GBM)$$

The interpretation here is that B_t is our bank account at time t – money invested risklessly at rate r, so growing exponentially. The risky stock S has a similar term, this time with growth-rate μ (which models the systematic part of the price dynamics), plus a second term which models the risky part. The uncertainty in the economic and financial climate is represented by the Brownian motion (BM) $W = (W_t)$; this is coupled to the stock-price dynamics via the paramater σ, the *volatility*, which measures how sensitive this particular risky stock is to changes in the overall economic climate. [**6**]
(b) Discounting the prices by e^{rt}, the discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t. \end{aligned}$$

Thus discounting changes the rate μ on the RHS of (GBM) to $\mu - r$. [**6**]
(c) Now use Girsanov's Theorem to change from the real probability measure P to an equivalent probability measure P^* under which the μdt in (GBM) is rdt. Then under P^*, the stock-price dynamics become

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t \qquad (\text{under } P^*).$$

Integrating, $\tilde{S}$ on the left is a stochastic integral w.r.t. Brownian motion – which is a martingale. This P^* is the *equivalent martingale measure (EMM)*, or *risk-neutral measure*. The EMM is that in the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure*. This leads to the Black-Scholes formula by direct probabilistic means, rather than via the Black-Scholes PDE. [**6**]
(d) In the Black-Scholes model, markets are complete. So the EMM is unique. This is a result of the representation theorem for Brownian martingales: *any* Brownian martingale can be represented as a stochastic integral w.r.t. BM. Completeness results from the *continuity* of the paths of BM. [**2**]
[Mainly seen – lectures]

N. H. Bingham