M3A22/M4A22 EXAMINATION SOLUTIONS 2015-16

Q1. (a) *Limited liability.*

A company engages in trading, which is risky; it may be unable to meet its obligations, and go *bankrupt*. The company is owned by its shareholders. They are *liable* for its debts, but only up to the value of their investment: their liability is *limited* (hence, plc [public limited company – earlier, '& Co. Ltd.']). Limited liability emerged in the mid-19th C. Before this, shareholders had unlimited liability, and could be sued for the whole of the loss suffered by a creditor. This made trading very dangerous [early merchants were called *merchant adventurers*; one could end up in a *debtors' prison* this way; Lloyds names had unlimited liability in the scandal of the 1990s]. **[5]**

(b) *Moral hazard.*

This is the danger that people are less careful with other people's money than they are with their own [hence the title of John Kay's book *Other people's money*, 2015]. This is most common with aggressively risky trading, which if it succeeds benefits the trader [bonus, etc.], but if it fails, the loss is born by others [the shareholders of the company]. Examples abound: the dot-com bubble; hedge funds; the behaviour of the banks before the Credit Crunch, etc. This is why there is pressure [from e.g. the Governor of the Bank of England] to make bankers etc. personally liable, under the criminal law, for misbehaviour 'on their watch'. **[5]**

(c) *Liquidity.*

Markets are *liquid* when one can buy or sell freely at the quoted prices. Typically, heavily traded stocks are liquid under normal market conditions. In a crisis, credit dries up [Credit Crunch – really a banking crisis], not enough cash is available to finance trades, and no one wants to trade. Rarely traded items are illiquid – and so are hard to value. **[5]**

(d) *Size of traders.*

Small economic agents are *price takers*. They have no power to influence prices, which they can either take or leave – but equally, do have the power to enter the market without thereby moving the market against them. By contrast, large economic agents are *price makers*. They do have the power to influence prices – but against this, are visible, and so are vulnerable, when forced to enter the market through weakness (examples: the financial authorities of a major country, defending the value of its currency by buying it on the market; a big company forced into a 'fire sale'). **[5]**

[Mainly seen in lectures; discussed in class]

Q2. *American options.* The discounting rate per unit time is $1+\rho$. With 'up' and 'down' factors $1+u$, $1+d$ and 'up' and 'down' probabilities q, $1-q$, the discounted price process is a martingale iff $(1+u)q+(1+d)(1-q) = 1+\rho$:

$$uq + d(1-q) = \rho; \qquad (u-d)q = \rho - d : \qquad q = \frac{\rho - d}{u - d}. \qquad \textbf{[2]}$$

To price the American put in this (Cox-Ross-Rubinstein) binomial-tree model:
1. Draw a binary tree showing the initial stock value S and with the right number, N, of time-intervals. **[2]**
2. Fill in the stock prices: after one time interval, these are Su (upper) and Sd (lower); after two, Su^2, Sud and Sd^2; after i time-intervals, $Su^j d^{i-j}$ at the node with j 'up' steps and $i-j$ 'down' steps. **[2]**
3. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) from the option at the terminal nodes (where the values of the European and American options coincide). **[2]**
4. Work back down the tree one time-step. Fill in (a) the 'European' value at the penultimate nodes as the discounted values of the terminal values, under the risk-neutral (P^*, Q) measure – 'q times upper right plus $1-q$ times lower right'; (b) the 'intrinsic' (early-exercise) value; (c) the American put value as the higher of these. **[2]**
5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'. **[2]**
6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the node set where the early-exercise value is the higher; the rest is the 'continuation region'. **[2]**

Connection with the Snell envelope.

Let $Z = (Z_n)_{n=0}^N$ be the payoff on exercising at time n. To price Z_n, by U_n say, so as to avoid arbitrage: we work backwards in time. Recursively:

$$U_N := Z_N, \qquad U_{n-1} := \max(Z_{n-1}, \frac{1}{1+\rho} E^*[U_n|\mathcal{F}_{n-1}]), \qquad \textbf{[2]}$$

the first alternative on the right corresponding to early exercise, the second to the discounted expectation under P^* (or Q), as usual. Let $\tilde{U}_n = U_n/(1+\rho)^n$ be the discounted price of the American option. Then

$$\tilde{U}_N = \tilde{Z}_N, \qquad \tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}]) : \qquad \textbf{[2]}$$

$(\tilde{U}_n)$ is the *Snell envelope* of the discounted payoff process $(\tilde{Z}_n)$. **[2]**
[Seen – lectures]

Q3. *Black-Scholes formula (BS).*
(a) The SDE for $GBM(\mu, \sigma)$ is $dS_t = S_t(\mu dt + \sigma dW_t)$ with $W = (W_t)$ BM. Its solution is $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$. **[4]**
(b) By Girsanov's Theorem, change probability measure from P to P^* and from $GBM(\mu, \sigma)$ to $GBM(r, \sigma)$, and from time-interval $[0, t]$ to $[t, T]$. With W a P^*-Brownian motion, we can write S_T explicitly as

$$S_T = S_t \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T - t)$, so $(W_T - W_t)/\sqrt{T - t} =: Z \sim N(0, 1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma Z\sqrt{T - t}\}, \quad s := S_t, \quad Z \sim N(0, 1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$C_t = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(T - t)^{\frac{1}{2}} x\} - K]_+ dx. \quad \mathbf{[6]}$$

(c) To derive BS, evaluate the integral. First, [...] > 0 where

$$S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} > K, \qquad (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x > \log(K/S_0):$$

$$x > [\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T} = c, \quad \text{say. So.}$$

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2 T}.\exp\{-\frac{1}{2}x^2 + \sigma\sqrt{T}x\} dx/\sqrt{2\pi} - Ke^{-rT}[1 - \Phi(c)],$$

and the last term is $Ke^{-rT}\Phi(-c) = Ke^{-rT}\Phi(d_-)$. The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x - \sigma\sqrt{T})^2\} dx/\sqrt{2\pi} = \int_{c-\sigma\sqrt{T}}^{\infty} \exp\{-\frac{1}{2}u^2\} du/\sqrt{2\pi}$$

$$= 1 - \Phi(c - \sigma\sqrt{T}) = \Phi(-c + \sigma\sqrt{T}) = \Phi(d_+),$$

as $-c + \sigma\sqrt{T} = d_+$ when $t = 0$. So the option price is given in terms of the initial price S_0, strike price K, expiry T, interest rate r and volatility σ by

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \quad d_\pm := [\log(S_0/K) + (r \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}. \quad //$$

[10]

[Seen – lectures]

Q4. *Real options.* (a) With starting value x, to solve the optimal stopping problem

$$V(x) := \max_{\tau} E[(X_\tau - I)e^{-r\tau}]$$

– buying an asset of value X for a cost I, at time τ chosen optimally. [**3**]
(b) If $\mu \leq 0$, the (mean) value of the project will decrease. So we invest immediately if $x > I$ (with immediate profit $x - I > 0$), and do not invest otherwise. If $\mu > r$, the (mean) growth will swamp the riskless interest rate (in the long run – Law of Large Numbers), so the investment is worthwhile: again invest immediately as there is no point in waiting. If $\mu = r$, there is no point in taking the risk of investing, so we should not invest. [**3**]
(c) There remains the case $0 < \mu < r$. Using the infinitesimal generator, one gets the differential equation (Bellman equation)

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV(x) = 0,$$

with $V(0) = 0$ (we get nothing from something worth nothing). A suitable trial solution is $V(x) = Cx^p$. This leads to a quadratic equation in p:

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. [**4**]
(d) The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$ we get $C_2 = 0$, so $V(x) = C_1 x^{p_1}$, or $V(x) = Cx^{p_1}$. If x^* is the critical value at which it is optimal to invest, 'value matching' and 'smooth pasting' give

$$V(x^*) = x^* - I, \qquad V'(x^*) = 1. \qquad [4]$$

From these two equations, we can find C and x^*:

$$V'(x^*) = Cp_1(x^*)^{p_1 - 1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then value matching gives

$$C(x^*)^{p_1} = x^* - I, \quad x^*/p_1 = x^* - I, \quad I = x^*.(1 - 1/p_1): \quad x^* = \frac{p_1}{(p_1 - 1)} I.$$

So we should not invest if the initial value x is below $x^* = qI$, where $q := p_1/(p_1 - 1)$ ("Tobin's q"). [**4**]
(e) Arbitrage arguments are absent here, as these depend on repeated trading either way, and this investment is a one-off, one way. [**2**]
[Seen, lectures, (a) - (d); (e) unseen]

Q5 *Mastery Question.*

(i) *Sharpe ratio.* The *Sharpe ratio* is $\theta := (\mu - r)/\sigma$: the excess return $\mu - r$ (the investor's reward for taking a risk), compared with the degree of risk as measured by σ. **[2]**

(ii) *Derivation of the Black-Scholes formula via Girsanov's Theorem.*

We summarise the main steps briefly as follows:

(a) Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$. **[1]**

(b) Discount: $d\tilde{S}_t = (\mu - r)\tilde{S} dt + \sigma \tilde{S} dW_t = \sigma \tilde{S}(\theta dt + dW_t)$. **[1]**

(c) Use Girsanov's Theorem to change μ to r, so $\theta := (\mu - r)/\sigma$ to 0: under P^*, $d\tilde{S}_t = \sigma \tilde{S} dW_t$. **[1]**

(d) With V the value process, H the strategy, h the payoff, $d\tilde{V}_t(H) = H_t d\tilde{S}_t = H_t.\sigma \tilde{S} dW_t$. Integrate: $\tilde{V}$ gives a P^*-mg, so has constant E^*-expectation. **[1]**

(e) This gives the Risk-Neutral Valuation Formula (RNVF). **[1]**

(f) From RNVF, we can obtain the Black-Scholes formula, by integration. **[1]**

(iii) *Hedging strategy.*

We seek a hedging strategy $H = (H_t^0, H_t)$ (H_t^0 for cash, H_t for stock) that replicates the value process $V = (V_t)$, given by RNVF:

$$V_t = H_t^0 + H_t S_t = E^*[e^{-r(T-t)} h | \mathcal{F}_t]. \qquad [2]$$

Now

$$M_t := E^*[e^{-rT} h | \mathcal{F}_t] \qquad [2]$$

is a (uniformly integrable) martingale under the filtration $\mathcal{F}_t$, that of the driving BM in (GBM), and the filtration is unchanged by the Girsanov change of measure. So by the Representation Theorem for Brownian Martingales, there is some adapted process $K = (K_t)$ with

$$M_t = M_0 + \int_0^t K_s dW_s \qquad (t \in [0, T]). \qquad [2]$$

Take

$$H_t := K_t/(\sigma \tilde{S}_t), \qquad H_t^0 := M_t - H_t \tilde{S}_t : \qquad [2]$$

$$dM_t = K_t dW_t = \frac{K_t}{\sigma \tilde{S}_t}.\sigma \tilde{S}_t dW_t = H_t d\tilde{S}_t, \qquad [2]$$

and the strategy K is self-financing.

(iv) *Limitations.*

This is of limited practical value: **[1]**

(a) the Representation Theorem does not give $K = (K_t)$ explicitly; **[1]**

(b) as Brownian paths have infinite variation, exact hedging in the Black-Scholes model is too rough to be practically possible. **[1]**

(v) *Practical implementation.*

In practice, one would work instead in *discrete time.* Divide the time-interval $[0, T]$ into a suitably large number N of equal intervals. For each interval, the hedging strategy may be calculated simply, as in the binary one-period model (two simultaneous linear equations in two unknowns). This is simple to programme, and simple to implement by computer. **[4]**

[(i) - (iv) seen – lectures; (v) unseen] N. H. Bingham