M5A22 EXAMINATION 2015-16

Six questions; do five; twenty marks each.

Q1. Discuss briefly:
(i) utility;
(ii) no arbitrage;
(iii) completeness.

Why is there no utility function in the Black-Scholes formula?

Q2. What is meant by saying that a process $C = (C_n)$ is *previsible* (or *predictable*)?

Define the martingale transform $C \bullet X$ of a process $X = (X_n)$ by a previsible process C, and give its financial interpretation.

Show that if X is a martingale and C is bounded and previsible, $C \bullet X$ is a martingale null at zero.

Q3. (i) What is meant by hedging? Who hedges, and why?
(ii) Discuss briefly the most important types of hedging.
(iii) Discuss briefly hedging in discrete time and in continuous time, and their contrasts.
(iv) Should one hedge partially, or completely, and why?

Q4. (a) Given the Black-Scholes formula for European call prices,

$$C_t := S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \qquad d_\pm := \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \quad (BS)$$

($S = S_t$ is the current stock price), show that (with $\tau := T - t$ the time to expiry and $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ the standard normal density)
(i) $\phi(d_-) = \phi(d_+).e^{d_+\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}$;
(ii) $e^{d_+\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}$;
(iii) $\phi(d_-) = \phi(d_+).(S/K).e^{r\tau}: \qquad Ke^{-r\tau}\phi(d_-) = S\phi(d_+)$.
Hence (or otherwise) show that vega (the partial derivative of the option price with respect to the volatility σ) is positive.
(b) Obtain the same result for European put option prices.
(c) Give the financial interpretation of these results.
(d) Do these results extend to American options? If so, prove them.

Q5. (a) Give the stochastic differential equation for $S = (S_t)$ geometric Brownian motion $GBM(\mu, \sigma)$ with parameters μ and σ. State its solution, without proof.
(b) State the risk-neutral valuation formula (in continuous time), applied to a European call option with stock price S_t at time $t \in [0, T]$, strike price K, riskless interest rate r, volatility σ and expiry T.
(c) Hence or otherwise derive the Black-Scholes formula for the price of the call at time $t = 0$:

$$c_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \quad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}. \quad (BS)$$

Q6. (a) Formulate the problem of real options as an optimal-stopping problem.
(b) Show that we may restrict to the case $0 < \mu < r$, where μ is the mean return on the investment and r is the riskless interest rate.
(c) Obtain the fundamental quadratic equation, with roots $p_2 < 0$, $1 < p_1$.
(d) Show that one should not invest the necessary capital I unless the initial value is at least qI, where $q := p_1/(p_1 - 1) > 1$.
(e) Why do arbitrage arguments play no role here?

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