m3a22soln3.tex

SOLUTIONS 3. 6.11.2015

Q1. (i) In spherical polar coordinates (r, θ, ϕ) (r: distance from centre, range 0 to ∞; θ: colatitude ($= \frac{1}{2}\pi$ - latitude), range 0 to π; ϕ longitude, range 0 to 2π): increase r to $r + dr$, etc. The element of volume dV is a (to first order) cuboid, of sides dr ("up"), $rd\theta$ ("South"), $r \sin\theta d\phi$ ("East") (draw a diagram – or consult a textbook if you need one!) So

$$dV = dr.rd\theta.r \sin\theta d\phi = r^2 \sin\theta dr d\theta d\phi.$$

So

$$V = \int_0^r r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = \frac{1}{3} r^3.2\pi[-\cos\theta]_0^{\pi} = \frac{2\pi}{3} r^3[-(-1)-(-1)] = 4\pi r^3/3.$$

(ii) Holding r fixed,

$$dS = r^2 \sin\theta.d\theta d\phi.$$

So

$$A = r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta = r^2.2\pi.2 = 4\pi r^2,$$

by above.
(iii) To first order,

$$dV = Sdr : \qquad S = dV/dr, \qquad V = \int_0^r Sdr$$

('flattening out' the spherical shell: volume = area $\times$ thickness: the curvature effects are second-order). So (i), (ii) are equivalent: ((ii) follows from (i) by differentiating, and (i) from (ii) by integrating.

Q2. This follows by the same method as the area of an ellipse $A = \pi ab$: wlog $a \geq b \geq c$. Compress [squash] the x- and y-axes in the ratios a/c, b/c, to get a sphere of radius c. This has volume $4\pi c^3/3$. Now dilate [unsquash] the x- and y-axes in the ratios a/c, b/c, to get volume

$$V = \frac{4\pi c^3}{3}.\frac{a}{c}.\frac{b}{c} = \frac{4\pi abc}{3}.$$

Q3. (i) Choose the vertex V as origin, and the z-axis vertical – the perpendicular from V to the horizontal base (with z going downwards, if we draw the tetrahedron the usual way). Slice the volume into thin horizontal slices. The area of the slice between z and $z + dz$ is $A(z/h)^2$, by similarity. So

$$V = \int_0^h A(z/h)^2 dz = Ah^{-2} \int_0^h z^2 dz = Ah^{-2}.h^3/3 :$$
$$V = Ah/3.$$

(ii) Similarly in the general case: the above does not use that the base is triangular.

Q4. (i) The range between x and $x + dx$ generates volume $dV = \pi y^2 dx = \pi f(x)^2 dx$. Integrate this from a to b.
(ii) The semicircle on base $[-r, r]$ is $y = f(x) = \sqrt{r^2 - x^2}$. This generates te sphere on revolution, giving

$$V = \int_{-r}^{r} \pi(r^2 - x^2)dx = \pi[r^2 x - \frac{1}{3}x^3]_{-r}^{r} = \pi r^3[1 - \frac{1}{3} - (-1) + (-\frac{1}{3})] = \pi r^3(2 - \frac{2}{3}) = 4\pi r^3/3.$$

Q5 (Georges BOULIGAND, 1935). *First Proof.* For the region S_1 with area A_1 with base the hypotenuse, side 1: use cartesian coordinates to approximate its area, arbitrarily closely, by decomposing it into small squares of area $dA_1 = dxdy$.

For each such small square on side 1, construct similar small squares on sides 2 and 3, of areas dA_2, dA_3.

By Pythagoras' theorem, $dA_1 = dA_2 + dA_3$.

Summing, we get $A_1 = A_2 + A_3$ arbitrarily closely, and so exactly.

Second Proof. Drop a perpendicular from the right-angled vertex to the hypotenuse. This splits the 'big figure' into two 'smaller figures', each similar to it. With l_1 the length of the hypotenuse and l_2, l_3 those of the other two sides, by similarity lengths scale by l_2/l_1, l_3/l_1 on going from the big figure to the smaller ones, so areas scale by $(l_2/l_1)^2$, $(l_3/l_1)^2$. So $A_2 + A_3 = A_1[(l_2/l_1)^2 + (l_3/l_1)^2] = A_1(l_2^3 + l_3^2)/l_1^2, = A_1$ by Pythagoras' theorem. //

NHB