m3a22prob8.tex

PROBLEMS 8. 4.12.2015

Q1. The *moment-generating function* $M_X(t)$ of a random variable X is defined as

$$M_X(t) := E[e^{tX}].$$

If X has the normal distribution $N(\mu, \sigma^2)$, with density

$$f(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\},$$

show (by completing the square, or otherwise) that

$$M_X(t) = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Q2. The *lognormal distribution* $logN(\mu, \sigma^2)$ is defined as the distribution of $X := e^Y$, where Y is $N(\mu, \sigma^2)$.
(i) By using the result of Question 1, or otherwise, show that X has mean

$$E[X] = \exp\{\mu + \frac{1}{2}\sigma^2\}.$$

(ii) Explain without proof why the prices of stocks in the Black-Scholes model are log-normally distributed.

NHB

Optional extra:

Q3. The *characteristic function* (CF) of a random variable X is defined as

$$\phi_X(t) := E[e^{itX}]$$

(this is the usual notation, despite the clash with $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ for the standard normal density – it's usually clear from context which is meant). Show that for $X \sim N(\mu, \sigma^2)$,

$$\phi_X(t) = \exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}.$$

Comment on the similarity with Q1.