M3A22 MOCK MASTERY QUESTION SOLUTION 2014

(i) In the continuous-time Black-Scholes model, the SDE of the stock price $S = (S_t)$ is that of *geometric Brownian motion (GBM)*:

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad (GBM)$$

with $W = (W_t)$ Brownian motion (BM), μ the mean return rate on the stock, σ the volatility of the stock. **[3]**

(ii) The solution to (GBM) is $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$. **[2]**

(iii) The paths $t \mapsto S_t$ of the solution in (ii) are continuous, as BM is. **[2]**

(iv)

$$C_t = e^{-r(T-t)} E^*[(S_T - K)_+ | \mathcal{F}_t],$$

where $\mathcal{F}_t$ is the information available at time t.

To obtain the Black-Scholes formula from this, one uses (ii) and Girsanov's theorem – which in effect replaces μ by r to get, under P^*,

$$\begin{aligned} S_T &= S_t \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\} \\ &= S_t \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{(T-t)}Z\}, \qquad Z \sim N(0,1). \end{aligned}$$

Combining, if $S_t = S$,

$$C_t = \int_{-\infty}^{\infty} [S \exp\{-\frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{(T-t)}x\} - K]_+ . \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

This can be evaluated explicitly, to give the Black-Scholes formula:

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)} K\Phi(d_-), \quad d_\pm := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

[4]

(v) The model is complete. This is a consequence of the Brownian Martingale Representation Theorem and the continuity of Brownian paths. **[2]**

(vi) Hedging is problematic, because it involves continuous rebalancing of the portfolio of cash and stock. This cannot be done in practice, as Brownian paths have finite quadratic variation, so infinite variation: one would need an infinite amount of rebalancing, which is impossible. **[4]**

(vii) To circumvent this, one could work in discrete time with a binomial tree model and the discrete Black-Scholes formula. Or, one could use a price process with jumps (e.g., from a Lévy process), at the cost of incompleteness and so non-uniqueness of prices. **[3]** NHB