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Lecture 7. 20.10.2016

Chapter II. PROBABILITY BACKGROUND.

1. Measure

The language of option pricing involves that of probability, which in turn involves that of *measure theory*. This originated with Henri LEBESGUE (1875-1941), in his 1902 thesis, *'Intégrale, longueur, aire'*. We begin with the simplest case.

Length. The length $\mu(I)$ of an interval $I = (a, b), [a, b], [a, b)$ or $(a, b]$ should be $b - a$: $\mu(I) = b - a$. The length of the disjoint union $I = \bigcup_{r=1}^{n} I_r$ of intervals I_r should be the sum of their lengths:

$$\mu\left(\bigcup_{r=1}^{n} I_r\right) = \sum_{r=1}^{n} \mu(I_r) \qquad \text{(finite additivity)}.$$

Consider now an infinite sequence $I_1, I_2, \ldots$(*ad infinitum*) of disjoint intervals. Letting $n \to \infty$ suggests that length should again be additive over disjoint intervals:

$$\mu\left(\bigcup_{r=1}^{\infty} I_r\right) = \sum_{r=1}^{\infty} \mu(I_r) \qquad \text{(countable additivity)}.$$

For I an interval, A a subset of length $\mu(A)$, the length of the complement $I \setminus A := I \cap A^c$ of A in I should be

$$\mu(I \setminus A) = \mu(I) - \mu(A) \qquad \text{(complementation)}.$$

If $A \subseteq B$ and B has length $\mu(B) = 0$, then A should have length 0 also:

$$A \subseteq B \;\&\; \mu(B) = 0 \;\Rightarrow\; \mu(A) = 0 \quad \text{(completeness)}.$$

Let $\mathcal{F}$ be the smallest class of sets $A \subset \mathbb{R}$ containing the intervals, closed under countable disjoint unions and complements, and complete (containing all subsets of sets of length 0 as sets of length 0). The above suggests – what Lebesgue showed – that length can be sensibly defined on the sets $\mathcal{F}$ on the line, but on no others. There are others – but they are hard to construct (in technical language: the Axiom of Choice (AC), or some variant of it such as Zorn's Lemma, is needed to demonstrate the existence of non-measurable sets – but all such proofs are highly non-constructive). So: some but not all subsets of the line have a length.[1] These are called the *Lebesgue-measurable sets*, and form the class $\mathcal{F}$ described above; length, defined on $\mathcal{F}$ is called *Lebesgue measure* μ (on the real line, $\mathbb{R}$).

Area. The area of a rectangle $R = (a_1, b_1) \times (a_2, b_2)$ – with or without any of its perimeter included – should be $\mu(R) = (b_1 - a_1) \times (b_2 - a_2)$. The area of a finite or countably infinite union of disjoint rectangles should be the sum of their areas:

$$\mu\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \mu(R_n) \qquad \text{(countable additivity).}$$

If R is a rectangle and $A \subseteq R$ with area $\mu(A)$, the area of the complement $R \setminus A$ should be

$$\mu(R \setminus A) = \mu(R) - \mu(A) \qquad \text{(complementation).}$$

If $B \subseteq A$ and A has area 0, B should have area 0:

$$A \subseteq B \ \& \ \mu(B) = 0 \ \Rightarrow \ \mu(A) = 0 \quad \text{(completeness).}$$

Let $\mathcal{F}$ be the smallest class of sets, containing the rectangles, closed under finite or countably infinite unions, closed under complements, and complete (containing all subsets of sets of area 0 as sets of area 0). Lebesgue showed that area can be sensibly defined on the sets in $\mathcal{F}$ and no others. The sets $A \in \mathcal{F}$ are called the *Lebesgue-measurable sets* in the plane $\mathbb{R}^2$; area, defined on $\mathcal{F}$, is called *Lebesgue measure* in the plane. So: some but not all sets in the plane have an area.

Volume. Similarly in three-dimensional space $\mathbb{R}^3$, starting with the volume of a cuboid $C = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ as

$$\mu(C) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot (b_3 - a_3).$$

Euclidean space. Similarly in k-dimensional Euclidean space $\mathbb{R}^k$. We start with

$$\mu\left(\prod_{i=1}^{k} (a_i, b_i\right) = \prod_{i=1}^{k} (b_i - a_i),$$

[1]There are alternatives to AC, under which *all sets are measurable*. So it is not a question of whether AC is *true* or not, but of what axioms of Set Theory we assume. Background: Model Theory in Mathematical Logic, etc.

and obtain the class $\mathcal{F}$ of *Lebesgue-measurable* sets in $\mathbb{R}^k$, and *Lebesgue measure* μ in $\mathbb{R}^k$.
Probability.

The unit cube $[0,1]^k$ in $\mathbb{R}^k$ has Lebesgue measure 1. It can be used to model the *uniform distribution* (density $f(x) = 1$ if $x \in [0,1]^k$, 0 otherwise), with probability = length/area/volume if $k = 1/2/3$.
Note. If a property holds everywhere except on a set of measure zero, we say it holds *almost everywhere* (a.e.) [French: *presque partout*, p.p.; German: *fast überall*, f.u.]. If it holds everywhere except on a set of probability zero, we say it holds *almost surely* (a.s.) [or, with probability one].

2 Integral.
1. Indicators. We start in dimension $k = 1$ for simplicity , and consider the simplest calculus formula $\int_a^b 1 \; dx = b - a$. We rewrite this as

$$I(f) := \int_{-\infty}^{\infty} f(x) \; dx = b - a \quad \text{if } f(x) = I_{[a,b)}(x),$$

the *indicator* function of $[a,b]$ (1 in $[a,b]$, 0 outside it), and similarly for the other three choices about end-points.
2. Simple functions. A function f is called *simple* if it is a finite linear combination of indicators: $f = \sum_{i=1}^n c_i f_i$ for constants c_i and indicator functions f_i of intervals I_i. One then extends the definition of the integral from indicator functions to simple functions by linearity:

$$I\left(\sum_{i=1}^n c_i f_i\right) := \sum_{i=1}^n c_i I(f_i)$$

for constants c_i and indicators f_i of intervals I_i.
3. Non-negative measurable functions. Call f a (*Lebesgue-*) *measurable function* if, for all c, the sets $\{x \; : f(x) \leq c\}$ is a Lebesgue-measurable set (§1). If f is a non-negative measurable function, we quote that it is possible to construct f as the increasing limit of a sequence of simple functions f_n:

$$f_n(x) \uparrow f(x) \quad \text{for all } x \in \mathbb{R} \quad (n \to \infty), \quad f_n \text{ simple}.$$

We then define the integral of f as

$$I(f) := \lim_{n\to\infty} I(f_n) \; (\leq \infty)$$

(we quote that this does indeed define $I(f)$: the value does not depend on *which* approximating sequence (f_n) we use). Since f_n increases in n, so does $I(f_n)$ (the integral is *order-preserving*), so either $I(f_n)$ increases to a finite limit, or diverges to ∞. In the first case, we say f is (*Lebesgue-*) *integrable* with (*Lebesgue-*) *integral* $I(f) = \lim I(f_n)$, or $\int f(x)\, dx = \lim \int f_n(x)\, dx$, or simply $\int f = \lim \int f_n$.

4. Measurable functions. If f is a measurable function that may change sign, we split it into its positive and negative parts, $f_\pm$:

$$f_+(x) := \max(f(x), 0), \quad f_-(x) := -\min(f(x), 0),$$
$$f(x) = f_+(x) - f_-(x), \quad |f(x)| = f_+(x) + f_-(x)$$

If both f_+ and f_- are integrable, we say that f is too, and define

$$\int f := \int f_+ - \int f_-.$$

Then, in particular, $|f|$ is also integrable, and

$$\int |f| = \int f_+ + \int f_-.$$

Note. The Lebesgue integral is, by construction, an *absolute integral*: f is integrable iff $|f|$ is integrable. Thus, for instance, the well-known formula

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

has no meaning for Lebesgue integrals, since $\int_1^\infty \frac{|\sin x|}{x}\, dx$ diverges to $+\infty$ like $\int_1^\infty \frac{1}{x}\, dx$. It has to be replaced by the limit relation

$$\int_0^X \frac{\sin x}{x}\, dx \to \frac{\pi}{2} \qquad (X \to \infty).$$

The class of (Lebesgue-) integrable functions f on $\mathbb{R}$ is written $L(\mathbb{R})$ or (for reasons explained below) $L_1(\mathbb{R})$ – abbreviated to L_1 or L.

Higher dimensions. In $\mathbb{R}^k$, we start instead from k-dimensional boxes. If f is the indicator of a box $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$, $\int f := \prod_{i=1}^k (b_i - a_i)$. We then extend to simple functions by linearity, to non-negative measurable functions by taking increasing limits, and to measurable functions by splitting into positive and negative parts.