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Lecture 4. 13.10.2016

Short-selling (ctd). Not only is short-selling both routine and necessary in some contexts, such as foreign exchange and commodities futures, it simplifies the mathematics. So we assume, unless otherwise specified, no restriction on short-selling. By extension, we call a portfolio, or position, *short* in an asset if the holding of the asset is negative, *long* if the holding of the asset is positive. It turns out that in some important contexts – such as the Black-Scholes theory of European and American calls – short-selling can be avoided. In such cases, it is natural and sensible to do so: see Ch. VI.

7. Put-Call Parity.

Just as long and short positions are diametrical opposites, so are call and put options. We now use arbitrage to show how they are linked.

Suppose there is a risky asset, value S (or S_t at time t), with European call and put options on it, value C, P (or C_t, P_t), with expiry time T and strike-price K. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for the value of this portfolio. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call).}$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ & \text{for a call, } C, \\ \max(K-S,0) & \text{or} \quad (K-S)_+ = (S-K)_- & \text{for a put, } P. \end{cases}$$

So the value of the above portfolio at expiry is K: for, it is

$$S + 0 - (S-K) = K \quad \text{if } S \geq K, \qquad S + (K-S) - 0 = K \quad \text{if } K \geq S.$$

Alternatively, use $x = x_+ - x_-$ and $(-x)_+ = x_-$ with $x = S - K$.

This portfolio thus guarantees a payoff K at time T. How much is it worth at time t?

Short answer (correct, and complete): $Ke^{-r(T-t)}$, because it is financially equivalent to cash K, so has the same time-t value as cash K.

Longer answer (included as an example of arbitrage arguments). The riskless way to guarantee a payoff K at time T is to deposit $Ke^{-r(T-t)}$ in the bank at time t and do nothing. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time T my

portfolio yields K (above), while my bank debt has grown to K. I clear my cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio short* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, my bank deposit has grown to K, and I again clear my cash account – use this to meet my obligation K on the portfolio I sold short, again locking in my earlier riskless profit. So the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make a riskless profit) – which they will take! Thus
(i) The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

This link between the prices of the underlying asset S and call and put options on it is called *put-call parity*.
(ii) The value of the portfolio $S + P - C$ is the *discounted value of the riskless equivalent*. This is a first glimpse at the central principle, or insight, of the entire subject of option pricing. But in general, we will have 'risk-neutral' in place of 'riskless'; see I.8 below, Ch. IV and Ch. VI.
(iii) Arbitrage arguments, although apparently qualitative, have quantitative conclusions, and allow one to calculate precisely the rational price – or *arbitrage price* – of a portfolio. The put-call parity argument above is the simplest example – though typical – of the *arbitrage pricing technique* (APT).
(iv) The APT is due to S. A. Ross in 1976-78 (details in [BK], Preface). Put-call parity has a long history (see Wikipedia).
Note. 1. History shows both that arbitrage opportunities exist (or are sought) in the real world and that the exploiting of them is a delicate matter. The collapse of Baring's Bank in 1995 (the UK's oldest bank, and bankers to HMQ) was triggered by unauthorised dealings by one individual, who tried and failed to exploit a fine margin between the Singapore and Osaka Stock Exchanges. The leadership of Baring's Bank at that time thought that the trader involved had discovered a clever way to exploit price movements in either direction between Singapore and Osaka. This is obviously impossible on theoretical grounds, to anyone who knows any Physics. See Problems 2 Q1 (key phrases: perpetual motion machine; Maxwell's demon; Second Law of Thermodynamics; entropy).

2. Major finance houses have an *arbitrage desk*, where their *arbs* work.

8. An Example: Single-Period Binary Model.

We consider the following simple example, taken from
[CRR] COX, J. C., ROSS, S. A. & RUBINSTEIN, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263.
For definiteness, we use the language of foreign exchange. Our risky asset will be the current price in Swiss francs (SFR) of (say) 100 US \$, supposed $X_0 = 150$ at time 0. Consider a call option with strike price $K = 150$ at time T. The simplest case is the binary model, with two outcomes: suppose the price X_T of 100 \$ at time T is (in SFR)

$$X_T = \begin{cases} 180 & \text{with probability } p \\ 90 & \text{with probability } 1-p. \end{cases}$$

The payoff H of the option will be $30 = 180 - 150$ with probability p, 0 with probability $1 - p$, so has expectation $EH = 30p$. This would seem to be the fair price for the option at $t = 0$, or allowing for an interest-rate r and discounting, we get the value

$$V_0 = E(\frac{H}{1+r}) = \frac{30p}{1+r}.$$

Take for simplicity $p = \frac{1}{2}$ and $r = 0$ (no interest): the naive, or expectation, value of the option at time 0 is

$$V_0 = 15.$$

The *Black-Scholes value* of the option, however, is different. To derive it, we follow the Black-Scholes prescription (Ch. IV, VI):
(i) First replace p by p^* so that the price, properly discounted, *behaves like a fair game*:

$$X_0 = E^*(\frac{X_T}{1+r}).$$

That is,

$$150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90);$$

for $r = 0$ this gives $60 = 90p^*$ or $p^* = 2/3$.
(ii) Now compute the fair price of the expected value in this new model:

$$V_0 = E^*(\frac{H}{1+r}) = \frac{30p^*}{1+r};$$

for $r = 0$ this gives the Black-Scholes value as $V_0 = 20$.
Justification: it works! – as the arbitrage constructed below shows. For simplicity, take $r = 0$.
We *sell* the option at time 0, for a price $\pi(H)$, say. We then prepare for the resulting contingent claim on us at time T by the option holder by using the following strategy:

Sell the option for $\pi(H)$	$+\pi(H)$
Buy \$33.33 at the present exchange rate of 1.50	-50
Borrow SFR 30	$+30$
Balance	$\pi(H) - 20.$

So our balance at time 0 is $\pi(H) - 20$. At time T, two cases are possible:
(i) The dollar has risen:

Option is exercised (against us)	-30
Sell dollars at 1.80	$+60$
Repay loan	-30
Balance	0.

(ii) The dollar has fallen:

Option is worthless	0.00
Sell dollars at 0.90	$+30$
Repay loan	-30
Balance	0.

So the balance at time T is zero in both cases. The balance $\pi(H) - 20$ at time 0 should thus also be zero, giving the Black-Scholes price $\pi(H) = 20$ as above. For, *any other price* gives an arbitrage opportunity. Argue as in put-call parity in §4: if the option is offered too cheaply, buy it; if it is offered too dearly, write it (the equivalent for options to 'sell it short' for stock). Thus any other price would offer an arbitrageur the opportunity to extract a riskless profit, by appropriately buying and selling (Swiss francs, US dollars and options) so as to exploit your mis-pricing.

The same argument with interest-rate r also applies: divide everything through by $1 + r$.
Note. This argument, and result, are **independent** of p, the 'real' probability, and depend instead **only** on this 'fictitious' new probability, p^* (which is called the *risk-neutral* or *risk-adjusted* probability.