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Lecture 31. 16.12.2016

Real options (continued).
For (i): this comes from the *generator* of the diffusion $GBM(r, \sigma)$ (cf. the *SDE* for $GBM(r, \sigma)$, and Black-Scholes *PDE*, VI.2); for details, see [DP Ch. 5], or Peskir & Shiryaev [PS, Ch. III]. For (ii) ("Nothing will come of nothing"): the GBM does not hit 0, but if it approaches 0, so will the value of the project, so (ii) follows from this by continuity). For (iii), this is the *value-matching condition*: on investment, the firm receives the net pay-off $x^* - I$. For (iv) (*smooth pasting*): think of a rope stretched tightly over a convex surface.

Again, the ODE (i) is *homogeneous* (cf. *Euler's theorem*). So we use a trial solution $V(x) = Cx^p$. So (i) gives that p satisfies the *fundamental quadratic*

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$, $C_2 = 0$, so $V(x) = C_1 x^{p_1}, = Cx^{p_1}$ say. With x^* the critical value at which it is optimal to invest, (iii) and (iv) give

$$V(x^*) = x^* - I, \qquad V'(x^*) = 1.$$

From these two equations, we can find C and x^*. The second is

$$V'(x^*) = Cp_1(x^*)^{p_1 - 1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then the first gives

$$C(x^*)^{p_1} = x^* - I, \qquad x^*/p_1 = x^* - I, \qquad x^* = \frac{p_1}{(p_1 - 1)} I.$$

The main feature here is the factor

$$q := p_1/(p_1 - 1) > 1$$

by which the value must exceed the investment cost I before investment should be made (q is used because this is related to "Tobin's q" in Economics). One can check that q increases with σ (the riskier the project, the more reluctant we are to invest), and also q increases with r (as then investing our capital risklessly becomes more attractive). Then the critical threshold above which it is optimal to invest is

$$x^* = qI.$$

Also

$$C = (qI)^{1-p_1}/p_1, \qquad V(x) = (qI)^{1-p_1} x^{p_1}/p_1.$$

The results above show that the traditional *net present value* (NPV – accountancy-based) approach to valuing real options is misleading – see [DP]. This is no surprise: our methods (*arbitrage pricing technique*, etc.) are superior to NPV!

7. Stochastic volatility (SV).

The Black-Scholes theory above – in discrete or continuous time – has involved the volatility – the parameter that describes the sensitivity of the stock price to new information, to the market's assessment of new information. Volatility is so important that it has been subjected to intensive scrutiny, in the light of much real market data. Alas, such detailed scrutiny reveals that volatility is not really constant at all – the Black-Scholes theory over-simplifies reality. (This is hardly surprising: real financial markets are more complicated than the contents of this course, as they involve *investor psychology*, rather than straight mathematics!) One way out is to admit that volatility is *random* (stochastic), and then try to model the stochastic process generating it. Volatility exhibits *clustering*, linked to *mean reversion*, so *Ornstein-Uhlenbeck* models are useful here. Such *stochastic volatility models* are topical today.

Stylised facts.

There are a number of *stylised facts* in mathematical finance. E.g.:

(i). Financial data show *skewness*. This is a result of the asymmetry between profit and loss (large losses are lethal!; large profits are just nice to have).

(ii). Financial data have much *fatter tails* than the normal (Gaussian). We have discussed this in I.5.

(iii) Financial data show *volatility clustering*. This is a result of the economic and financial environment, which is extremely complex, and which moves between good times/booms/upswings and bad times/slumps/downswings. Typically, the market 'gets stuck', staying in its current state for longer than is objectively justified, and then over-correcting. As investors are highly sensitive to losses (see (i) above), downturns cause widespread nervousness, which is reflected in higher volatility. The upshot is that good times are associated with periods of growth but low volatility; downturns spark extended periods of high volatility (and economic stagnation, or shrinkage).

ARCH and GARCH.

We turn to models that can incorporate such features.

The model equations are (with Z_t ind. $N(0,1)$)

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2, \tag{$ARCH(p)$}$$

while in $GARCH(p,q)$ the σ_t^2 term becomes

$$\sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2 + \sum_1^q \beta_j \sigma_{t-j}^2. \tag{$GARCH(p,q)$}$$

The names stand for (generalised) autoregressive conditionally heteroscedastic (= variable variance). These are widely used in Econometrics, to model *volatility clustering* – the common tendency for periods of high volatility, or variability, to cluster together in time. They were introduced in 1987 by Robert Engle (1942) and C. W. J. (Sir Clive) Granger (1934-2009), who received the Nobel Prize for this in 2003. From Granger's obituary (*The Times*, 1.6.2009): "Following Granger's arrival at UCSD in La Jolla, he began the work with his colleague Robert F. Engle for which he is most famous, and for which they received the Bank of Sweden Nobel Memorial Prize in Economic Sciences in 2003. They developed in 1987 the concept of cointegration. Cointegrated series are series that tend to move together, and commonly occur in economics. Engle and Granger gave the example of the price of tomatoes in N. and S. Carolina .... Cointegration may be used to reduce non-stationary situations to stationary ones, which are much easier to handle statistically and so to make predictions for. This is a matter of great economic importance, as most macroeconomic time series are non-stationary, so temporary disturbances in, say, GDP may have a long-lasting effect, and so a permanent economic cost. The Engle-Granger approach helps to separate out short-term effects, which are random and unpredictable, from long-term effects, which reflect the underlying economics. This is invaluable for macroeconomic policy formulation, on matters such as interest rates, exchange rates, and the relationship between incomes and consumption."

Volatility Modelling

In the standard Black-Scholes theory we have developed, volatility σ is *constant*. Thus a graph of volatility against strike K (or stock price S) should be flat. But typically it isn't, and displays curvature. Such volatility curves often turn upwards at both ends ('volatility smile'); there may well be asymmetry ('volatility smirk').

As above, it may be useful to model volatility stochastically, and use an SV model. However, the driving noise in this model will have a volatility of its own ('vol of vol'), etc. Practitioners often use *computer graphics* to represent *volatility surfaces* – the three-dimensional equivalents of graphs, where e.g. σ is graphed against K and S. The subject is too big to pursue further here; there is a good account (mixing theory with practice) in

J. GATHERAL: *The volatility surface: A practitioner's guide*. Wiley 2006.

Volatility is rough.

This is the title of an influential paper by Gatheral, Jaisson and Rosenbaum in 2014. The message there is that (log-)volatility is not only rough, it is *rougher than Brownian motion*. The reasons are the obvious ones: high-frequency trading, and order splitting. There is a family, *fractional Brownian motion*, with a parameter controlling the roughness (called the Hurst index): BM is 'in the middle'. This is highly topical today: there is a lot going on in this area.

Postscript.

1. One recent book on Financial Mathematics describes the subject as being composed of three strands:
arbitrage – the core economic concept, which we have used throughout;
martingales – the key probabilistic concept (Ch. III on);
numerics. Finance houses in the City use *models*, which they need to *calibrate to data* – a task involving both statistical and numerical skills, and in particular an ability to *programme*.

2. You will probably already have experience with at least one general/mathematical programming language (e.g., Matlab, Python) (if not: get it, a.s.a.p.!), and for Statistics, R. You may also know some Numerical Analysis, the theory behind computation. You may have encountered *simulation*, also known as *Monte Carlo*, and/or a branch of Probability and Statistics called *Markov Chain Monte Carlo (MCMC)* – computer-intensive methods for numerical solutions to problems too complicated to solve analytically. The leaders of R & D teams in the City need to be expert at both stochastic modelling (e.g., to propose new products), and simulation (to evaluate how these perform). Most of the ones I know use Matlab for this. At a lower level, quantitative analysts (quants) working under them need expertise in a computer language; C++ is the industry standard. If you are thinking of a career in Mathematical Finance, learn C++, as soon as possible, and for academic credit.

3. This course deals with *equity markets* – with *stocks*, and financial derivatives of them – options on stocks, etc. The relevant mathematics is *finite-dimensional*. Lurking in the background are *bond markets* ('money markets': bonds, gilts etc., where *interest rates* dominate), and the relevant options – *interest-rate derivatives*, and *foreign exchange* between different currencies ('forex'). The resulting mathematics (which is highly topical, and so in great demand in the City!) is *infinite-dimensional*, and so much harder than the equity-market theory we have done. However, the underlying principles are basically the same. One has to learn to walk before one learns to run, and equity markets serve as a preparation for money markets.

4. The aim of this lecture course is simple. It is to familiarize the student with the basics of Black-Scholes theory, as the core of modern finance, and with the mathematics necessary to understand this. The motivation driving the ever-increasing study of this material is the financial services industry and the City. I hope that any of you who seek City careers will find this introduction to the subject useful in later life. NHB, 2016