m3f22l30.tex
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§5. Infinite time-horizon; American puts

We sketch here the theory of the American option (one can exercise at any time), over an infinite time-horizon. We deal first with a *put* option (see VI.6 below under *Real options* for the corresponding 'call option') – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we have to take the decision whether or not to stop at τ based on information already available – no access to the future, no insider trading. As above, we pass to the risk-neutral measure.

Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \qquad (GBM_r)$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_{\tau} E_x[e^{-r\tau}(K - X_\tau)^+]$$

where the sup is taken over all stopping times τ and $X_0 = x$ under P_x.

The process X satisfying (GBM_r) – a *diffusion* – is specified by a second-order linear differential operator, called its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \partial/\partial x.$$

Now the closer X gets to 0, the less likely we are to gain by continuing. This suggests that our best strategy is to stop when X gets too small: to stop at $\tau = \tau_b$, where

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\},$$

for some $b \in (0, K)$ (the only range in which we would want to exercise an option to sell at K). This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* $b \in (0, K)$:

$$L_X V = rV \qquad \text{for } x > b; \qquad (i)$$

$$V(x) = (K - x)_+ = K - x \qquad \text{for } x = b; \qquad (ii)$$

$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)}; \qquad (iii)$$

$$V(x) > (K-x)^+ \qquad \text{for } x > b; \tag{iv}$$

$$V(x) = (K-x)^+ \qquad \text{for } 0 < x < b. \tag{v}$$

Writing $d := \sigma^2/2$ ('d for diffusion'), (i) is

$$dx^2 V'' + rxV' - rV = 0. \tag{i*}$$

This ODE is *homogeneous.* So (Euler's theorem): use trial solution:

$$V(x) = x^p.$$

Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

Trial solution: $V(x) = x^p$. Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0: \qquad (p-1)(p + r/d) = 0.$$

One root is $p = 1$; the other is $p = -r/d$. So the general solution is $V(x) = C_1 x + C_2 x^{-r/d}$. But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!): $V(x)$ is bounded. Taking x large ($x < b$ is covered by (v)), we must have $C_1 = 0$. So (with $C := C_2$)

$$V(x) = Cx^{-r/d}; \qquad V'(x) = -\frac{r}{d}.C^{-r/d-1}. \tag{$*$}$$

From (ii),

$$Cb^{-r/d} = K - b,$$

while from (iii),

$$-\frac{r}{d}.Cb^{-r/d}.\frac{1}{b} = -1: \qquad Cb^{-r/d} = \frac{bd}{r}.$$

Equating, this gives C and b:

$$\frac{bd}{r} = K - b, \qquad K = b(1 + d/r), \quad b = K/(1 + d/r).$$

Then ($*$) and (iii) give

$$C = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}.$$

So

$$
\begin{aligned}
V(x) &= \frac{d}{r}\Big(\frac{K}{1+d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b,\infty) \\
&= K - x \qquad \text{if } x \in (0,b].
\end{aligned}
$$

This is in fact the full and correct solution to the problem; see [P&S], §25.1.

The 'smooth fit' in (iii) is characteristic of free boundary problems. For a heuristic analogy: imagine trying to determine the shape of a rope, tied to the ground on one side of a convex body, stretched over the body, then pulled tight and tied to the ground on the other side. We can see on physical grounds that the rope will be:
straight to the left of the convex body;
continuously in contact with the body for a while, then
straight to the right of the body, and
there should be no kink in the rope at the points where it makes and then leaves contact with the body. This corresponds to 'smooth fit' in (iii).

6. Real options (Investment options).

For background and details, see e.g.
[DP] Avinash K. DIXIT and Robert S. PINDYCK: *Investment under uncertainty*. Princeton University press, 1994;
[PS] G. PESKIR and A. N. SHIRYAEV: *Optimal stopping and free-boundary problems*. Birkhäuser, 2006.

The options considered above concern financial *derivatives* (so called because they derive from the underlying fundamentals such as stock). We turn now to options of another kind, concerned with business decision-making. Typically, we shall be concerned with the decision of whether or not to make a particular investment, and if so, when. Because these options concern the real economy (of manufacturing, etc.) rather than financial markets such as the stock market, such options are often called *real options*. But because they typically concern investment decisions, they are also often called *investment options*. There is a good introductory treatment in [DP].

The key features are as follows. We are contemplating making some major investment – buying or building a factory, drilling an oil well, etc. While if the decision goes wrong it may be possible to recoup some of the cost, much or most of it will usually be irrecoverable (a *sunk cost* – as with an oil well). So the investment is *irreversible* – at least in part. Just as stock prices are uncertain – so we model them as random, using some stochastic process

– here too, the future profitability of the proposed investment is *uncertain*. Finally, we do not have to act now, or indeed at all. So we have an open-ended – or infinite – time-horizon, $T = \infty$.

We may choose to delay investment,
(a) to gather more information, to help us assess the project, or
(b) to continue to generate interest on the capital we propose to invest.
So we must recognize, and feed into the decision process, the value of *waiting for further information*. When we commit ourselves and make the decision to invest, it is not just the sunk cost that we lose – we lose the valuable option to wait for new information.

This situation is really that of an *American call* option with an infinite time-horizon. With such an American *call*, we have the right to buy at a specified price at a time of our choosing (or indeed, not to buy). Following Dixit & Pindyck [DP, Ch. 5], we formulate an *optimal stopping problem*, and solve it as a *free boundary problem*, using the *principle of smooth fit*.

We suppose the cost of the investment is I, and that the value of the project is given by a GBM, $X = (X_t) \sim GBM(\mu, \sigma)$ (the value of a project is uncertain for the same reasons that stock prices are uncertain; we model them both as stochastic processes; GBM is the default option here, just as in the BS theory of Ch. IV). If we invest at time τ, we want to maximize

$$V(X) := \max_{\tau} \; E[(X_\tau - I)e^{-r\tau}],$$

with r the riskless rate (discount rate) as before. Now if $\mu \leq 0$ the value of the project will fall, so we should invest immediately if $X_0 > I$ and not invest if not. If $\mu > r$, the growth of X will swamp the investment cost I and more than offset the discounting, so we should invest and there is no point in waiting. So we take $\mu \in (0, r]$. We invest iff the value x^* at the time of investment is *large enough*; finding x^* is part of the problem; x^* is a *free boundary* (between the continuation region and the investment region).

We need the following four conditions:

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV = 0, \qquad (i)$$

$$V(0) = 0, \qquad (ii)$$

$$V(x^*) = x^* - I, \qquad (iii)$$

$$V'(x^*) = 1 \qquad \text{(smooth pasting)}. \qquad (iv)$$