m3f22l29.tex
Lecture 29 12.12.2016

Now form a portfolio based on two assets: the underlying stock and the option (recall that options are also assets in their own right – they have a value (Black-Scholes formula), and are traded (in large quantities)). Let the *relative portfolio* in stock S and derivative Π be (U_t^S, U_t^Π). Then the dynamics for the value V of the portfolio are given by

$$\begin{aligned} dV_t/V_t &= U_t^S dS_t/S_t + U_t^\Pi d\Pi_t/\Pi_t \\ &= U_t^S(\mu dt + \sigma dW_t) + U_t^\Pi(\mu_\Pi dt + \sigma_\Pi dW_t) \\ &= (U_t^S \mu + U_t^\Pi \mu_\Pi)dt + (U_t^S \sigma + U_t^\Pi \sigma_\Pi)dW_t, \end{aligned}$$

by above. Now both brackets are linear in U^S, U^Π, and $U^S + U^\Pi = 1$ as proportions sum to 1. This is one linear equation in the two unknowns U^S, U^Π, and we can obtain a second one by eliminating the driving Wiener term in the dynamics of V – for then, the portfolio is *riskless*. So it must have return r, the riskless interest rate, to avoid arbitrage. We thus solve the two equations

$$\begin{aligned} U^S + U^\Pi &= 1 \\ U^S\sigma + U^\Pi\sigma_\Pi &= 0. \end{aligned}$$

The solution of the two equations above is

$$U^\Pi = \frac{\sigma}{\sigma - \sigma_\Pi}, \qquad U^S = \frac{-\sigma_\Pi}{\sigma - \sigma_\Pi},$$

which as $\sigma_\Pi = \sigma S F_2/F$ gives the portfolio explicitly as

$$U^\Pi = \frac{F}{F - SF_2}, \qquad U^S = \frac{-SF_2}{F - SF_2}.$$

With this choice of relative portfolio, the dynamics of V are given by

$$dV_t/V = (U_t^S \mu + U_t^\Pi \mu_\Pi)dt,$$

which has no driving Wiener term. So, no arbitrage as above implies that the return rate is the short interest rate r:

$$U_t^S \mu + U_t^\Pi \mu_\Pi = r.$$

Now substitute the values (obtained above)

$$\mu_\Pi = (F+\mu SF_2+\frac{1}{2}\sigma^2 S^2 F_{22})/F, \quad U^S = (-SF_2)/(F-SF_2), \quad U^\Pi = F/(F-SF_2).$$

Substituting the values above in the no-arbitrage relation gives

$$\frac{-SF_2}{F-SF_2}.\mu + \frac{F}{F-SF_2}.\frac{F_1+\mu SF_2+\frac{1}{2}\sigma^2 F_{22}}{F} = r.$$

So

$$-SF_2\mu + F_1 + \mu SF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} = rF - rSF_2,$$

giving the Black-Scholes PDE as required:

$$F_1 + rSF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} - rF = 0. \qquad (BS) \quad //$$

Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic, and so a relative of the *heat equation*. After some months' work, they were able to transform it into the heat equation. The solution to this is known classically.[1] On transforming back, they obtained the Black-Scholes formula.

Theorem (Feynman-Kac Formula). The solution $F(t,x)$ to the PDE

$$F_1(t,x) + \mu(t,x)F_2(t,x) + \frac{1}{2}\sigma^2(t,x)F_{22}(t,x) = g(t,x) \qquad (PDE)$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = E_{t,x}h(X_T) - E_{t,x}\int_t^T g(s,X_s)ds, \qquad (FK)$$

where X satisfies the SDE

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s \qquad (t \le s \le T) \qquad (SDE)$$

[1]See e.g. the link to MPC2 (Mathematics and Physics for Chemists, Year 2) on my website, Weeks 4, 9. The solution is in terms of *Green functions*. The Green function for (fundamental solution of) the heat equation has the form of a *normal density* (*heat kernel*). This reflects the close link between the mathematics of the heat equation (Fourier in 1807) and the mathematics of Brownian motion (Wiener in 1923) noted earlier (Kakutani, 1944 – Potential Theory).

with initial condition $X_t = x$.

Proof. Consider a SDE, with initial condition (IC), of the form

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \le s \le T), \tag{SDE}$$

$$X_t = x. \tag{IC}$$

For suitably well-behaved functions μ, σ, this SDE has a unique solution $X = (X_s : t \le s \le T)$, a diffusion. We refer for details on solutions of SDEs and diffusions to an advanced text such as [RW2], [RY], [KS §5.7]. Uniqueness of solutions of the SDE is related to *completeness*, and uniqueness of prices (Representation Theorem for Brownian Martingales, above). (This is as in the FTAP of Ch. IV, but the continuous-time case is harder – here we have to quote uniqueness rather than prove it.)

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma, as above,

$$dF = F_1 ds + F_2 dX + \frac{1}{2} F_{22} (dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2 (dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22}) ds + \sigma F_2 dW_s. \tag{$*$}$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t, x) + \mu(t, x) F_2(t, x) + \frac{1}{2}\sigma^2 F_{22}(t, x) = g(t, x) \tag{PDE}$$

$$F(T, x) = h(x). \tag{BC}$$

Then $(*)$ gives

$$dF = g ds + \sigma F_2 dW_s,$$

which can be written in stochastic-integral form as

$$F(T, X_T) = F(t, X_t) + \int_t^T g(s, X_s) ds + \int_t^T \sigma(s, X_s) F_2(s, X_s) dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, writing $E_{t,x}$

for expectation with value x and starting-time t, and the price at expiry T as $h(X_T)$ as before, taking $E_{t,x}$ gives the Feynman-Kac formula:

$$E_{t,x}h(X_T) = F(t,x) + E_{t,x}\int_t^T g(s,X_s)ds. \qquad //$$

Re-derivation of the Black-Scholes formula via the Black-Scholes PDE and the Feynman-Kac formula.

Now replace $\mu(t,x)$ by rx, $\sigma(t,x)$ by σx, g by rF in the Feynman-Kac formula above. The SDE becomes that for $GBM(r,\sigma)$:

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (**)$$

– the same as for a risky asset with mean return-rate r (the short interest-rate for a riskless asset) in place of μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = rF, \qquad (BS)$$

the Black-Scholes PDE. So by the Feynman-Kac formula,

$$dF = rFds + \sigma F_2 dW_s, \qquad F(T,s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s,X_s) := e^{-rs}F(s,X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs}Fds + e^{-rs}dF = e^{-rs}(dF - rFds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a mg: *the discounted price process* $G(s,X_s) = e^{-rs}F(s,X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(**)$. This is the measure P we started with, *except* that μ has been changed to r. Thus, G has constant P^*-expectation: with $X_t = x$,

$$E^*_{t,x}G(t,X_t) = E^*_{t,x}e^{-rt}F(t,X_t) = e^{-rt}F(t,x) = E^*_{T,x}e^{-rT}F(T,X_T) = e^{-rT}h(X_T).$$

This gives the Black-Scholes formula, as before. //

The route of §3 via Girsanov's theorem is more direct and probabilistic; that here via the Black-Scholes PDE and Feynman-Kac is more traditional applied mathematics.