m3f22l28tex
Lecture 28 9.12.2016

Comments.
1. *Risk-neutral measure.* We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem, which gives the Radon-Nikodym derivative showing equivalence), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing (FTAP).* The above continuous-time result may be summarised just as the FTAP in discrete time: to get the no-arbitrage price of a contingent claim, take the discounted expected value under the equivalent mg (risk-neutral) measure.
3. *Completeness.* In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to Girsanov's Theorem, which we have not proved in full. Completeness questions are linked to the Representation Theorem for Brownian Martingales, below.

Theorem (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \leq t \leq T)$ be a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \leq t \leq T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \leq t \leq T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof.

The economic relevance of the Representation Theorem is that it shows (see e.g. [KS, I.6], and below) that the Black-Scholes model is *complete* – that is, that EMMs are unique, and so that *Black-Scholes prices are unique* (we know this already, from FTAP/RNVF above). Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of BM are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

Hedging.

To find a hedging strategy $H = (H_t^0, H_t)$ (H_t^0 for cash, H_t for stock) that replicates the value process $V = (V_t)$, itself given by RNVF (VI.3 L28):

$$V_t = H_t^0 + H_t S_t = E^*[e^{-r(T-t)} h | \mathcal{F}_t].$$

Now

$$M_t := E^*[e^{-rT} h | \mathcal{F}_t]$$

is a martingale (indeed, a uniformly integrable mg: IV.4, V.2) under the filtration $\mathcal{F}_t$, that of the driving BM in (GBM) (VI.1, VI.2), and the filtration is unchanged by the Girsanov change of measure (we quote this). So by the Representation Theorem for Brownian Martingales, there is some adapted process $K = (K_t)$ with

$$M_t = M_0 + \int_0^t K_s dW_s \qquad (t \in [0, T]).$$

Take

$$H_t := K_t / (\sigma \tilde{S}_t), \qquad H_t^0 := M_t - H_t \tilde{S}_t.$$

Then

$$dM_t = K_t dW_t = \frac{K_t}{\sigma \tilde{S}_t} . \sigma \tilde{S}_t dW_t = H_t d\tilde{S}_t,$$

and the strategy given by K is self-financing, by VI.2. This is of limited practical value:

(a) the Representation Theorem does not give $K = (K_t)$ explicitly – it is merely an existence proof;

(b) we already know that, as Brownian paths have infinite variation, exact hedging in the Black-Scholes model is too rough to be practically possible.

So to hedge in practice, we need to go back to discrete time, where we can compute things and where such roughness questions do not arise. But this is familiar by now (and is why we have Chapters III, IV in discrete time and Chapters V, VI in continuous time). We need to go back and forth at will between *continuous* time – where we can do *calculus*, in particular, Itô calculus – and *discrete* time – where we can *calculate*, using *computers.*

Comments on the Black-Scholes formula.

1. The Black-Scholes formula transformed the financial world. Before it (see Ch. I), the expert view was that asking what an option is worth was (in effect) a silly question: the answer would necessarily depend on the attitude to risk of the individual considering buying the option. It turned out that – at least approximately (i.e., subject to the restrictions to perfect – frictionless – markets, including No Arbitrage – an over-simplification of reality) there *is* an option value. One can see this in one's head, without doing any mathematics, if one knows that the Black-Scholes market is *complete* (above). So, every contingent claim (option, etc.) can be *replicated*, by a suitable combination of cash and stock. Anyone can price this: (i) count the cash, and count the stock; (ii) look up the current stock price; (iii) do the arithmetic.

2. The programmable pocket calculator was becoming available around this time. Every trader immediately got one, and programmed it, so that he could price an option (using the Black-Scholes model!) in real time, from market data.

3. The missing quantity in the Black-Scholes formula is the *volatility*, σ. But, the price is continuous and strictly increasing in σ (options like volatility!). So there is *exactly one* value of σ that gives the price at which options are being currently traded. This – the *implied volatility* – is the value that the market currently judges σ to be, and the one that traders use.

4. Because the Black-Scholes model is the benchmark model of mathematical finance, and gives a value for σ at the push of a button, it is widely used.

5. This is *despite* the fact that no one actually believes the Black-Scholes model! It is an over-simplified approximation to reality. Indeed, Fischer Black himself famously once wrote a paper called *The holes in Black-Scholes*.

6. This is an interesting example of theory and practice interacting!

7. Black and Scholes has considerable difficulty in getting their paper published! It was ahead of its time. When published, and its importance understood, it changed its times.

8. Black-Scholes theory and its developments, plus the internet (a global network of fibre-optic cables – using *photons* rather than *electrons*), were important contributory factors to *globalization*. Enormous sums of money can be transported round the world at the push of a button, and are every day. This has led to *financial contagion* – "one country's economic problem becomes the world's economic problem". (The Ebola virus comes to mind here.) The resulting problems of *systemic stability* are very important, and still largely unsolved; they dominate the agenda at international meetings.

4. BS via the Black-Scholes PDE and the Feynman-Kac formula

Theorem (Black-Scholes PDE, 1973). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \end{aligned}$$

let a contingent claim be tradable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t,x) + rxF_2(t,x) + \frac{1}{2}x^2\sigma^2(t,x)F_{22}(t,x) - rF(t,x) = 0, \qquad (BS)$$

$$F(T,x) = h(x). \qquad (BC)$$

Proof. By Itô's Lemma ($\Pi = F$, $d\Pi = dF$),

$$d\Pi_t = F_1 dt + F_2 dS_t + \frac{1}{2}F_{22}(dS_t)^2$$

(since t has finite variation, the F_{11}- and F_{12}-terms are absent as $(dt)^2$ and $dtdS_t$ are negligible with respect to the terms retained)

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\mu S_t dt + \sigma S_t dW_t)^2$$

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\sigma S_t dW_t)^2$$

(the contribution of the FV terms in dt are negligible, as above)

$$= (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 {S_t}^2 F_{22})dt + \sigma S_t F_2 dW_t$$

(as $(dW_t)^2 = dt$). Now $\Pi = F$, so (multiplying by Π, dividing by F)

$$d\Pi_t = \Pi_t(\mu_\Pi(t)dt + \sigma_\Pi(t)dW_t),$$

where

$$\mu_\Pi(t) := (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 S_t^2 F_{22})/F, \qquad \sigma_\Pi(t) := \sigma S_t F_2/F.$$