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Lecture 26 5.12.2016

Return intervals.

We saw above (VI.1 L25) that log-prices and returns being (approximately) normal is bound up with

$$\log(1+x) \sim x, \qquad (1+x/n)^n \to e^x$$

– which we can now recognise as being bound up with the passage from discrete time (time-interval Δt, small but finite, as in IV) to continuous time (time-interval dt, infinitesimal, and the SDE for GBM as above). Now in investment, there are many possible time-scales, corresponding to how often we observe prices; we single out the main three (cf. [BK, §2.9]).

1. *Long (macroscopic).*

Here we are investing over a time-scale of months (say), and observe prices daily (say). As the price-change over the month is the sum of price-changes over the days, and these are independent (as Brownian increments are), the reason we get normality is the *Central Limit Theorem (CLT)*: if we sum many independent random variables with finite mean and variance, we get normality (in the limit) after centring and scaling. This is the phenomenon of *aggregational Gaussianity*. Note that Gaussian (normal) tails are *extremely thin* ('minus log-density' grows quadratically). The 'rule of thumb' is that 16 trading days suffice here.

2. *Intermediate (mesoscopic).*

If our investment time-frame is, say, a day (there are 'day traders' out there!), aggregational Gaussianity does not set in, and the tails observed are *much fatter* – typically, 'minus log-density' grows linearly. One model commonly used here is that of *hyperbolic distributions* (see e.g. [BK, §2.12]).

3. *Short (microscopic).*

With the development of the Internet and the intensive computerisation of trading, high-frequency data – '*tick data*' – is available; here the interval may be of the order of seconds or much smaller. Here, the picture is different again: the tails are *much fatter still*: tails decay like a power, so 'minus log-density' grows logarithmically. Distributions used include Student t and stable (see e.g. [BK, §2.9]).

Note. The world's most famous investor, Warren Buffett, the Sage of Omaha, famously invests *right*, and over a time-frame of *many years*.

§2. The Black-Scholes Model

For the purposes of this section only, it is convenient to be able to use the 'W for Wiener' notation for Brownian motion/Wiener process, thus liberating B for the alternative use 'B for bank [account]'. Thus our driving noise terms will now involve dW_t, our deterministic [bank-account] terms dB_t.

We now consider an investor constructing a trading strategy in continuous time, with the choice of two types of investment:
(i) riskless investment in a bank account paying interest at rate $r > 0$ (the *short rate* of interest): $B_t = B_0 e^{rt} \quad (t \geq 0)$ [we neglect the complications involved in possible failure of the bank – though *banks do fail* – witness Barings 1995, or AIB 2002!];
(ii) risky investment in stock, one unit of which has price modelled as above by $GMB(\mu, \sigma)$. Here the volatility $\sigma > 0$; the restriction $0 < r < \mu$ on the short rate r for the bank and underlying rate μ for the stock are economically natural (but not mathematically necessary); the stock dynamics are thus

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Notation. Later, we shall need to consider several types of risky stock - d stocks, say. It is convenient, and customary, to use a *superscript* i to label stock type, $i = 1, \cdots, d$; thus $S^1, \cdots, S^d$ are the risky stock prices. We can then use a superscript 0 to label the bank account, S^0. So with one risky asset as above, the dynamics are

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t. \end{aligned}$$

We shall focus on pricing at time 0 of options with expiry time T; thus the index-set for time t throughout may be taken as $[0, T]$ rather than $[0, \infty)$.

We proceed as in the discrete-time model of IV.1. A *trading strategy* H is a vector stochastic process

$$H = (H_t : 0 \leq t \leq T) = ((H_t^0, H_t^1, \cdots, H_t^d)) : 0 \leq t \leq T)$$

which is *previsible*: each H_t^i is a previsible process (so, in particular, $(\mathcal{F}_{t-})$-adapted) [we may simplify with little loss of generality by replacing previsibility here by *left-continuity* of H_t in t]. The vector $H_t = (H_t^0, H_t^1, \cdots, H_t^d)$

is the *portfolio* at time t. If $S_t = (S_t^0, S_t^1, \cdots, S_t^d)$ is the vector of *prices* at time t, the *value* of the portfolio at t is the scalar product

$$V_t(H) := H_t.S_t = \Sigma_{i=0}^d H_t^i S_t^i.$$

The *discounted value* is

$$\tilde{V}_t(H) = \beta_t(H_t.S_t) = H_t.\tilde{S}_t,$$

where $\beta_t := 1/S_t^0 = e^{-rt}$ (fixing the scale by taking the initial bank account as 1, $S_0^0 = 1$), so

$$\tilde{S}_t = (1, \beta_t S_t^1, \cdots, \beta_t S_t^d)$$

is the vector of discounted prices.

Recall that
(i) in IV.1 H is a self-financing strategy if $\Delta V_n(H) = H_n.\Delta S_n$, i.e. $V_n(H)$ is the martingale transform of S by H,
(ii) stochastic integrals are the continuous analogues of mg transforms.
We thus define the strategy H to be *self-financing*, $H \in SF$, if

$$dV_t = H_t.dS_t = \Sigma_0^d H_t^i dS_t^i.$$

The discounted value process is

$$\tilde{V}_t(H) = e^{-rt} V_t(H)$$

and the interest rate is r. So

$$d\tilde{V}_t(H) = -re^{-rt}dt.V_t(H) + e^{-rt}dV_t(H)$$

(since e^{-rt} has finite variation, this follows from integration by parts,

$$d(XY)_t = X_t dY_t + Y_t dX_t + \frac{1}{2}d\langle X, Y\rangle_t$$

– the quadratic covariation of a finite-variation term with any term is zero)

$$= -re^{-rt}H_t.S_t dt + e^{-rt}H_t.dS_t = H_t.(-re^{-rt}S_t dt + e^{-rt}dS_t) = H_t.d\tilde{S}_t$$

($\tilde{S}_t = e^{-rt}S_t$, so $d\tilde{S}_t = -re^{-rt}S_t dt + e^{-rt}dS_t$ as above).

Summarising: for H self-financing,

$$dV_t(H) = H_t.dS_t, \qquad d\tilde{V}_t(H) = H_t.d\tilde{S}_t,$$

$$V_t(H) = V_0(H) + \int_0^t H_s dS_s, \qquad \tilde{V}_t(H) = \tilde{V}_0(H) + \int_0^t H_s d\tilde{S}_s.$$

Now write $U_t^i := H_t^i S_t^i / V_t(H) = H_t^i S_t^i / \Sigma_j H_t^j S_t^j$ for the *proportion* of the value of the portfolio held in asset $i = 0, 1, \cdots, d$. Then $\Sigma U_t^i = 1$, and $U_t = (U_t^0, \cdots, U_t^d)$ is called the *relative portfolio*. For H self-financing,

$$dV_t = H_t.dS_t = \Sigma H_t^i dS_t^i = V_t \Sigma \frac{H_t^i S_t^i}{V_t}.\frac{dS_t^i}{S_t^i} : \qquad dV_t = V_t \Sigma U_t^i dS_t^i / S_t^i.$$

Dividing through by V_t, this says that the return dV_t/V_t is the weighted average of the returns dS_t^i/S_t^i on the assets, weighted according to their proportions U_t^i in the portfolio – as one would expect.
Note. Having set up this notation (that of [HP]) – in order to be able if we wish to have a basket of assets in our portfolio – we now prefer – for simplicity – to specialise back to the simplest case, that of one risky asset. Thus we will now take $d = 1$ until further notice.

§3. The (continuous) Black-Scholes formula (BS): derivation via Girsanov's Theorem

The Sharpe ratio.

There is no point in investing in a risky asset with mean return rate μ, when cash is a riskless asset with return rate r, unless $\mu > r$. The excess return $\mu - r$ (the investor's reward for taking a risk) is compared with the risk, as measured by the volatility σ, via the *Sharpe ratio*

$$\theta := (\mu - r)/\sigma,$$

also known as the *market price of risk*. This is important, both here (see below), in CAPM (I.3, L2), and in asset allocation decisions.

Consider now the Black-Scholes model, with dynamics

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t = -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t = \sigma \tilde{S}_t (\theta dt + dW_t). \end{aligned}$$