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Lecture 25 2.12.2016

Itô's Lemma (ctd).

This important result may be summarised as follows: use Taylor's theorem formally, together with the rule

$$(dt)^2 = 0, \qquad dtdB = 0, \qquad (dB)^2 = dt.$$

Itô's Lemma extends to higher dimensions, as does the rule above:

$$df = (f_0 + \Sigma_{i=1}^d U_i f_i + \frac{1}{2}\Sigma_1^d V_i^2 f_{ii})dt + \Sigma_1^d V_i f_i dB_i$$

(where U_i, V_i, B_i denote the ith coordinates of vectors U, V, B, f_i, f_{ii} denote partials as above); here the formal rule is

$$(dt)^2 = 0, \qquad dtdB_i = 0, \qquad (dB_i)^2 = dt, \qquad dB_i dB_j = 0 \qquad (i \neq j).$$

Corollary. $E[f(t, X_t)] = f_0 + \int_0^t E[f_1 + Uf_2 + \frac{1}{2}V^2 f_{22}]dt$.

Proof. $\int_0^t V f_2 dB$ is a stochastic integral, so a martingale, so its expectation is constant ($= 0$, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's Lemma really comes into its own in the more general setting of semimartingales. It says there that if X is a semimartingale and f is a smooth function as above, then $f(t, X(t))$ is also a semimartingale. The ordinary differential dt gives rise to the bounded-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.

Example: The Ornstein-Uhlenbeck Process.

The most important example of a SDE for us is that for geometric Brownian motion (VI.1 below). We close here with another example.

Consider now a model of the velocity V_t of a particle at time t ($V_0 = v_0$), moving through a fluid or gas, which exerts

(i) a frictional drag, assumed propertional to the velocity,

(ii) a noise term resulting from the random bombardment of the particle by the molecules of the surrounding fluid or gas. The basic model is the SDE

$$dV = -\beta V dt + cdB, \qquad (OU)$$

whose solution is called the *Ornstein-Uhlenbeck* (velocity) process with *relaxation time* $1/\beta$ and *diffusion coefficient* $D := \frac{1}{2}c^2/\beta^2$. It is a stationary Gaussian Markov process (not stationary-increments Gaussian Markov like Brownian motion), whose limiting (ergodic) distribution is $N(0, \beta D)$ (the *Maxwell-Boltzmann distribution* of Statistical Mechanics) and whose limiting correlation function is $e^{-\beta|\cdot|}$.

If we integrate the OU velocity process to get the OU *displacement process*, we lose the Markov property (though the process is still Gaussian). Being non-Markov, the resulting process is much more difficult to analyse.

The OU process is the prototype of processes exhibiting *mean reversion*, or a *central push*: frictional drag acts as a restoring force tending to push the process back towards its mean. It is important in many areas, including
(i) statistical mechanics, where it originated,
(ii) mathematical finance, where it appears in the *Vasicek model* for the term-structure of interest-rates (the mean represents the 'natural' interest rate),
(iii) *stochastic volatility* models, where the volatility σ itself is now a stochastic process σ_t, subject to an SDE of OU type.

Theory of interest rates.

This subject dominates the mathematics of *money markets*, or *bond markets*. These are more important in today's world than stock markets, but are more complicated, so we must be brief here. The area is crucially important in *macro-economic policy*, and in political decision-making, particularly after the financial crisis ("credit crunch"). Government policy is driven by fear of speculators in the bond markets (rather than aimed at inter-governmental cooperation against them). The mathematics is infinite-dimensional (at each time-point t we have a whole *yield curve* over future times), but reduces to finite-dimensionality: bonds are only offered at discrete times, with a *tenor* structure (a finite set of maturity times).

Chapter VI. MATHEMATICAL FINANCE IN CONTINUOUS TIME

§1. Geometric Brownian Motion (GBM)

As before, we write B for standard Brownian motion. We write $B_{\mu,\sigma}$ for Brownian motion with *drift* μ and *diffusion coefficient* σ: the path-continuous Gaussian process with independent increments such that

$$B_{\mu,\sigma}(s+t) - B_{\mu,\sigma}(s) \text{ is } N(\mu t, \sigma^2 t).$$

This may be realised as

$$B_{\mu,\sigma}(t) = \mu t + \sigma B(t).$$

Consider the process

$$X_t = f(t, B_t) := x_0.\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\} : \ \log X_t = const + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t. \qquad (*)$$

Here, since

$$f(t,x) = x_0.\exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\},$$

$$f_1 = (\mu - \frac{1}{2}\sigma^2)f, \qquad f_2 = \sigma f, \qquad f_{22} = \sigma^2 f.$$

By Itô's Lemma (Ch. V: $dX_t = U_t dt + V_t dB_t$ and f smooth implies $df = (f_1 + U f_2 + \frac{1}{2}V^2 f_{22})dt + V f_2 dB_t$) we have (taking $U = 0, \quad V = 1, \quad X = B$),

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t :$$

$$dX_t = \mu f dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE

$$dX_t = X_t(\mu dt + \sigma dB_t) : \qquad dX_t/X_t = \mu dt + \sigma dB_t, \qquad (GBM)$$

and is called *geometric Brownian motion* (GBM). We turn to its economic meaning, and the role of the two parameters μ and σ, below. It will be used to model *price processes* in the Black-Scholes model of VI.2. But note that in $(*)$, *log-prices* $\log X_t$ *are normally distributed.*

Note that for $\mu = 0$, (GBM) shows that X is a martingale (see VI.3, in connection with Girsanov's theorem).

We recall the model of Brownian motion from Ch. V. It was developed (by Brown, Einstein, Wiener, ...) in *statistical mechanics*, to model the irregular, random motion of a particle suspended in fluid under the impact of collisions with the molecules of the fluid.

The situation in *economics* and *finance* is analogous: the price of an asset depends on many factors (a share in a manufacturing company depends on, say, its own labour costs, and raw material prices for the articles it manufactures. Together, these involve, e.g., foreign exchange rates, labour costs, transport costs, etc. – all of which respond to the unfolding of events – economic data/political events/the weather/technological change/labour, commercial and environmental legislation/ ... in time. There is also the effect of individual transactions in the buying and selling of a traded asset on the asset price. The analogy between the buffeting effect of molecules on a particle in the statistical mechanics context on the one hand, and that of this continuous flood of new price-sensitive information on the other, is highly suggestive. The first person to use Brownian motion to model price movements in economics was Bachelier in his celebrated thesis of 1900.

Bachelier's seminal work was not definitive (indeed, not correct), either mathematically (it was pre-Wiener) or economically. In particular, Brownian motion itself is inadequate for modelling prices, as
(i) it attains negative levels, and
(ii) one should think in terms of *return*, rather than prices themselves.
However, one can allow for both of these by using *geometric*, rather than ordinary, Brownian motion as one's basic model. This has been advocated in economics from 1965 on by Samuelson[1] – and was Itô's starting-point for his development of Itô or stochastic calculus in 1944 – and is now standard.

Returning now to (GBM), the SDE above for geometric Brownian motion driven by Brownian noise, we can see how to interpret it. We have a risky asset (stock), whose price at time t is X_t; $dX_t = X(t + dt) - X(t)$ is the change in X_t over a small time-interval of length dt beginning at time t; dX_t/X_t is the gain per unit of value in the stock, i.e. the *return*. This is a sum of two components:
(i) a deterministic component μdt, equivalent to investing the money risklessly in the bank at interest-rate μ (> 0 in applications), called the *underlying return rate* for the stock,
(ii) a random, or noise, component σdB_t, with *volatility* parameter $\sigma > 0$ and driving Brownian motion B, which models the market uncertainty, i.e. the effect of noise. Note that dB_t is a Brownian increment, so is normally distributed. So: *returns are normally distributed.*
Note. That both log-prices and returns are normally distributed just reflects

$$\log(1 + x) \sim x \qquad (x \to 0),$$

or equivalently (as in I.1, L1),

$$\left(1 + \frac{x}{n}\right) \to e^x \qquad (n \to \infty).$$

[1]Paul A. Samuelson (1915-2009), American economist; Nobel Prize in Economics, 1970