m3f22l22tex

Lecture 22 25.11.2016

Filtrations and insider trading (ctd).

The regulatory authorities (Securities and Exchange Commission – SEC – in US; Financial Conduct Authority (FCA) and Prudential Regulation Authority (PRA, part of the Bank of England (BoE) in UK) monitor all trading electronically. Their software alerts them to patterns of suspicious trades. The software design (necessarily secret, in view of its value to criminals) involves all the necessary elements of Mathematical Finance in exaggerated form: economic and financial insight, plus: mathematics, probability and stochastic processes; statistics (especially pattern recognition, data mining and machine learning); numerics and computation.

§3. Classes of Processes.

1. *Martingales.*

The martingale property in continuous time is as in discrete time:

$$E[X_t|\mathcal{F}_s] = X_s \qquad (s < t),$$

and similarly for submartingales and supermartingales. There are regularization results, under which one can take X_t right-continuous in t. The convergence results, and UI mgs – important as they occur in RNVF – are similar. Among the contrasts: the Doob-Meyer decomposition, easy in discrete time (III.8), is deep in continuous time. For background, see e.g.
MEYER, P.-A. (1966): *Probabilities and potentials.* Blaisdell
- and subsequent work by Meyer and the French school (Dellacherie & Meyer, *Probabilités et potentiel*, I-V, etc.

2. Gaussian Processes.

Recall the multivariate normal distribution $N(\mu, \mathbf{\Sigma})$ in n dimensions. If $\mu \in \mathbb{R}^n$, $\mathbf{\Sigma}$ is a non-negative definite $n \times n$ matrix, $\mathbf{X}$ has distribution $N(\mu, \mathbf{\Sigma})$ if it has characteristic function

$$\phi_{\mathbf{X}}(\mathbf{t}) := E\exp\{i\mathbf{t}^T.\mathbf{X}\} = \exp\{i\mathbf{t}^T.\mu - \frac{1}{2}\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}\} \qquad (\mathbf{t} \in \mathbb{R}^n).$$

If further $\mathbf{\Sigma}$ is positive definite (so non-singular), $\mathbf{X}$ has density (*Edgeworth's Theorem* of 1893: F. Y. Edgeworth (1845-1926), English statistician)

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{1}{2}n}|\Sigma|^{\frac{1}{2}}}\exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\}.$$

A process $X = (X_t)_{t \geq 0}$ is *Gaussian* if all its finite-dimensional distributions are Gaussian. Such a process can be specified by:
(i) a measurable function $\mu = \mu(t)$ with $EX_t = \mu(t)$,
(ii) a non-negative definite function $\sigma(s, t)$ with

$$\sigma(s, t) = cov(X_s, X_t).$$

Gaussian processes have many interesting properties. Among these, we quote *Belayev's dichotomy*: with probability one, the paths of a Gaussian process are either continuous, or extremely pathological: for example, unbounded above and below on any time-interval, however short. Naturally, we shall confine attention in this course to continuous Gaussian processes.

3. Markov Processes.

X is *Markov* if for each t, each $A \in \sigma(X_s : s > t)$ (the 'future') and $B \in \sigma(X_s : s < t)$ (the 'past'),

$$P(A|X_t, B) = P(A|X_t).$$

That is, if you know where you are (at time t), how you got there doesn't matter so far as predicting the future is concerned – equivalently, past and future are conditionally independent given the present.

The same definition applied to Markov processes in discrete time.

X is said to be *strong Markov* if the above holds with the *fixed* time t replaced by a *stopping time* T (a random variable). This is a real restriction of the Markov property in continuous time (though not in discrete time) – another instance of the difference between the two.

4. Diffusions.

A *diffusion* is a path-continuous strong-Markov process such that for each time t and state x the following limits exist:

$$\mu(t, x) := \lim_{h \downarrow 0} \frac{1}{h} E[(X_{t+h} - X_t)|X_t = x],$$

$$\sigma^2(t, x) := \lim_{h \downarrow 0} \frac{1}{h} E[(X_{t+h} - X_t)^2|X_t = x].$$

Then $\mu(t, x)$ is called the *drift*, $\sigma^2(t, x)$ the *diffusion coefficient*. Then $p(t, x, y)$, the density of transitions from x to y in time t, satisfies the parabolic PDE

$$Lp = \partial p/\partial t, \qquad L := \frac{1}{2}\sigma^2 D^2 + \mu(x) D, \qquad D := \partial/\partial x.$$

The (2nd-order, linear) differential operator L is called the *generator*. *Brownian motion* is the case $\sigma = 1$, $\mu = 0$, and gives the *heat equation* ($L = \frac{1}{2}D^2$ in one dimension, half the *Laplacian* Δ in higher dimensions).

It is not at all obvious, but it is true, that this definition does indeed capture the nature of physical diffusion. Examples: heat diffusing through a metal; smoke diffusing through air; dye diffusing through liquid; pollutants diffusing through air or liquid.

§4. Quadratic Variation (QV) of Brownian Motion; Itô's Lemma

Recall that for $\xi\ N(\mu, \sigma^2)$, ξ has moment-generating function (MGF)

$$M(t) := E\exp\{t\xi\} = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Take $\mu = 0$ below; for $\xi\ N(0, \sigma^2)$,

$$\begin{aligned} M(t) := E\exp\{t\xi\} &= \exp\{\frac{1}{2}\sigma^2 t^2\} \\ &= 1 + \frac{1}{2}\sigma^2 t^2 + \frac{1}{2!}\left(\frac{1}{2}\sigma^2 t^2\right)^2 + O(t^6) \\ &= 1 + \frac{1}{2!}\sigma^2 t^2 + \frac{3}{4!}\sigma^4 t^4 + O(t^6). \end{aligned}$$

So as the Taylor coefficients of the MGF are the moments (hence the name!),

$$E(\xi^2) = var\xi = \sigma^2, \qquad E(\xi^4) = 3\sigma^4, \qquad \text{so} \qquad var(\xi^2) = E(\xi^4) - [E(\xi^2)]^2 = 2\sigma^4.$$

For $B\ BM$, this gives in particular

$$EB_t = 0, \qquad varB_t = t, \qquad E[(B_t)^2] = t, \qquad var[(B_t)^2] = 2t^2.$$

In particular, for $t > 0$ *small*, this shows that the variance of B_t^2 is negligible compared with its expected value. Thus, the *randomness* in B_t^2 is negligible compared to its mean for t small.

This suggests that if we take a fine enough partition $\mathcal{P}$ of $[0, T]$ – a finite set of points

$$0 = t_0 < t_1 < \cdots < t_k = T$$

with $|\mathcal{P}| := \max|t_i - t_{i-1}|$ small enough – then writing

$$\Delta B(t_i) := B(t_i) - B(t_{i-1}), \qquad \Delta t_i := t_i - t_{i-1},$$

$\Sigma(\Delta B(t_i))^2$ will closely resemble $\Sigma E[(\Delta B(t_i)^2]$, which is $\Sigma \Delta t_i = \Sigma(t_i - t_{i-1}) = T$. This is in fact true a.s.:

$$\Sigma(\Delta B(t_i))^2 \to \Sigma \Delta t_i = T \qquad \text{as} \qquad \max |t_i - t_{i-1}| \to 0.$$

This limit is called the *quadratic variation* V_T^2 of B over $[0, T]$:

Theorem. The quadratic variation of a Brownian path over $[0, T]$ exists and equals T, a.s.

For details of the proof, see e.g. [BK], §5.3.2, SP L22, SA L7,8.

If we increase t by a small amount to $t + dt$, the increase in the QV can be written symbolically as $(dB_t)^2$, and the increase in t is dt. So, formally we may summarise the theorem as

$$(dB_t)^2 = dt.$$

Suppose now we look at the *ordinary* variation $\Sigma|\Delta B_t|$, rather than the *quadratic* variation $\Sigma(\Delta B_t)^2$. Then instead of $\Sigma(\Delta B_t)^2 \sim \Sigma \Delta t \sim t$, we get $\Sigma|\Delta B_t| \sim \Sigma\sqrt{\Delta t}$. Now for Δt small, $\sqrt{\Delta t}$ is of a larger order of magnitude that Δt. So if $\Sigma \Delta t = t$ converges, $\Sigma\sqrt{\Delta t}$ diverges to $+\infty$. This suggests – what is in fact true – the

Corollary. The paths of Brownian motion are of infinite variation - their variation is $+\infty$ on every interval, a.s.

The QV result above leads to Lévy's 1948 result, the Martingale Characterization of BM. Recall that B_t is a continuous martingale with respect to its natural filtration $(\mathcal{F}_t)$ and with QV t. There is a remarkable converse; we give two forms.

Theorem (Lévy; Martingale Characterization of Brownian Motion). If M is any continuous local $(\mathcal{F}_t)$-martingale with $M_0 = 0$ and quadratic variation t, then M is an $(\mathcal{F}_t)$-Brownian motion.

Theorem (Lévy). If M is any continuous $(\mathcal{F}_t)$-martingale with $M_0 = 0$ and $M_t^2 - t$ a martingale, then M is an $(\mathcal{F}_t)$-Brownian motion.

For proof, see e.g. [RW1], I.2.