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Lecture 20 21.11.2016

American puts (ctd).
5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'.
6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the node set where the early-exercise value is the higher; the rest is the 'continuation region'.
Note. The above procedure is simple to describe and understand, and simple to programme. It is laborious to implement numerically by hand, on examples big enough to be non-trivial. Numerical examples are worked through in detail in [H1], 359-360 and [CR], 241-242.

Mathematically, the task remains of describing the *continuation region* - the part of the tree where early exercise is not optimal. This is a classical *optimal stopping problem*. No explicit solution is known (and presumably there isn't one). We will, however, connect the work above with that of III.7 [L13] on the Snell envelope. Consider the pricing of an American put, strike price K, expiry N, in discrete time, with discount factor $1+r$ per unit time as earlier. Let $Z = (Z_n)_{n=0}^N$ be the payoff on exercising at time n. We want to price Z_n, by U_n say (to conform to our earlier notation), so as to avoid arbitrage; again, we work backwards in time. The recursive step is

$$U_{n-1} = \max(Z_{n-1}, \frac{1}{1+r} E^*[U_n | \mathcal{F}_{n-1}]),$$

the first alternative on the right corresponding to early exercise, the second to the discounted expectation under P^*, as usual. Let $\tilde{U}_n = U_n/(1+r)^n$ be the discounted price of the American option. Then

$$\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n | \mathcal{F}_{n-1}]) :$$

$(\tilde{U}_n)$ is the *Snell envelope* (III.7) of the discounted payoff process $(\tilde{Z}_n)$. So:
(i) a P^*-supermartingale,
(ii) the smallest supermartingale dominating $(\tilde{Z}_n)$,
(iii) the solution of the optimal stopping problem for $\tilde{Z}$.
Note. One can use the Snell envelope to prove Merton's theorem (equivalence of American and European calls) without using arbitrage arguments. For details see e.g. [BK, Th. 4.7.1 and Cor. 4.7.1].

P-measure and P^*- (or $Q-$) measure.

We use P and P^* in the above, as E and E^* are convenient, but P and Q when the emphasis is on Q, for brevity.

The measure P, the *real* (or real-world) probability measure, models the uncertainty driving prices, which are indeed uncertain, thus allowing us to bring mathematics to bear on financial problems. But P is difficult to get at directly. By contrast, Q is more accessible: the *market* tells us about Q, or more specifically, *trading* does. In addition, trading also tells us about the *volatility* σ, via implied volatility, which we can infer from observing the prices at which options are traded. So Q is certainly more accessible than P. There is thus a sense in which it is Q, rather than P, which is the more real.

It is as well to bear all this in mind when looking at specific problems, particularly numerical ones. Now that we know the CRR binomial-tree model, which gives us the Black-Scholes formula in discrete time (and hence also, by the limiting argument above, the Black- Scholes formula in continuous time, the main result of the course), we can recognise the 'one-period, up or down' model ($/SFr in I.8 L5, price of gold in Problems 5), though clearly artificial and stylised, as a workable 'building block' of the whole theory. Because P itself does not occur in the Black-Scholes formula(e), from a purely financial point of view there is little need to try to construct more realistic, and so more complicated, models of P. Instead, one can exploit what one can infer about Q, which does occur in Black-Scholes, from seeing the prices at which options trade.

From the economic point of view, it is the real world, the real economy, and so the real probability measure P, that matters. The 'Q-measure-eye view of the world' has a degree of artificiality, in so far as options do. One can eat food, and needs to. One can't eat options.

A fuller discussion of Q-measure involves *Arrow-Debreu* prices, equilibria etc., but we omit this for lack of time.

Where we are.

The course splits neatly into three parts: Ch. I, II [L 1-10] on background, Ch. III, IV [L 11-20] on discrete time, and Ch. V, VI [L 20-30] on continuous time. We have already seen the main ideas – and proved nearly everything seen so far. In V, VI we gain the tremendous power of Itô (stochastic) calculus (calculus is our most powerful weapon, in mathematics and science!), and the ability to work in continuous time. What we lose is the ability to prove so much and to see what is happening so clearly and so concretely.

Chapter V. STOCHASTIC PROCESSES IN CONTINUOUS TIME

§1. Brownian Motion.

The Scottish botanist Robert Brown observed pollen particles in suspension under a microscope in 1828 and 1829 (though this had been observed before),[1] and observed that they were in constant irregular motion.

In 1900 L. Bachelier considered Brownian motion a possible model for stock-market prices:
BACHELIER, L. (1900): Théorie de la spéculation. *Ann. Sci. Ecole Normale Supérieure* **17**, 21-86
– the first time Brownian motion had been used to model financial or economic phenomena, and before a mathematical theory had been developed.

In 1905 Albert Einstein considered Brownian motion as a model of particles in suspension, and used it to estimate *Avogadro's number* ($N \sim 6 \times 10^{23}$), based on the diffusion coefficient D in the *Einstein relation*

$$var X_t = Dt \qquad (t > 0).$$

In 1923 Norbert WIENER defined and constructed Brownian motion rigorously for the first time. The resulting stochastic process is often called the *Wiener process* in his honour, and its probability measure (on path-space) is called *Wiener measure*.

We define *standard Brownian motion* on $\mathbb{R}$, BM or $BM(\mathbb{R})$, to be a stochastic process $X = (X_t)_{t \geq 0}$ such that
1. $X_0 = 0$,
2. X has *independent increments*: $X_{t+u} - X_t$ is independent of $\sigma(X_s : s \leq t)$ for $u \geq 0$,
3. X has *stationary increments*: the law of $X_{t+u} - X_t$ depends only on u,
4. X has *Gaussian increments*: $X_{t+u} - X_t$ is normally distributed with mean 0 and variance u,

$$X_{t+u} - X_t \sim N(0, u),$$

5. X has *continuous paths*: X_t is a continuous function of t, i.e. $t \mapsto X_t$ is continuous in t.

For time t in a finite interval – $[0, 1]$, say – we can use the following filtered space: (i) $\Omega = C[0, 1]$, the space of all continuous functions on $[0, 1]$; (ii) the points $\omega \in \Omega$ are thus random functions, and we use the coordinate mappings:

[1]The Roman author Lucretius observed this phenomenon in the gaseous phase – dust particles dancing in sunbeams – in antiquity: *De rerum natura*, c. 50 BC.

X_t, or $X_t(\omega)$, $= \omega_t$; (iii) the filtration is given by $\mathcal{F}_t := \sigma(X_s : 0 \le s \le t)$, $\mathcal{F} := \mathcal{F}_1$; (iv) P is the measure on $(\Omega, \mathcal{F})$ with finite-dimensional distributions specified by the restriction that the increments $X_{t+u} - X_t$ are stationary independent Gaussian $N(0, u)$.

Theorem (WIENER, 1923). Brownian motion exists.

The best way to prove this is by construction, and one that reveals some properties. The result below is originally due to Paley, Wiener and Zygmund (1933) and Lévy (1948), but is re-written in the modern language of *wavelet* expansions. We omit the proof; for this, see e.g. [BK] 5.3.1, or SP L20-22. The *Haar system* $(H_n) = (H_n(.))$ is a complete orthonormal system (cons) of functions in $L^2[0,1]$. The *Schauder system* $\Delta_n)$ is obtained by integrating the Haar system. Consider the triangular function (or 'tent function')

$$\Delta(t) := 2t \quad \text{on} \quad [0, \frac{1}{2}), \qquad 2(1-t) \quad \text{on } [\frac{1}{2}, 1], \qquad 0 \quad \text{else.}$$

With $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, define the nth *Schauder function* Δ_n by

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \ge 1).$$

Note that Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval, which is small for n, j large). We see that

$$\int_0^t H(u)du = \frac{1}{2}\Delta(t),$$

and similarly

$$\int_0^t H_n(u)du = \lambda_n \Delta_n(t),$$

where $\lambda_0 = 1$ and for $n \ge 1$,

$$\lambda_n = \frac{1}{2} \times 2^{-j/2} \qquad (n = 2^j + k \ge 1).$$

The Schauder system (Δ_n) is again a complete orthogonal system on $L^2[0,1]$. We can now formulate the next result; for proof, see the references above.