m3f22l19tex
**Lecture 19 18.11.2016**
*Vega.*

From the Black-Scholes formula (which gives the price explicitly as a function of $\sigma$), one can check by calculus (Problems 7) that

$$\partial c/\partial \sigma > 0,$$

and similarly for puts (or, use the result for calls and put-call parity). In sum: *options like volatility.* This fits our intuition. The more uncertain things are (the higher the volatility), the more valuable protection against adversity – or insurance against it – becomes (the higher the option price).
(v) *rho* is $\partial c/\partial r$, the sensitivity to interest rates.

## §8. American Options.

We now consider an American call option (value $C$), in the simplest case of a stock paying no dividends. The following result goes back (at least) to R. C. MERTON in 1973.

**Theorem (Merton's theorem)**. It is never optimal to exercise an American call option early. That is, the American call option is equivalent to the European call, so has the same value:

$$C = c.$$

*First Proof.* Consider the following two portfolios:
I: one American call option plus cash $Ke^{-rT}$; II: one share.
The value of the cash in I is $K$ at time $T$, $Ke^{-r(T-t)}$ at time $t$. If the call option is exercised early at $t < T$, the value of Portfolio I is then $S_t - K$ from the call, $Ke^{-r(T-t)}$ from the cash, total

$$S_t - K + Ke^{-r(T-t)}.$$

Since $r > 0$ and $t < T$, this is $< S_t$, the value of Portfolio II at $t$. So Portfolio I is *always* worth less than Portfolio II if exercised *early.*

If however the option is exercised instead at expiry, $T$, the American call option is then the same as a European call option. Then at time $T$, Portfolio I is worth $\max(S_T, K)$ and Portfolio II is worth $S_T$. So:

$$\begin{array}{ll} \text{before} \quad T, & I < II, \\ \text{at} \quad T, & I \geq II \text{ always, and } > \text{sometimes.} \end{array}$$

This direct comparison with the underlying [the share in Portfolio II] shows that early exercise is never optimal. Since an American option at expiry is the same as a European one, this completes the proof. //
*Second Proof.* One can instead use the bounds of §7.1. For details, see e.g. [BK, Th. 4.7.1].

*Financial Interpretation.*

There are two reasons why an American call should not be exercised early:
1. *Insurance.* Consider an investor choosing to hold a call option instead of the underlying stock. He does not care if the share price falls below the strike price (as he can then just discard his option) – but if he held the stock, he would. Thus the option *insures* the investor against such a fall in stock price, and if he exercises early, he loses this insurance.
2. *Interest on the strike price.* When the holder exercises the option, he buys the stock and pays the strike price, $K$. Early exercise at $t < T$ loses the interest on $K$ between times $t$ and $T$: the later he pays out $K$, the better.
*Economic Note.* Despite Merton's theorem, and the interpretation above, there are plenty of real-life situations where early exercise of an American call might be sensible, and indeed done routinely. Consider, for example, a manufacturer of electrical goods, in bulk. He needs a regular supply of large amounts of copper. The danger is future price increases; the obvious precaution is to hedge against this by buying call options. If the expiry is a year but copper stocks are running low after six months, he would exercise his American call early, to keep an adequate inventory of copper, his crucial raw material. This ensures that his main business activity – manufacturing – can continue unobstructed. Neither of the reasons above applies here:
*Insurance.* He doesn't care if the price of copper falls: he isn't going to sell his copper stocks, but use them.
*Interest.* He doesn't care about losing interest on cash over the remaining six months. He is in manufacturing to use his money to make things, and then sell them, rather than put it in the bank.

This neatly illustrates the contrast between *finance* (money, options etc.) and *economics* (the real economy – goods and services).

*Put-Call Symmetry.*

The BS formulae for puts and calls resemble each other, with stock price $S$ and discounted strike $K$ interchanged. Results of this type are called *put-call symmetry.*

**American Puts.**

Recall the put-call parity of Ch. I (valid only for European options): $c - p = S - Ke^{-rT}$. A partial analogue for American options is given by the inequalities below:

$$S - K < C - P < S - Ke^{-rT}.$$

For proof (as above) and background, see e.g. Ch. 8 (p. 216) of [H1].

We now consider how to evaluate an American put option, European and American call options having been treated already. First, we will need to work in discrete time. We do this by dividing the time-interval $[0, T]$ into $N$ equal subintervals of length $\Delta t$ say. Next, we take the values of the underlying stock to be discrete: we use the binomial model of §5, with a slight change of notation: we write $u, d$ ('up', 'down') for $(1 + b), (1 + a)$: thus stock with initial value $S$ is worth $Su^i d^j$ after $i$ steps up and $j$ steps down. Consequently, after $N$ steps, there are $N + 1$ possible prices, $Su^i d^{N-i}$ $(i = 0, \cdots, N)$. It is convenient to display the possible paths followed by the stock price as a binomial tree, with time going left to right and two paths, up and down, leaving each node in the tree, until we reach the $N + 1$ terminal nodes at expiry. There are $2^N$ possible paths through the tree. It is common to take $N$ of the order of 30, for two reasons:
(i) typical lengths of time to expiry are measured in months (9 months, say); this gives a time-step around the corresponding number of days,
(ii) $2^{30}$ paths is about the order of magnitude that can be easily handled by computers (recall that $2^{10} = 1,024$, so $2^{30}$ is somewhat over a billion).

We now return to the binomial model in §§5,6, with a slight change of notation. Recall that in §5 (discrete time) we used $1 + r$ for the discount factor. Now call this $1 + \rho$ instead, freeing $r$ for its usual use as the short rate of interest in continuous time. Thus $1 + \rho = e^{r\Delta t}$, and the risk-neutrality condition $p^* = (b - r)/(b - a)$ of §5 becomes

$$p^* = (u - e^{r\Delta t})/(u - d).$$

Now recall (§7) $(1+a)/(1+r) = \exp(-\sigma/\sqrt{N})$, $(1+b)/(1+r) = \exp(\sigma/\sqrt{N})$. We replaced $\sigma^2$ by $\sigma^2 T$ (to make $\sigma$ the volatility per unit time), and $T = N.\Delta t$, so $\sigma/\sqrt{N}$ becomes $\sigma\sqrt{T}/\sqrt{N} = \sigma\sqrt{\Delta t}$. So now

$$u/e^{r\Delta t} = e^{\sigma/\sqrt{\Delta t}}, \qquad d/e^{r\Delta t} = e^{-\sigma\sqrt{\Delta t}} : \qquad ud = e^{2r\Delta t}.$$

Since $\sqrt{\Delta t}$ is small, its square $\Delta t$ is a second-order term. So to first order, $ud = 1$, which simplifies filling in the terminal values in the binary tree.

We begin again: define our up and down factors $u, d$ so that

$$ud = 1; \qquad (*)$$

define the risk-neutral probability $p^*$ so as to have

$$p^* = (u - e^{r\Delta t})/(u - d)$$

(to get the mean return from the risky stock the same as that from the riskless bank account), and the volatility $\sigma$ to get the variance of the stock price $S'$ after one time-step when it is worth $S$ initially as $S^2\sigma^2\Delta t$:

$$S^2\sigma^2\Delta t = p^* S^2 u^2 + (1 - p^*)S^2 d^2 - S^2[p^* u + (1 - p^*)d]^2$$

(using $var S' = E(S'^2) - [ES']^2$). Then to first order in $\sqrt{\Delta t}$ (which is all the accuracy we shall need), one can check that we have as before

$$u = \exp(\sigma\sqrt{\Delta t}), \qquad d = \exp(-\sigma\sqrt{\Delta t}). \qquad (**)$$

We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree (this method of backward recursion in time is a form of the Dynamic Programming [DP] technique (Richard Bellman (1920-84) in 1953, book, 1957), which is important in many areas of optimization and Operational Research (OR)).

1. Draw a binary tree showing the initial stock value and having the right number, $N$, of time-intervals.

2. Fill in the stock prices: after one time interval, these are $Su$ (upper) and $Sd$ (lower); after two time-intervals, $Su^2$, $S$ and $Sd^2 = S/u^2$; after $i$ time-intervals, these are $Su^j d^{i-j} = Su^{2j-i}$ at the node with $j$ 'up' steps and $i - j$ 'down' steps (the '$(i, j)$' node).

3. Using the strike price $K$ and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) from the option at the terminal nodes (where, at expiry, the values of the European and American options coincide) underneath the terminal prices.

4. Work back down the tree one time-step. Fill in the 'European' value at the penultimate nodes as the discounted values of the upper and lower right (terminal node) values, under the risk-neutral measure – '$p^*$ times lower right plus $1 - p^*$ times upper right' [notation of IV.6 L18]. Fill in the 'intrinsic' (or early-exercise) value - the value if the option is exercised. Fill in the American put value as the higher of these.