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Lecture 18 17.11.2016

Proof of the Black-Scholes formula.

It suffices to take $t = 0$ – so T is the *remaining* time to expiry.

We use the Lemma, with $\mu = -\frac{1}{2}\sigma^2 T$ (in our new notation). In (1), we have $Y_N \to Y$ in distribution and (replacing R in the Lemma by r, as above) $(1 + \frac{rT}{N})^{-N} \to e^{-rT}$ as $N \to \infty$. This suggests

$$C_0^{(N)} \to C_0 := E_Y[(S_0 e^Y - e^{-rT} K)_+] = e^{-rT} E_Y[(S_0 e^{rT+Y} - K)_+],$$

where E_Y is the expectation for the distribution of Y, which is $N(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$ (in our current notation). This can be justified, by standard properties of convergence in distribution (see e.g. [W], Ch. 18). So if $Z := (Y + \frac{1}{2}\sigma^2 T)/(\sigma\sqrt{T})$, $Z \sim N(0,1)$, $Y = -\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} Z$, and

$$C_0 = e^{-rT} E_Z[(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z} - K)_+]$$

$$= e^{-rT} \int_{-\infty}^{\infty} [S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} - K]_+ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

Similarly, with payoff h, the time-0 price of the claim, or option is

$$e^{-rT} \int_{-\infty}^{\infty} h(S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\}) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx. \qquad (*)$$

To evaluate the integral, note first that $[...] > 0$ where

$S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} > K: \quad x > [\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T} = c,$ say. So

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2 T}.\exp\{-\frac{1}{2}x^2 + \sigma\sqrt{T}x\} dx/\sqrt{2\pi} - K e^{-rT}[1 - \Phi(c)],$$

and the last term is $K e^{-rT}\Phi(-c) = K e^{-rT}\Phi(d_-)$ (L17: $-c = [\log(S/K) + (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T} = d_-$, when $t = 0$). The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x - \sigma\sqrt{T})^2\} dx/\sqrt{2\pi} = \int_{c-\sigma\sqrt{T}}^{\infty} \exp\{-\frac{1}{2}u^2\} du/\sqrt{2\pi}$$

$$= 1 - \Phi(c - \sigma\sqrt{T}) = \Phi(-c + \sigma\sqrt{T}) = \Phi(d_+),$$

as $-c + \sigma\sqrt{T} = d_- + \sigma\sqrt{T} = d_+$ when $t = 0$. So the option price is given in terms of the initial price S_0, strike price K, expiry T, interest rate r and volatility σ by

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \qquad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}. \ //$$

Note. 1. *Normal approximation to binomial.* The proof above starts from a binomial distribution and ends with a normal distribution. The binomial distribution is that of a sum of independent Bernoulli random variables. That sums (or averages) of independent random variables with finite means and variances gives a normal limit is the content of the *Central Limit Theorem* or CLT (the *Law of Errors*, as physicists would say). This form of the CLT is the *de Moivre-Laplace limit theorem.*

The picture for this is familiar. The Binomial distribution $B(n, p)$ has a histogram with $n + 1$ bars, whose heights peak at the mode and decrease to either side. For large n, one can draw a smooth curve through the histogram. This curve is the relevant approximating normal density.

2. The Cox-Ross-Rubinstein binomial model above goes over in the passage to the limit to the geometric Brownian motion model of VI.1. We will later re-derive the continuous Black-Scholes formula in Ch. VI, using continuous-time methods (Itô calculus), rather than, as above, deriving the discrete BS formula and going to the limit on the *formula*, rather than the *model.*

3. For similar derivations of the discrete Black-Scholes formula and the passage to the limit to the continuous Black-Scholes formula, see e.g. [CR], §5.6.

4. One of the most striking features of the Black-Scholes formula is that it does **not** involve the mean rate of return μ of the stock – only the riskless interest-rate r and the volatility of the stock σ. Mathematically, this reflects the fact that the *change of measure* involved in the passage to the risk-neutral measure involves a *change of drift.* This eliminates the μ term; see Ch. VI.

5. *Volatility.* The volatility σ can be estimated in two ways:

a. *Historic volatility.* Directly from the movement of a stock price in time, using Time Series methods in discrete time [see Ch. VI for continuous time].

b. *Implied volatility.* From the observed market prices of options: if we know everything in the Black-Scholes formula (including the price at which the option is traded) *except* the volatility σ, we can solve for σ. Since σ appears inside the argument of the normal distribution function Φ as well as outside it, this is a transcendental equation for σ and has to be solved numerically by iteration (Newton-Raphson method). We quote (see 'The Greeks' below, and

Problems 7) that the Black-Scholes price is a monotone (increasing) function of the volatility (more volatility doesn't make us 'more likely to win', but when we do win, we 'win bigger'), so there is a unique root of the equation.

In practice, one sees discrepancies between historic and implied volatility, which show limitations to the accuracy of the Black-Scholes model. But it is the standard 'benchmark model', and useful as a first approximation.

The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account periods when the markets are open and closed [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This suggests that *trading itself is one of the main causes of volatility.*

Note. This observation has deep implications for the macro-prudential and regulatory issues discussed in Ch. 1. The real economy cannot afford too much volatility. Volatility is (at least partly) caused by trading. Conclusion: there is too much trading. Policy question: how can we reduce the volume of trading (much of it speculative, designed to enrich traders, and not serving a more widely useful economic purpose)? One answer is the so-called *Tobin tax* (also known as the "Robin Hood tax") (James Tobin (1918-2002), American economist; Nobel Prize for Economics, 1981). This would levy a small charge (e.g. 0.01%) on *all* financial transactions. This would both provide a major and useful source of tax revenue, and – more importantly – would discourage a lot of speculative trading, thereby (shrinking the size of the financial services industry, but) diminishing volatility, to the benefit of the general economy (Problems 7 again).

If the Black-Scholes model were perfect, historic and implied volatility estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model.

Volatility estimation is a major topic, both theoretically and in practice. We return to this in IV.7.3-4 below and VI.7.5-8. But we note here:
(i) trading is itself one of the major causes of volatility, as above;
(ii) options like volatility [i.e., option prices go up with volatility].
Recalling Ch. I, this shows that volatility is a 'bad thing' from the point of view of the real economy (uncertainty about, e.g., future material costs is nothing but a nuisance to manufacturers), but a 'good thing' for financial markets (trading increases volatility, which increases option prices, which generates more trade ...) – at the cost of increased instability.

§7. More on European Options

1. *Bounds.* We use the notation above. We also write c, p for the values of European calls and puts, C, P for the values of the American counterparts.

Obvious upper bounds are $c \leq S, C \leq S$, where S is the stock price (we can buy for S on the market without worrying about options, so would not pay more than this for the right to buy). For puts, one has correspondingly the obvious upper bounds $p \leq K, P \leq K$, where K is the strike price: one would not pay more than K for the right to sell at price K, as one would not spend more than one's maximum return. For lower bounds:

$c_0 \geq \max(S_0 - Ke^{-rT}, 0)$.

Proof. Consider the following two portfolios:

I: one European call plus Ke^{-rT} in cash; II: one share. Show "I $\geq$ II".

$p_0 \geq \max(Ke^{-rT} - S_0, 0)$ (proof: by above and put-call parity).

2. *Dependence of the Black-Scholes price on the parameters.*

Recall the Black-Scholes formulae for the values c_t, p_t for the European call and put: with

$$d_{\pm} := [\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{(T-t)},$$

$$c_t = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \qquad p_t = Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+),$$

(a). S. As the stock price S increases, the call option becomes more likely to be exercised. As $S \to \infty$, $d_{\pm} \to \infty$, $\Phi(d_{\pm}) \to 1$, so $c_t \sim S_t - Ke^{-r(T-t)}$. This has a natural economic interpretation: as the value of a *forward contract* with *delivery price* K (Hull [H1] Ch. 3, [H2] Ch. 3).

(b). σ. When the volatility $\sigma \to 0$, the stock becomes riskless, and behaves like money in the bank. Again, $d_{\pm} \to \infty$, as above, with the same economic interpretation.

3. *The Greeks.*

These are the partial derivatives of the option price with respect to the input parameters. They have the interpretation of *sensitivities.*

(i) For a call, say, $\partial c/\partial S$ is called the *delta*, Δ. Adjusting our holdings of stock to eliminate our portfolio's dependence on S is called *delta-hedging.*

(ii) Second-order effects involve *gamma* $:= \partial(\Delta)/\partial S$.

(iii) Time-dependence is given by *Theta* is $\partial c/\partial t$.

(iv) Volatility dependence is given by *vega* $:= \partial c/\partial \sigma$.[1]

[1] Of course, vega is not a letter of the Greek alphabet! (it is the Spanish word for 'meadow', as in Las Vegas) – presumably so named for "v for volatility, variance and vega", and because vega sounds quite like beta, etc.