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Lecture 11 28.10.2016
Chapter III. STOCHASTIC PROCESSES IN DISCRETE TIME.

§1. Filtrations.

The Kolmogorov triples $(\Omega, \mathcal{F}, P)$, and the Kolmogorov conditional expectations $E(X|\mathcal{B})$, give us all the machinery we need to handle *static* situations involving randomness. To handle *dynamic* situations, involving randomness which unfolds with *time*, we need further structure.

We may take the initial, or starting, time as $t = 0$. Time may evolve discretely, or continuously. We postpone the continuous case to Ch. V; in the discrete case, we may suppose time evolves in integer steps, $t = 0, 1, 2, \cdots$ (say, stock-market quotations daily, or tick data by the second). There may be a final time T, or *time horizon*, or we may have an infinite time horizon (in the context of option pricing, the time horizon T is the expiry time).

We wish to model a situation involving randomness unfolding with time. We suppose, for simplicity, that information is never lost (or forgotten): thus, as time increases we learn more. Recall that σ-fields represent information or knowledge. We thus need a sequence of σ-fields $\{\mathcal{F}_n : n = 0, 1, 2, \cdots\}$, which are increasing:

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \qquad (n = 0, 1, 2, \cdots),$$

with $\mathcal{F}_n$ representing the information, or knowledge, available to us at time n. We shall always suppose all σ-fields to be *complete* (this can be avoided, and is not always appropriate, but it simplifies matters and suffices for our purposes). Thus $\mathcal{F}_0$ represents the initial information (if there is none, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field). On the other hand,

$$\mathcal{F}_\infty := lim_{n \to \infty} \mathcal{F}_n$$

represents all we ever will know (the 'Doomsday σ-field'). Often, $\mathcal{F}_\infty$ will be $\mathcal{F}$ (the σ-field from Ch. II, representing 'knowing everything'. But this will not always be so; see e.g. [W], §15.8 for an interesting example.

Such a family $\{\mathcal{F}_n : n = 0, 1, 2, \cdots\}$ is called a *filtration*; a probability space endowed with such a filtration, $\{\Omega, \{\mathcal{F}_n\}, \mathcal{F}, \mathcal{P}\}$ is called a *filtered probability space*. (These definitions are due to P.- A. MEYER of Strasbourg; Meyer and the Strasbourg (and more generally, French) school of probabilists have been responsible for the ‘general theory of [stochastic] processes’, and for much of the progress in stochastic integration, since the 1960s.) Since the filtration is so basic to the definition of a stochastic process, the more modern term for a filtered probability space is a *stochastic basis.*

§2. Discrete-Parameter Stochastic Processes.

A *stochastic process* $X = \{X_t : t \in I\}$ is a family of random variables, defined on some common probability space, indexed by an index-set I. Usually (always in this course), I represents *time* (sometimes I represents *space*, and one calls X a spatial process). Here, $I = \{0, 1, 2, \cdots, T\}$ (finite horizon) or $I = \{0, 1, 2, \cdots\}$ (infinite horizon – as in VI.6 L30, Real/Investment options).

The (stochastic) process $X = (X_n)_{n=0}^{\infty}$ is said to be *adapted* to the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ if

$$X_n \text{ is } \mathcal{F}_n - \text{measurable.}$$

So if X is adapted, we will know the value of X_n at time n. If

$$\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$$

we call $(\mathcal{F}_n)$ the *natural filtration* of X. Thus a process is always adapted to its natural filtration. A typical situation is that

$$\mathcal{F}_n = \sigma(W_0, W_1, \cdots, W_n)$$

is the natural filtration of some process $W = (W_n)$. Then X is adapted to $(\mathcal{F}_n)$, i.e. each X_n is $\mathcal{F}_n$- (or $\sigma(W_0, \cdots, W_n)$-) measurable, iff

$$X_n = f_n(W_0, W_1, \cdots, W_n)$$

for some measurable function f_n (non-random) of $n + 1$ variables.
Notation.

For a random variable X on $(\Omega, \mathcal{F}, P)$, $X(\omega)$ is the value X takes on ω (ω represents the randomness). Often, to simplify notation, ω is suppressed - e.g., we may write $E[X] := \int_\Omega X dP$ instead of $E[X] := \int_\Omega X(\omega) dP(\omega)$.

For a stochastic process $X = (X_n)$, it is convenient (e.g., if using suffices, n_i say) to use X_n, $X(n)$ interchangeably, and we shall feel free to do this. With ω displayed, these become $X_n(\omega)$, $X(n, \omega)$, etc.

§3. Discrete-Parameter Martingales.

We summarise what we need; for details, see [W], or e.g. [N]

Definition.

A process $X = (X_n)$ is called a *martingale* (mg for short) relative to $((\mathcal{F}_n), P)$ if

(i) X is adapted (to $(\mathcal{F}_n)$),

(ii) $E[|X_n|] < \infty$ for all n,

(iii) $E[X_n|\mathcal{F}_{n-1}] = X_{n-1} \qquad P-a.s. \qquad (n \geq 1)$;

X is a *supermartingale* if in place of (iii)

$$E[X_n|\mathcal{F}_{n-1}] \leq X_{n-1} \qquad P-a.s. \qquad (n \geq 1);$$

X is a *submartingale* if in place of (iii)

$$E[X_n|\mathcal{F}_{n-1}] \geq X_{n-1} \qquad P-a.s. \qquad (n \geq 1).$$

Thus: a mg is 'constant on average', and models a *fair* game;
a supermg is 'decreasing on average', and models an *unfavourable game*;
a submg is 'increasing on average', and models a *favourable* game.

Note. 1. Martingales have many connections with harmonic functions in probabilistic potential theory. The terminology in the inequalities above comes from this: supermartingales correspond to superharmonic functions, submartingales to subharmonic functions.

2. X is a submg [supermg] iff $-X$ is a supermg [submg]; X is a mg iff it is both a submg and a supermg.

3. (X_n) is a mg iff $(X_n - X_0)$ is a mg. So we may without loss of generality take $X_0 = 0$ when convenient.

4. If X is a mg, then for $m < n$

$$\begin{aligned} E[X_n|\mathcal{F}_m] &= E[E(X_n|\mathcal{F}_{n-1})|\mathcal{F}_m] \qquad \text{(iterated conditional expectations)} \\ &= E[X_{n-1}|\mathcal{F}_m] \qquad a.s. \qquad \text{(martingale property)} \\ &= \cdots = E[X_m|\mathcal{F}_m] \qquad a.s. \qquad \text{(induction on } n\text{)}, \\ &= X_m \qquad (X_m \text{ is } \mathcal{F}_m\text{-measurable}) \end{aligned}$$

and similarly for submartingales, supermartingales.

5. Examples of a mg include: sums of independent, integrable zero-mean random variables [submg: positive mean; supermg: negative mean].

From the OED: martingale (etymology unknown)
1. 1589. An article of harness, to control a horse's head.
2. Naut. A rope for guying down the jib-boom to the dolphin-striker.
3. A system of gambling which consists in doubling the stake when losing in order to recoup oneself (1815).
Thackeray: 'You have not played as yet? Do not do so; above all avoid a martingale if you do.'
Problem. Analyse this strategy.

Gambling games have been studied since time immemorial - indeed, the Pascal-Fermat correspondence of 1654 which started the subject was on a problem (de Méré's problem) related to gambling.

The doubling strategy above has been known at least since 1815.

The term 'mg' in our sense is due to J. VILLE (1939). Martingales were studied by Paul LÉVY (1886-1971) from 1934 on [see obituary, *Annals of Probability* **1** (1973), 5-6] and by J. L. DOOB (1910-2004) from 1940 on. The first systematic exposition was Doob's book [D], Ch. VII.
Example: Accumulating data about a random variable ([W], 96, 166-167).
If $\xi \in L_1(\Omega, \mathcal{F}, \mathcal{P})$, $M_n := E(\xi|\mathcal{F}_n)$ (so M_n represents our best estimate of ξ based on knowledge at time n), then

$$\begin{aligned} E[M_n|\mathcal{F}_{n-1}] &= E[E(\xi|\mathcal{F}_n)|\mathcal{F}_{n-1}] \\ &= E[\xi|\mathcal{F}_{n-1}] \qquad \text{(iterated conditional expectations)} \\ &= M_{n-1}, \end{aligned}$$

so (M_n) is a mg. One has the convergence

$$M_n \to M_\infty := E[\xi|\mathcal{F}_\infty] \qquad a.s. \qquad \text{and in } L_1;$$

see II.4 below.

§4. Martingale Convergence.

A supermartingale is 'decreasing on average'. Recall that a decreasing sequence [of real numbers] that is bounded below converges (decreases to its greatest lower bound or infimum). This suggests that a supermartingale which is bounded below converges a.s. This is so [Doob's Forward Convergence Theorem: [W], §§11.5, 11.7].