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M3F22/4A22/5F22 MATHEMATICAL FINANCE

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Imperial College London, 6 October – 16 December 2016

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Course website: My homepage.

Mon 10-11, 340; Th 1-2, 340; Fri 9-10, 139,

+ Fri 10-11 [MSc in Math. Finance students only], 139

There will be no lecture on Thursday 3 November.

Books.

Course text: Ch. 1-6 of

[BK] N. H. BINGHAM and Rüdiger KIESEL: *Risk-neutral valuation: Pricing and hedging of financial derivatives*, 2nd ed., CUP, 2004 (1st ed. 1998).

Alternatives:

S. E. SHREVE: *Stochastic calculus for finance. Vol. I: The binomial asset pricing model; Vol. II: Continuous-time models*, Springer, 2004.

T. MIKOSCH: *Elementary stochastic calculus, with finance in view*, World Scientific, 1998.

Course Website: M3F22 link on my home-page (Imperial College > Mathematics Department > Staff > Staff List > Bingham > Homepage – or Google Nick (or Nicholas, or N H) Bingham: favouritize this to get it in one click).

Other relevant links (on my home-page):

[SP] Stochastic Processes [30 hours, MSc, Mathematatial Finance];

[SA] Stochastic Analysis [20 hours, MSc].

[LTCC] Measure-theoretic probability theory [10 hours; MSc].

For background:

[PfS] Probability for Statistics;

[SMF] Statistical Methods for Finance;

[Math482] – a course along these lines I gave at Liverpool.

We shall make systematic use of *conditioning* (informally: using what we know). For background, see e.g.

[BF] N. H. BINGHAM and J. M. FRY: *Regression: Linear models in statistics*. Springer Undergraduate Mathematics Series (SUMS), Springer, 2010.

Mathematical finance, for reference

[CR] John C. COX and Mark RUBINSTEIN: *Options markets*. Prentice Hall, 1985.

[H1] HULL, J. (1995): *Introduction to futures and options markets* (2nd ed), Prentice-Hall, ('baby Hull'), or

[H2] HULL, J. (1993): *Options, futures and other derivative securities* (2nd ed.), Prentice-Hall ('Hull').

Background and general interest

[K1] John KAY, *The Truth about Markets: Their Genius, their Limits, their Follies*. Penguin/Allen Lane, 2003.

[K2] John KAY, *Other People's Money: Finance: Masters of the Universe or Servants of the People?* Profile Books, 2015, £16.99).

[AH] Anat R. ADMATI & Martin HELLWIG, *The Bankers' New Clothes: What's Wrong with Banking and What to Do about it*, Princeton UP, 2013.

[G] Alan GREENSPAN, *The age of turbulence*. Penguin, 2007.

I think the Kay books are essential reading for anyone thinking of working in the financial services industry – maybe [AH] too. I thoroughly recommend [G] – but get the latest edition of it that you can. The author was Chairman of the US Federal Reserve (Fed) 1987-2006. His views up to 2007 were largely Panglossian optimism (markets know best, and are self-correcting, etc.). The ongoing problems since have forced a re-think; see the epilogues to later editions, his evidence to the House Committee, etc.

Mathematics, for reference

[D] J. L. DOOB: *Stochastic processes*, Wiley, 1953.

[N] J. NEVEU: *Discrete-parameter martingales*, North-Holland, 1975.

[KS] KARATZAS, I. & SHREVE, S. (1988): *Brownian motion and stochastic calculus*. Graduate Texts in Math. **113**, Springer.

[RY] REVUZ, D. & YOR, M. (1999): *Continuous martingales and Brownian motion*. Grundlehren der math. Wiss. **293**, Springer, 3rd ed. (1st ed. 1991, 2nd ed. 1994,).

[RW1] ROGERS, L. C. G. & WILLIAMS, D. (1994): *Diffusions, Markov processes and martingales, Volume 1: Foundation*, 2nd ed.

[RW2] ROGERS, L. C. G. & WILLIAMS, D. (1987): *Diffusions, Markov processes and martingales, Volume 2: Itô calculus*. Wiley.

Mathematics for reference: Insurance Mathematics

[A] S. ASMUSSEN, *Applied probability and queues*, 2nd ed., Springer, 2003 [1st ed. Wiley 1987].
[AA] S. ASMUSSEN and H. ALBRECHER, *Ruin probabilities*, 2nd ed., World Scientific, 2010 [1st ed., S. Asmussen, 2000].
[Kyp] Andreas E. Kyprianou, *Fluctuations of Lévy processes with applications: Introductory lectures*, 2nd ed., Universitext, Springer, 2014 [1st ed. 2006].
[RSST] T. ROLSKI, H. SCHMIDLI, V. SCHMIDT and J. L. TEUGELS, *Stochastic processes for insurance and finance*, Wiley, 1999.

Assessed Coursework: One assignment, 10% credit, Week 6 (due Week 7).

CONTENTS

§2. Discrete-parameter stochastic processes [L11]
§3. Discrete-parameter martingales [L11]
§4. Martingale convergence [L11-12]
§5. Martingale transforms [L12]
§6. Stopping times and optional stopping [L12-13]
§7. The Snell envelope and optimal stopping [L13]
§8. Doob decomposition [L13]
§9. Examples [L13]

IV. MATHEMATICAL FINANCE IN DISCRETE TIME [$6\frac{1}{2}$ h: L14-20].
§1. The model [L14]
§2. Viability: existence of equivalent martingale measures (EMMs) [L14-15]
§3. Complete markets: uniqueness of equivalent mg measures [L15-16]
§4. The Fundamental Th. of Asset Pricing: Risk-Neutral Valuation [L16]
§5. European options. The discrete Black-Scholes formula [L16-17]
§6. Continuous-time limit of the binomial model [L17-18]
§7. More on European options [L18-19]
§8. American options [L19-20]

V. STOCHASTIC PROCESSES IN CONTINUOUS TIME [5 h: L20-25].
§1. Brownian motion [L20-21]
§2. Filtrations; finite-dimensional distributions [L21-22]
§3. Classes of processes [L22]
§4. Quadratic variation (QV) of Brownian motion; Itô's Lemma [L22-23]
§5. Stochastic integrals; Itô calculus [L23-24]
§6. Stochastic differential equations (SDEs); Itô's Lemma [L24-25]

VI. MATH. FINANCE IN CONTINUOUS TIME [$6\frac{1}{2}$ h: L25-31].
§1. Geometric Brownian motion (GBM) [L25]
§2. The Black-Scholes model [L26]
§3. The (continuous) Black-Scholes formula (BS): derivation via Girsanov's theorem [L26-28]
§4. BS via the Black-Scholes PDE and the Feynman-Kac formula [L28-29]
§5. Infinite time-horizon; American puts [L29-30]
§6. Real options (Investment options) [L30-31]
§7. Stochastic volatility (SV); Postscript [L31]

VII. INSURANCE MATHEMATICS [MSc in Financial Mathematics students only; Fridays, 10-11, as time left over from questions, Problems and Solutions allows]

§1. Insurance background.
§2. The Poisson process; compound Poisson processes.
§3. Renewal theory.
§4. The ruin problem.
§5. Complements.

Exam
MSc in Financial Mathematics students.
Syllabus: Ch. I-VII above.
Exam in January. Same format in previous years: 6 questions, do 5.

You will be taking this course in tandem with Tom Cass's M5F3 Stochastic Processes course; Tom and I will be liaising on content and timing.
Everyone else: 3rd year BSc, 4th year MSci, 5th year, other MScs.
Syllabus: Ch. I-VI above.
Exam in the summer. Again, same format as before: M3F22, 4 questions; M4F22/M5F22, five questions; Q5 Mastery Question.

About the course.

The course is an introduction to mathematical finance and option pricing – Black-Scholes theory via martingales. The Black-Scholes formula (of 1973, which enabled the pricing of options on an industrial scale) changed the world, and is certainly the most important new formula of the last part of the last century. It is basic to the financial services industry. So: do the course either for academic interest, or if you have any career interest in the financial services industry – or both.

This is not a get-rich-quick course (if you wanted to get rich quick, why did you choose mathematics?).

On the 'finance' side, you will learn something about Economics (below); this is very important, and will be new to many of you. On the 'mathematical' side, you will learn *stochastic* (or Itô, or martingale) *calculus.* Calculus is the most powerful weapon we have – in mathematics, or in science generally. You learn it at school, where it is a revelation; you re-learn it doing Mathematics at university (less fun at first, for most people). The first obviously new and obviously powerful kind of calculus you learn here is Complex Analysis in Semester 4. Think of stochastic calculus as a sequel to that.

About the course: Content.

There are two things here, one pure mathematical, one non-mathematical.

(a) *Measure Theory.*

'Grown-up probability', needed here (Itô calculus, etc.) is measure-theoretic. Ideal preparation would be a full course in Measure Theory (such as M3P19 Measure and Integral, Autumn Term), and then Probability (M3P6, Spring Term). But only a minority of students attending this course will have had these. So, we deal with necessary measure-theoretic preliminaries in Chapter II. In the time available, one cannot prove the guts of technical measure theory – the key approximation arguments. So we quote these, confining proofs to what goes before and what comes afterwards (both much easier).

(b) *Economics.*

Finance is a small and specialised part of Economics. Ideal preparation would also include a good grounding in Economics. There would not be room for this in the Maths curriculum here, and most of you will not have an Economics qualification from school. So again, we have to take a lot for granted; we cover the necessary economic and financial background, in Chapter I.

In this regard, please bear three things in mind:

1. Anything important enough becomes political (M. Maurice Couve de Murville). This stuff is certainly important.
2. Politics in not an exact science (Bismarck). But,
3. Mathematics is an exact science.

We will be doing lots of mathematics – in particular, we derive the Black-Scholes formula. We will extend calculus, the most powerful single weapon we have, to become probabilistic (Itô calculus) and apply it to these problems. But, there are limits to which finance, economics, or anything involving human psychology, is mathematicisable. As always in Applied Mathematics, we have to be on guard: if we don't simplify enough, we can't do anything; if we over-simplify, we can do things, but can't trust our conclusions.

Just as important as the technical mathematics, you need to think about the systemic faults at the geofinancial/economic/political level thrown up by the crisis of 2007-08 on (Credit Crunch, etc.). Any prospective employer in the financial services industry should ask you questions about this, and your views on it, in interview. We spend half of one lecture on such things (I.10, L6 – not examinable). The rest is down to you.

For a range of views here, see e.g.

Quantitative Finance 15 no. 4 (2015), Special Issue on Interlinkages & Systemic Risk, esp. Dempster's review of the Admati-Hellwig book, 579-582;
N. H. BINGHAM: The Crash of 2008: A mathematician's view. *Significance* **5** no. 4 (2008), 173-175 [on my home-page, under Papers]. NHB