M5MF22 MATHEMATICAL FINANCE: EXAM SOLUTIONS 2014-15

Q1. (i) *Size of market participants.* We assume in mathematical finance that market participants are price takers and not price makers. This is true to a first approximation, depending on the size of the trade and the degree of detail. The price is the level at which markets clear – i.e., at which supply and demand currently balance. But each trade itself alters the current balance of supply and demand, and moves the price.

For small market participants, the assumption of 'price taker, not price maker' is a reasonable approximation. For large market participants, it is not: large trades visibly move prices, at any level of detail. **[5]**

(ii) *Normal v. crisis market conditions.* For heavily traded liquid stocks under normal market conditions, we observe a large number of individually small trades, with the price discontinuous in detail but continuous viewed 'from a distance'. This phenomenon is called *jitter*. We can model it using a continuous process (e.g. geometric BM, as in the Black-Scholes (BS) model – complete), or using, e.g. an infinite-activity Lévy process (incomplete).

Financial and economic crises create *shocks*, which affect prices of all assets. We can model these using e.g. a compound Poisson process.

Markowitzian diversification breaks down under crisis conditions. We cannot diversify: prices of all assets fall, and the negative correlation in a balanced portfolio that protects us under normal market conditions ('what we lose on the swings we gain on the roundabouts') no longer works. **[5]**

(iii) *Continuous v. jump price processes.* Continuous price processes are convenient: they occur in the benchmark BS model, and give a reasonable approximation. The BS model is complete, so prices are unique. Problems include the symmetry and ultra-light tails of the BS model, neither realistic: real prices show *skew*, and much fatter tails than normal. Jump price processes avoid these, but give incomplete markets, where prices are non-unique ('bid-ask spread'). **[5]**

(iv) *Discrete v. continuous time.* Time has both discrete and continuous aspects, depending how we measure it. Continuous time allows us to use Itô calculus and BS. But as BM has infinite variation, hedging is now problematic. Discrete time is more realistic, in that trading can only be done discretely in practice. But the mathematics is more complicated: e.g., discrete BS (binomial sum) is more complicated than continuous BS. **[5]**

[Mainly seen – lectures]

Q2: *Copper option.* (i) *Pricing.* With H the payoff of the call option C in a year's time: H is 100 if the copper price goes up, 0 if it goes down.

We determine the risk-neutral probability p^* so as to make the option a fair game [martingale]:

$$6,720 = p^*.6,820 + (1-p^*).6,570 = 250p^* + 6,570 : \quad 150 = 250p^* :$$

$$p^* = 3/5 = 0.6.$$

The value of the option at time 0 is

$$V_0 = E^*[H] = p^*.100 + (1-p^*).0 = 100.(0.6) = 60. \qquad [\mathbf{6}]$$

(ii) *Hedging.* The call C is financially equivalent to a portfolio Π consisting of a combination of cash and copper, as the binomial model is *complete* – all contingent claims (options etc.) can be *replicated.* To find *which* combination (ϕ_0, ϕ_1) of cash and copper, we solve two simultaneous linear equations, for the 'up' and 'down' states:

$$100 = \phi_0 + 6,820\phi_1,$$
$$0 = \phi_0 + 6,570\phi_1.$$

Subtract: $100 = 250\phi_1$: $\phi_1 = 2/5 = 0.4$.
Substitute: $\phi_0 = -6,570 \times 0.4 = -\$2,628$.
So the option is equivalent to the portfolio $\Pi = (-2,628, 2/5)$: *long*, 2/5 tonne copper, *short*, \$ 2,628.
Check: in a year's time,
Copper up: Π is worth (0.4).6820 - 2628 = 2728 - 2628 = 100, as H is;
Copper down: Π is worth (0.4).6570 - 2628 = 2628 - 2628 = 0, as H is. [**6**]
Arbitrage. By (i) and (ii), you know C and Π are worth 60 now.
(iii) If you see C being traded (= bought and sold) for *more* than it is worth, *sell* it, for 80. You can buy it, or equivalently the hedging portfolio Π, for 60. Pocket the risk-free profit \$ 20 now. The hedge enables you to meet your obligations to the option holder, at zero net cost. [**2**]
(iv) If you see C being traded for *less* than it is worth, *buy* it, for 40. You can sell it, equivalently Π, for 60. Pocket the risk-free profit \$ 20 now. Again, the option payoff enables you to clear your trading account, at zero net cost. [**2**]
(v) Such options are bought by manufacturers using copper (e.g., electrical goods), as an insurance policy against prices going up. [**2**]
(vi) The corresponding put options are bought by producers of copper, as an insurance policy against prices going down. [**2**]
[Similar seen: lectures and problems]

Q3 *Doubling strategy.* (i) With N the number of losses before the first win:

$$P(N = k) = P(L, L, \cdots, L(k \text{ times}), W) = (\frac{1}{2})^k . \frac{1}{2} = (\frac{1}{2})^{k+1}.$$

That is, N is geometrically distributed with parameter 1/2. As

$$\sum_{k=0}^{\infty} P(N = k) = \sum_0^{\infty} (\frac{1}{2})^{k+1} = \frac{1}{2}/(1 - \frac{1}{2}) = 1,$$

$P(N < \infty) = 1$: $N < \infty$ a.s. So one is certain to win eventually. **[4]**
(ii) Let S_n be one's fortune at time n. When $N = k$, one has losses at trials $1, 2, 3, \ldots, k$, with losses $1, 2, 4, \ldots, 2^{k-1}$, followed by a win at trial $k + 1$ (of 2^k). So one's fortune then is

$$2^k - (1 + 2 + 2^2 + \ldots + 2^{k-1}) = 2^k - (2^k - 1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is $+1$ (which, by (i), one is certain to win eventually). **[4]**
(iii) N has PGF

$$P(s) := E[s^N] = \sum_{n=0}^{\infty} s^k P(N = k) = \sum_0^{\infty} s^k . (\frac{1}{2})^{k+1}$$

$$= \frac{1}{2} \sum_0^{\infty} (\frac{1}{2} s)^k = \frac{1}{2}/(1 - \frac{1}{2} s) = 1/(2 - s) :$$

$$P'(s) = E[N s^{N-1}] = (2 - s)^{-2}; \qquad P'(1) = E[N] = 1.$$

So the mean number of losses is 1, and the mean time the game lasts is 2. **[4]**
(iv) As with the simple random walk: this is an impossible strategy to use in reality, for two reasons:
(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky. **[4]**
(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice. **[4]**
[Seen – Problems]

Q4. (i) A *filtration* $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is an increasing family of σ-fields $\mathcal{F}_t$, right-continuous ($\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$) and complete (containing all subsets of P-null sets as P-null sets). A filtration models the *information flow* of a dynamic model (stochastic process). **[4]**

(ii) A *uniformly integrable martingale* is a martingale $M = (M_t)$ satisfying the uniform integrability condition

$$\sup_t \inf_{|X_t|>A} |X_t| dP \to 0 \qquad (A \to \infty). \tag{UI}$$

Such a mg is of the form

$$X_t = E[X_T | \mathcal{F}_t],$$

where T (finite or infinite) is the time-horizon (e.g., the expiry time of an option). It gives the best estimate available at time t of a payoff H known only at T, by 'progressive revelation'. Such a UI mg converges:

$$X_t \to X_T \qquad (t \uparrow T) \qquad \text{a.s. and in } L_1. \qquad \textbf{[4]}$$

(iii) The *risk-neutral measure (RNM)* (or *equivalent martingale measure (EMM)*) P^* makes discounted asset prices martingales. The RNM *exists* if the market has *no arbitrage*; it is *unique* if the market is *complete*. So (Fundamental Theorem of Asset Pricing, FTAP) the RNM exists and is unique iff the market is arbitrage-free and complete. **[4]**

(iv) The *risk-neutral valuation formula (RNV)* is

$$V_t = E^*[e^{-r(T-t)} H | \mathcal{F}_t] \qquad (0 \leq t \leq T), \tag{RNV}$$

with E^* the P^*-expectation. The value process $V = (V_t)$ of the option with payoff H at expiry T, given by (RNV), is a UI mg. **[4]**

(v) *Insider trading.* The crux of (RNV) is that prices of options on publicly quoted risky assets at time t depend only on publicly available information at time t. Price-sensitive information is often available to numbers of people (typically, those involved in the planning and execution of mergers and acquisitions (M&A) – e.g., a hostile take-over). Such *insiders* have access to a *bigger filtration* than traders using only publicly available information. Such people are prohibited by law from seeking to profit by such inside information by trading on their own account. This is rightly regarded as theft at the expense of the market. The regulatory authorities monitor trading, and their software flags suspicious trades, which can be investigated for such insider trading. **[4]**

[Seen – lectures]

Q5. *The Ornstein-Uhlenbeck process.* (i) The Ornstein-Uhlenbeck SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[2]**
(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta C e^{-\beta t} dt + e^{-\beta t} dC = -\beta V dt + e^{-\beta t} dC,$$

so V is a solution of (OU) if $e^{-\beta t} dC = \sigma dW$, $dC = \sigma e^{\beta t} dW$, $C = c + \int_0^t e^{\beta u} dW$. So with initial velocity v_0,

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u. \qquad [4]$$

(iii) V is Gaussian, as it is obtained from the Gaussian process W by linear operations.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u} dW_u] = \int_0^t e^{\beta u} E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u)^2] = \sigma^2 \int_0^t (e^{-\beta t + \beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta u} du = \sigma^2 e^{-2\beta t} [e^{2\beta t} - 1]/(2\beta) = \sigma^2 [1 - e^{-2\beta t}]/(2\beta).$$

So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$. **[4]**
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which (subtracting off $v_0 e^{-\beta t}$ as we may) is

$$\sigma^2 E[e^{-\beta t} \int_0^t e^{\beta v} dW_v . e^{-\beta(t+u)} (\int_0^t + \int_t^{t+u}) e^{\beta w} dW_w].$$

By independence of Brownian increments, the $\int_t^{t+u}$ term contributes 0, leaving as before

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u} [1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty). \quad [4]$$

(v) The process V is Markov (a diffusion), being the solution of the SDE (OU). **[3]**
(vi) The process shows *mean reversion*, and the financial relevance is to the *Vasicek model* of interest-rate theory. **[3]**
[Seen – lectures and problems]

Q6. (i) In the continuous-time Black-Scholes model, the SDE of the stock price $S = (S_t)$ is that of *geometric Brownian motion (GBM)*:

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad (GBM)$$

with $W = (W_t)$ Brownian motion (BM), μ the mean return rate on the stock, σ the volatility of the stock. **[3]**

(ii) The solution to (GBM) is $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$. **[2]**

(iii) The paths $t \mapsto S_t$ of the solution in (ii) are continuous, as BM is. **[2]**

(iv)

$$C_t = e^{-r(T-t)} E^*[(S_T - K)_+ | \mathcal{F}_t],$$

where $\mathcal{F}_t$ is the information available at time t.

To obtain the Black-Scholes formula from this, one uses (ii) and Girsanov's theorem – which in effect replaces μ by r to get, under P^*,

$$\begin{aligned} S_T &= S_t \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\} \\ &= S_t \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{(T-t)}Z\}, \qquad Z \sim N(0,1). \end{aligned}$$

Combining, if $S_t = S$,

$$C_t = \int_{-\infty}^{\infty} [S \exp\{-\frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{(T-t)}x\} - K]_+ . \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

This can be evaluated explicitly, to give the Black-Scholes formula:

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-), \quad d_\pm := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

[4]

(v) The model is complete. This is a consequence of the Brownian Martingale Representation Theorem and the continuity of Brownian paths. **[2]**

(vi) Hedging is problematic, because it involves continuous rebalancing of the portfolio of cash and stock. This cannot be done in practice, as Brownian paths have finite quadratic variation, so infinite variation: one would need an infinite amount of rebalancing, which is impossible. **[4]**

(vii) To circumvent this, one could work in discrete time with a binomial tree model and the discrete Black-Scholes formula. Or, one could use a price process with jumps (e.g., from a Lévy process), at the cost of incompleteness and so non-uniqueness of prices. **[3]**

[Seen – lectures] NHB